1. Prove L’Hospital’s rule: Suppose that $f(z)$ and $g(z)$ are analytic functions defined in a neighbourhood of $z = a$. Show that if

$$f(a) = f^{(1)}(a) = \cdots = f^{(m-1)}(a) = 0$$

and if

$$g(a) = g^{(1)}(a) = \cdots = g^{(m-1)}(a) = 0 \text{ and } g^{(m)}(a) \neq 0,$$

then

$$\lim_{z \to a} \frac{f(z)}{g(z)}$$

exists and equals $f^{(m)}(a)/g^{(m)}(a)$.

Power series expand both about $z = a$:

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n \text{ and } g(z) = \sum_{n=0}^{\infty} d_n (z - a)^n.$$

Since

$$c_n = \frac{f^{(n)}(a)}{n!} \text{ and } d_n = \frac{g^{(n)}(a)}{n!}$$

the assumptions imply

$$f(z) = c_m (z - a)^m/m! + c_{m+1}(z - a)^{m+1} + \cdots = (z - a)^m \phi(z)$$

where $\phi(z)$ is analytic at $z = a$ and $\phi(a) = c_m$, and

$$g(z) = d_m (z - a)^m/m! + d_{m+1}(z - a)^{m+1} + \cdots = (z - a)^m \psi(z)$$

where $\psi(z)$ is analytic at $z = a$ and $\psi(a) = d_m \neq 0$. So in a neighborhood of $z = a$.

$$\frac{f(z)}{g(z)} = \frac{\phi(z)}{\psi(z)}.$$  

Since $\psi(a) \neq 0$, the right hand side is analytic at $z = a$, which means that the left hand side has a removable singularity at $z = a$. So

$$\lim_{z \to a} \frac{f(z)}{g(z)} = \frac{\phi(a)}{\psi(a)} = \frac{c_m}{d_m} = \frac{f^{(m)}(a)}{g^{(m)}(a)}.$$  

2. Explain why $1/(e^{z^2} - 1)$ has an isolated singularity at $z = 0$. Is this singularity removable, a pole or an essential singularity? If it is a pole, what is its order?
Set \( f(z) = e^{z^2} - 1 \). Then
\[
 f(z) = z^2 + z^4/2! + z^6/3! + \cdots = z^2(1 + z^2/2! + z^4/3! + \cdots) = z^2 h(z)
\]
where
\[
 h(z) := 1 + z^2/2! + z^4/3! + \cdots.
\]
Since \( h(z) \) is analytic at \( z = 0 \) and since \( h(0) = 1 \neq 0 \), \( f(z) \) has a zero of order 2 at \( z = 0 \) and
\[
 1/h(z) = \frac{z^2}{e^{z^2} - 1}
\]
is analytic and non-vanishing at \( z = 0 \). It follows that \( 1/(e^{z^2} - 1) \) has a pole of order 2 at \( z = 0 \).

3. Consider the function \( f(z) = z^3 - z^2 \).
   (i) Does \( f(z) \) have an analytic square root in a neighbourhood of \( z = 0 \)? Explain.
   
   Yes. To see this, note that \( f(z) = z^2(z-1) \). Since \( z-1 \) does not vanish in the unit disk \( \mathbb{D} \), it has an analytic logarithm in \( \mathbb{D} \) and thus an analytic square root \( r(z) \) in \( \mathbb{D} \). So
   \[
   f(z) = z^2 r(z)^2 = (zr(z))^2.
   \]
   Note: Similar reasoning says that an analytic function defined in a disk about \( z = 0 \) has a \( d \)th root in a neighbourhood of \( z = 0 \) if and only if \( f(z) \) has a zero of order divisible by \( d \). (This includes the case where the order of vanishing is zero.)
   (ii) Does \( f(z) \) have an analytic logarithm in a neighbourhood \( D(0, r) \) of \( z = 0 \)? Explain.
   
   No. The reason is that \( f \) vanishes at 0, so cannot have a logarithm in any neigbourhood of \( z = 0 \).

