Math 333
Solutions to Assignment 1

1. Suppose that \( n \) is a positive integer and that
\[
\zeta = \cos(2\pi/n) + i\sin(2\pi/n).
\]

(i) Suppose that \( k \) is an integer. Show that \( \zeta^k = 1 \) if and only if \( n \) divides \( k \). (That is, \( k/n \) is an integer.)

Since \( \text{Arg}(zw) = \text{Arg}(z) + \text{Arg}(w) \), and since \( \text{Arg}(\zeta) = 2\pi/n \), we have \( \text{Arg}(\zeta^k) = 2\pi k/n \). This is zero if and only if \( k/n \) is an integer. That is, if and only if \( n \) divides \( k \). Since \( j\zeta^kj = 1 \), \( \zeta^k = 1 \) if and only if \( \text{Arg}(\zeta^k) = 0 \). So \( \zeta^k = 1 \) if and only if \( n \) divides \( k \).

(ii) Show that if \( k \) is not divisible by \( n \), then
\[
1 + \zeta^k + \zeta^{2k} + \cdots + \zeta^{(n-1)k} = 0.
\]

Hint: Factor \( t^n - 1 \) then take \( t \) to be an appropriate power of \( \zeta \).

Setting \( t = \zeta^k \) in the identity
\[
t^n - 1 = (t - 1)(1 + t + t^2 + \cdots + t^{n-1})
\]
we see that
\[
(\zeta^{kn} - 1)(1 + \zeta^k + \zeta^{2k} + \cdots + \zeta^{(n-1)k}) = 0.
\]

Since \( n \) does not divide \( k \), \( (\zeta^{kn} - 1) \neq 0 \), so we have
\[
1 + \zeta^k + \zeta^{2k} + \cdots + \zeta^{(n-1)k} = 0.
\]

2. Suppose that \( z_1, \ldots, z_n \in \mathbb{C} \) satisfy \( |z_j| \leq 1 \) and that \( t_1, \ldots, t_n \) are non-negative real numbers satisfying \( t_1 + t_2 + \cdots + t_n = 1 \). Show that
\[
|t_1 z_1 + t_2 z_2 + \cdots + t_n z_n| \leq 1.
\]

We have:
\[
|t_1 z_1 + t_2 z_2 + \cdots + t_n z_n| \leq |t_1 z_1| + |t_2 z_2| + \cdots + |t_n z_n| \quad \text{(triangle inequality)}
\leq t_1 |z_1| + t_2 |z_2| + \cdots + t_n |z_n| \quad \text{(as each } t_j \geq 0 \text{)}
\leq t_1 + t_2 + \cdots + t_n \quad \text{(as each } |z_j| \leq 1 \text{)}
\leq 1.
\]
3. Show that if the sequence \( \{z_n\} \) converges to \( z \in \mathbb{C} \), then there is a positive real number \( A \) such that \( |z_n| \leq A \) for all \( n \geq 1 \). Show that if \( z \neq 0 \), then there exists \( B > 0 \) and a positive integer \( N \) such that \( |z_n| \geq B \) for all \( n \geq N \).

We can take \( \epsilon = 1 \) in the definition of convergence. Since \( \{z_n\} \) converges to \( z \), there is \( N > 0 \) such that \( |z_n - z| < 1 \) all \( n \geq N \). So, by the triangle inequality, we have

\[
|z_n| = |(z_n - z) + z| \leq 1 + |z|
\]

Set

\[
A = \max\{|z_1|, \ldots, |z_{N-1}|, 1 + |z|\}.
\]

Then \( |z_n| \leq A \) for all \( n \geq 1 \).

Now suppose that \( z \neq 0 \). Set \( \epsilon = |z|/2 \). There is an \( N > 0 \) such that \( |z_n - z| < \epsilon \). Then, when \( n \geq N \),

\[
|z_n| = |z - (z - z_n)| \geq ||z| - |z - z_n|| \geq |z| - |z|/2 = |z|/2.
\]

So, in the second part, we can take \( B = |z|/2 \).