Math 333
Solutions to Assignment 2

1. Is there an analytic function \( f(z) = u(x, y) + iv(x, y) \), where \( z = x + iy \), defined on some open subset of \( \mathbb{C} \) with \( u = x^3 - 3xy^2 - 2x^2 + 2y^2 + 1 \)? If so, find all such \( f(z) \).

If there is such an analytic function \( f(z) \), its real and imaginary parts \( u(x, y) \) and \( v(x, y) \) will satisfy the Cauchy–Riemann equations

\[ u_x = v_y \text{ and } u_y = -v_x. \]

So, to see if there is such an \( f \), we need to see if we can find \( v(x, y) \) which satisfies

\[ v_x = -u_y = 6xy - 4y \text{ and } v_y = u_x = 3x^2 - 3y^2 - 4x. \]

The general solution of the first equation is

\[ v(x, y) = 3x^2y - 4xy + \phi(y), \]

where \( \phi(y) \) is an arbitrary differentiable function of \( y \). To find \( \phi \), we differentiate this with respect to \( y \) and plug it into the second equation to obtain the equation

\[ v_y = 3x^2 - 4x + \phi'(y) = 3x^2 - 3y^2 - 4x. \]

This implies that \( \phi'(y) = -3y^2 \), so that \( \phi = -y^3 + c \), where \( c \) is an arbitrary complex constant. Thus

\[ v = 3x^2y - 4xy - y^3 + c. \]

There are no other solutions. Since \( u \) and \( v \) satisfy the Cauchy–Riemann equations on the entire complex plane, \( f = u + iv \) is analytic on all of \( \mathbb{C} \).

Remark: In fact

\[
\begin{align*}
\quad f(x + iy) &= x^3 - 3xy^2 - 2x^2 + 2y^2 + 1 + i(3x^2y - 4xy - y^3 + c) \\
&= (x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3) - 2(x^2 + 2x(iy) + (iy)^2) + (1 + ic) \\
&= (x + iy)^3 - 2(x + iy)^2 + C.
\end{align*}
\]

So all such \( f(z) \) are \( f(z) = z^3 - 2z^2 + C \), where \( C = 1 + ic \in \mathbb{C} \).

2. Suppose that \( S \) is an open subset of \( \mathbb{C} \) and that \( f : S \to \mathbb{C} \) is an analytic function whose derivative \( f'(z) \) is non-zero for all \( z \in S \). Write \( f = u + iv \), where \( u, v : S \to \mathbb{R} \).
(i) Show that each level set \( u(x, y) = c \) of \( u \) and that each level set \( v(x, y) = d \) is a smooth curve in \( S \).

The level set \( u = c \) is smooth at \( z = a + ib \) when the gradient \( \nabla u \) of \( u \) is non-zero at \( (a, b) \). If \( f \) is analytic at \( z_o = a + ib \), then

\[ f'(z_o) = u_x(a, b) + iv_x(a, b) \quad \text{and} \quad if'(z_o) = u_y(a, b) + iv_y(a, b). \]

So, if \( f'(z_o) \neq 0 \), then by the CR-equations,

\[ \nabla u(a, b) = (u_x(a, b), u_y(a, b)) \neq 0 \quad \text{and} \quad \nabla v(a, b) = (v_x(a, b), v_y(a, b)) \neq 0. \]

The implicit function theorem implies that the level curves \( u(x, y) = u(a, b) \) and \( v(x, y) = v(a, b) \) of \( u \) and \( v \) that pass through \( (a, b) \) are smooth at \( (a, b) \) and that \( \nabla u(a, b) \) and \( \nabla v(a, b) \) are normal vectors to the level curves.

(ii) Show that the intersection of a level set \( u = c \) with a level set \( v = d \) is orthogonal.

The CR-equations implies that

\[ \nabla u(a, b) \cdot \nabla v(a, b) = u_x(a, b)v_x(a, b) + u_y(a, b)v_y(a, b) \]

\[ = -u_x(a, b)u_y(a, b) + u_y(a, b)u_x(a, b) \]

\[ = 0. \]

(iii) Sketch the level sets of the real and imaginary parts of \( f(z) = z^2 \).