4. Find the Laurent expansion of
   \[
   f(z) = \frac{z^2}{(z - 1)^2(z - 2)}
   \]
in the annulus \( 1 < |z| < 2 \).
   We can ignore the \( z^2 \) as we can multiply by it at the end. The partial fraction decomposition of \( 1/(z - 1)^2(z - 2) \) is
   \[
   \frac{1}{(z - 1)^2(z - 2)} = \frac{1}{z - 2} - \frac{z}{(z - 1)^2} = \frac{1}{z - 2} - \frac{1}{z - 1} - \frac{1}{(z - 1)^2}.
   \]
On $|z| < 2$, we have
\[
\frac{1}{z - 2} = -\frac{1}{2} \cdot \frac{1}{1 - z/2} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = -\frac{1}{2} - \frac{z}{2^2} - \frac{z^2}{2^3} + \cdots
\]

On $|z| > 1$, we have
\[
\frac{1}{z - 1} = \frac{1}{z} \cdot \frac{1}{1 - 1/z} = \sum_{n=-\infty}^{-1} z^n = z^{-1} + z^{-2} + z^{-3} + \cdots
\]
and
\[
\frac{1}{(z - 1)^2} = -\frac{d}{dz} \left( \frac{1}{z - 1} \right) = -\frac{d}{dz} \left( \sum_{n=-\infty}^{-1} z^n \right) = -\sum_{n=-\infty}^{-1} nz^{n-1} = z^{-2} + 2z^{-3} + 3z^{-4} + \cdots
\]
Therefore
\[
\frac{z}{(z - 1)^2} = -\sum_{n=-\infty}^{-1} nz^n = z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + \cdots
\]
So
\[
\frac{1}{(z - 1)^2(z - 2)} = \frac{1}{z - 2} - \frac{z}{(z - 1)^2} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} + \sum_{n=-\infty}^{-1} nz^n
\]
and
\[
f(z) = \frac{z^2}{(z - 1)^2(z - 2)} = -\sum_{n=0}^{\infty} \frac{z^{n+2}}{2^{n+1}} + \sum_{n=-\infty}^{-1} nz^{n+2}
= -\sum_{m=-\infty}^{1} (m - 2) z^m + \sum_{m=2}^{\infty} \frac{z^m}{2^{m-1}}
\]
5. Suppose that $S$ is an open subset of $\mathbb{C}$ and that $f : S \to \mathbb{C}$ is analytic and has a finite number of zeros $a_1, \ldots, a_r$ in $S$. Show that if $S$ is simply connected, then $f$ has an $n$th root in $S$ if and only if $n$
divides the order of $f$ at each $a_j$. Does this hold when $S$ is not simply connected? (Either prove this or give a counter example.)

Suppose that the order of $f$ at $a_j$ is $m_j$. Then the function $g : S - \{a_1, \ldots, a_r\} \to \mathbb{C}$ defined by

$$g(z) = \frac{f(z)}{(z - a_1)^{m_1}(z - a_2)^{m_2} \cdots (z - a_r)^{m_r}}$$

has a removable singularity at each $a_j$ and is thus extends to an analytic function $g : S \to \mathbb{C}$. This function has no zeros. Since $S$ is simply connected, $g$ has a logarithm in $S$, and therefore an $n$th root $h : S \to \mathbb{C}$. If $n$ divides each $m_j$, then the function

$$h(z) = \frac{g(z)}{(z - a_1)^{m_1/n}(z - a_2)^{m_2/n} \cdots (z - a_r)^{m_r/n}}$$

is an analytic $n$th root of $f$ on $S$.

Conversely, if $f(z)$ has an $n$th root in $S$, it has an $n$th root in a disk $D(a_j, \epsilon)$ about $a_j$. By a result from class, $n$ must divide the order of $f$ at $a_j$. This completes the proof of the first part.

The result does not hold when $S$ is not simply connected. The simplest example is when $S = \mathbb{C}^*$ and $f(z) = z(z - 1)^5$. This has a zero of order 5 at $z = 1$, but it does not have an analytic 5th root in $S$. If it did have one, $h(z)$, then $h(z)/(z - 1)$ would be a 5th root of $z$ in $S$. But $z$ does not have a 5th root in $\mathbb{C}^*$. 