1. Expand

\[ f(z) = \frac{1}{z^3} \]

as a power series about \( z = -2 \). What is the radius of convergence of this power series?

We need to expand \( f(z) \) as a power series in \( z + 2 \). You can either set \( w = z + 2 \) (perhaps the safest) or else write \( z = (z + 2) - 2 \). I'll do the first. First \( z = w - 2 \). So

\[
\frac{1}{z} = \frac{1}{w - 2} = -\frac{1}{2} \frac{1}{1 - w/2} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{w^n}{2^n} = -\sum_{n=0}^{\infty} \frac{(z+2)^n}{2^{n+1}}.
\]

Since this is a geometric series, it converges if and only \( |w|/2 < 1 \), or \( |w| < 2 \). So the radius of convergence is \( R = 2 \).

Now

\[
\frac{1}{z^3} = \frac{1}{2} \left( \frac{1}{z} \right)'' = -\frac{1}{2} \frac{d^2}{dz^2} \sum_{n=0}^{\infty} \frac{(z+2)^n}{2^{n+1}}
\]

\[
= -\sum_{n=0}^{\infty} n(n-1) \frac{(z+2)^{n-2}}{2^{n+2}}
\]

\[
= -\sum_{k=0}^{\infty} (k+2)(k+1) \frac{(z+2)^k}{2^{k+4}}
\]

This series also has radius of convergence \( R = 2 \) as differentiation does not change the radius of convergence.

2. For what integers \( n \geq 0 \) (if any) is

\[
\frac{z^n}{z - \sin z}
\]

analytic at \( z = 0 \)?
First note that
\[
z - \sin z = z - \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \right) \\
= \frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \cdots \\
= z^3 \left( \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \cdots \right) \\
= z^3 h(z)
\]
where
\[
h(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n + 3)!}.
\]
This series has radius of convergence \( R = \infty \) as the series for \( \sin z \) has infinite radius of convergence and dividing by \( z^3 \) does not change the convergence when \( z \neq 0 \). Thus \( h(z) \) is entire. Since \( h(0) = 1/6 \), we know that \( 1/h(z) \) is analytic at \( z = 0 \).

Next,
\[
\frac{z^n}{z - \sin z} = \frac{z^{n-3}}{h(z)}.
\]
Since \( 1/h(0) = 6 \neq 0 \), this is analytic at \( z = 0 \) if and only if \( n - 3 \geq 0 \), or \( n \geq 3 \). To make this absolutely clear, you can see this by writing
\[
1/h(z) = \sum_{k=0}^{\infty} c_k z^k,
\]
where \( c_0 = 6 \). This will converge in some disk about \( z = 0 \). Then
\[
\frac{z^m}{h(z)} = \sum_{k=m}^{\infty} c_{k-m} z^k = \frac{1}{6} z^m + c_1 z^{m+1} + \cdots.
\]
This is analytic at \( z = 0 \) if and only if the coefficients of the \( z^k \) vanish when \( k < 0 \). That is, if and only if \( m \geq 0 \).

3.

(i) Show that \( \cos z = 0 \) if and only if \( z \) is an odd multiple of \( \pi/2 \).

The easiest way to do this is to use the definition:
\[
\cos z = \frac{e^{iz} + e^{-iz}}{2}.
\]
From this we see that \( \cos z = 0 \) if and only if \( e^{iz} = -e^{-iz} \). But since \( e^{i\pi} = -1 \), we see that \( \cos z = 0 \) if and only if
\[
e^{iz} = e^{i\pi} e^{-iz} = e^{i(\pi-z)}.
\]
Equivalently, $e^{i(2z-\pi)} = 1$. Since $e^w = 1$ if and only if $w = 2\pi in$, where $n$ is an integer, we see that $\cos z = 0$ if and only if there is an integer $n$ such that $2z - \pi = 2n\pi$. That is, $z = (2n+1)\pi/2$.

(ii) Compute

$$\oint_{|z-z_0|=1} \frac{\cos z}{z-\pi} \, dz$$

when (a) $z_0 = 0$; (b) $z_0 = \pi$, and (c) $z_0 = 3$.

A good way to understand these integrals is to first expand $\cos z$ about $z = \pi$:

$$\cos z = -1 + \sum_{n=1}^{\infty} c_n (z-\pi)^n = -1 + (z-\pi)g(z)$$

where

$$g(z) = c_1 + c_2(z-\pi) + c_3(z-\pi)^2 + \cdots = \sum_{k=0}^{\infty} c_{k+1}(z-\pi)^k.$$ 

These series converge of all $z \in \mathbb{C}$ as $\cos z$ is entire. In particular $g(z)$ is entire. So

$$\frac{\cos z}{z-\pi} = \frac{-1}{z-\pi} + g(z).$$

Since $g(z)$ is entire, $\oint_C g(z) \, dz = 0$ for all closed $C$. So

$$\oint_{|z-z_0|=1} \frac{\cos z}{z-\pi} \, dz = -\oint_{|z-z_0|=1} \frac{dz}{z-\pi} = \begin{cases} 0, & z_0 = 0, \\ -2\pi i, & z_0 = \pi, \\ -2\pi i, & z_0 = 3. \end{cases}$$

4. Suppose that $S$ is an open set that contains the closed unit disk $|z| \leq 1$. Show that if $f : S \to \mathbb{C}$ an analytic function that maps the unit disk into itself and if $f(z) = \sum_{n=0}^{\infty} c_n z^n$, then $|c_n| \leq 1$ for all $n$. Hint: use the formula for $c_n$.

Recall that if $f(z) = \sum_{n=0}^{\infty} c_n z^n$ when $|z| < R$, then for all $r$ satisfying $0 < r < R$,

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(w)}{w^{n+1}} \, dw.$$ 

Here we can take $r = 1$. Since $f$ takes the unit disk into itself, $|f(z)| < 1$ whenever $|z| < 1$. But since $f$ is analytic (and thus continuous) on $|z| \leq 1$, this implies that $|f(z)| \leq 1$ when $|z| \leq 1$. So

$$|c_n| = \left| \frac{1}{2\pi} \oint_{|z|=1} \frac{f(w)}{w^{n+1}} \, dw \right| \leq \frac{1}{2\pi} \oint_{|z|=1} |dw| \leq 1.$$
5.

(i) Let $S$ be the upper half plane $\text{Im}(z) > 0$. Show that the exponential function $z \mapsto e^{iz}$ maps $S$ into the unit disk.

Write $z = x + iy$, where $x$ and $y$ are real. Then $z$ lies in the upper half plane if and only if $y > 0$. Since

$$|e^{iz}| = |e^{ix-y}| = |e^{ix}e^{-y}| = e^{-y},$$

$|e^{iz}| < 1$ whenever $y > 0$. That is, $e^{iz}$ maps the half plane $y > 0$ into the unit disk $|w| < 1$.

(ii) Show that if $f$ is an entire function whose image lies in any half plane, then $f$ is constant. Hint: Use the first part and Liouville’s Theorem.

Every half plane is of the form $\text{Im}(az+b) > 0$ for some $a, b \in \mathbb{C}$, where $a \neq 0$. So if the image of $f(z)$ lies in $\text{Im}(az+b) > 0$, then $\text{Im}(af(z) + b) > 0$ for all $z \in \mathbb{C}$. In other words, the image of the entire function $h(z) := af(z) + b$ lies in the upper half plane. By the first part, this implies that $e^{ih(z)}$ is an entire function whose image lies in the unit disk. In particular, $e^{ih(z)}$ is a bounded entire function, and thus constant. It follows that $h(z)$ and $f(z)$ are constant.