Since \((x + iy)^2 = (x^2 - y^2) + 2ixy\), \( u = x^2 - y^2 \) and \( v = 2xy \).
3. Suppose that $S$ is a subset of $\mathbb{C}$. Show that $S \cup \partial S$ is closed in $\mathbb{C}$ and that it is the smallest closed subset of $\mathbb{C}$ that contains $S$.

First observe that if a disk $D(a, r)$ contains a point $b$ of $\partial S$, then $D(a, r)$ contains a point of $S$. This is because $D(a, r)$ is open, and thus contains a disk $D(b, \epsilon)$ about $b$ for some $\epsilon > 0$. But, since $b \in \partial S$, this disk contains a point of $S$.

Next we show that $S \cup \partial S$ is closed. If $z \in \mathbb{C}$ and $D(z, r) \cap \partial S$ is non-empty for all $r > 0$, then, by the previous paragraph, $D(z, r) \cap S$ is non-empty for all $r > 0$. So either $z \in S$ or $z \in \partial S$. Consequently, if $z \notin S \cup \partial S$, then there exists $r > 0$ such that $D(z, r) \cap (S \cup \partial S)$ is empty. That is, $\mathbb{C} - (S \cup \partial S)$ is open, and $S \cup \partial S$ is closed.

Finally, we show that $S \cup \partial S$ is contained in all other closed sets that contain $S$. Suppose that $F$ is a closed set that contains $S$. To prove the assertion, we only need show that $\partial S$ is contained in $F$. We will do this by showing that $\partial S \cap (\mathbb{C} - F)$ is empty. Since $F$ is closed, its complement $\mathbb{C} - F$ is open. Thus, for each $z \in \mathbb{C} - F$, there is an $r > 0$ such that $D(z, r)$ is contained in $\mathbb{C} - F$. Equivalently, $D(z, r) \cap F$ is empty. Since $S \subseteq F$, this implies that $D(z, r) \cap S$ is empty. The definition of $\partial S$ implies that $z$ cannot be in $\partial S$. So $\partial S \subseteq F$.

4. Suppose that if $f(z) = u(x, y) + iv(x, y)$ is an analytic function defined on an open subset $S$ of $\mathbb{C}$, where $u, v : S \to \mathbb{R}$. Assume that the second partials

$$u_{xx}, u_{xy}, u_{yx}, u_{yy}, v_{xx}, v_{xy}, v_{yx}, v_{yy}$$

exist and are continuous in $S$. Show that $u$ and $v$ are harmonic on $S$. That is,

$$u_{xx} + u_{yy} = 0 \text{ and } v_{xx} + v_{yy} = 0$$

for all $(x, y) \in S$.

Since $f$ is analytic, $u$ and $v$ satisfy the CR-equations: $v_x = -u_y$ and $v_y = u_x$. So

$$u_{xx} + u_{yy} = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0$$

as second partials commute (provided they are continuous). (Strictly speaking, we do not yet know that if $f$ is analytic, then $f$ has all complex derivatives $f^{(k)}$ except when $f$ is defined locally by a power series. Here we implicitly used the fact that $f''$ exists and is continuous.)
5. Compute the limit of the series

\[ \sum_{n=0}^{\infty} n(n-1)z^n \]

and its radius of convergence.

\[
\sum_{n=0}^{\infty} n(n-1)z^n = z^2 \sum_{n=0}^{\infty} n(n-1)z^{n-2}
\]
\[
= z^2 \frac{d}{dz} \left( \sum_{n=0}^{\infty} nz^{n-1} \right)
\]
\[
= z^2 \frac{d^2}{dz^2} \left( \sum_{n=0}^{\infty} z^n \right)
\]
\[
= z^2 \frac{d^2}{dz^2} \left( \frac{1}{1-z} \right)
\]
\[
= \frac{2z^2}{(1-z)^3}.
\]

At each stage, the radius of convergence does not change. Since the geometric series on line 3 has radius of convergence \( R = 1 \), the original series also has radius of convergence \( R = 1 \). It converges to \( 2z^2/(1-z)^3 \).