Rough Notes on Real Analysis

A very quick and dirty construction of the real numbers is that a real number is a decimal expression

\[ \pm a_k a_{k-1} \ldots a_0 a_{-1} b_1 b_2 b_3 \ldots \]

where each \( a_j, b_m \in \{0, 1, \ldots, 9\} \).

This is to be thought of as

\[ \pm a_k 10^k + a_{k-1} 10^{k-1} + \ldots + a_0 + b_1/10 + b_2/10^2 + b_3/10^3 + \ldots \]

One has to introduce relations such as

not 9

\[ a_k \ldots a_0 \cdot b_1 \ldots b_k \ 9 \ 9 \ 9 \ldots \]

\[ = a_k \ldots a_0 \cdot b_1 \ldots (b_k+1) 0 \ 0 \ 0 \ldots \]

Equivalently

\[ 0.9999\ldots = 1.0 \]

2.

The important property of the real numbers \( \mathbb{R} \) is that every Cauchy sequence in \( \mathbb{R} \) converges. This has many consequences, such as the "least upper bound property".

Least upper bounds:

An upper bound of a subset \( A \) of \( \mathbb{R} \) is a real number \( M \) such that

\[ a < M \quad \text{all} \ a \in A. \]

Not every subset \( A \) of \( \mathbb{R} \) has an upper bound.

Definition: A real number \( M \) is a least upper bound of a subset \( A \) of \( \mathbb{R} \) if:

(i) \( M \) is an upper bound of \( A \);
(2) No smaller $m \in \mathbb{R}$ is an upper bound of $A$. This means that if $m < M$, then there exists $a \in A$ such that $m < a \leq M$.

Note that if $M$ is a least upper bound of $A$, then $M$ may or may not be in $A$.

Eq: (i) $A = \{ x \in \mathbb{R} : x < 0 \}$

$M = 0$, $M \not\in A$

(ii) $A = \{ x \in \mathbb{R} : x \leq 0 \}$

$M = 0$, $M \in A$.

A fundamental property of $\mathbb{R}$ is:

$\forall^{\text{non-empty}} A \in \mathbb{R}$ is bounded above, then it has a least upper bound in $\mathbb{R}$.

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**Proof (Sketch).**

Let $x_0$ be the smallest integer that is an upper bound of $A$. This exists as:

(i) There is certainly an integer upper bound $N$ of $A$.

(ii) If $a \in A$ ($A$ is non-empty!)
there is only a finite number of integers $n$ satisfying $a \leq n \leq N$.

Let $x_1 =$ smallest integer multiple of $\frac{1}{2}$ that is an upper bound of $A$.

It exists by a similar argument.

Note that $x_0 \geq x_1$.

Continue in this way:

$x_k =$ smallest integer multiple of $\frac{1}{2^k}$ that is an upper bound of $A$.
5. Then

\[ x_0 \geq x_1 \geq x_2 \geq \cdots \]

and \[ |x_k - x_{k+1}| \leq \frac{1}{2^{k+1}} \]

This implies that \( k \geq 0 \)

\[ |x_k - x_{k+1}| \leq \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{k+1}} \]

\[ < \frac{1}{2^{k+1}} \]

\[ < \frac{1}{2^k} \quad \text{geometric series} \]

It follows that \( \{x_k\} \) is a Cauchy sequence, and therefore converges.

Let \( M = \lim_{k \to \infty} x_k \).

Then \( M \) is a least upper bound of \( A \).

Exercise: Prove this.

6. Similarly, every non-empty subset \( B \) of \( \mathbb{R} \) that is bounded below has a "greatest lower bound."

Remark: The set of rational numbers \( \mathbb{Q} \) does not have the least upper bound property.

\[ A = \{ y \in \mathbb{Q} : y^2 < 2 \} \]

This does not have a least upper bound in \( \mathbb{Q} \). Its least upper bound in \( \mathbb{R} \) is \( \sqrt{2} \), which is irrational.

\textit{Nested interval property}

Suppose that

\[ I_n = [a_n, b_n] = \{ x \in \mathbb{R} : a_n \leq x \leq b_n \} \]

and that \( I_0 \supset I_1 \supset I_2 \supset \cdots \)
That is, for all \( n \geq 0 \)
\[
\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \beta_n \leq \cdots \leq \beta_2 \leq \beta_1.
\]

**Proposition:** \( \bigcap_{n=0}^{\infty} I_n \neq \emptyset \)

That is, there is an \( x \in \mathbb{R} \) that is in every \( I_n \).

**proof.** The set \( A = \{a_n\} \) is bounded above by all \( \beta_m \). It therefore has a least upper bound \( a \in \mathbb{R} \). It has to be smaller than each \( \beta_m \), as each \( \beta_m \) is an upper bound.

The set \( B \) is bounded below by \( a \). It therefore has a greatest lower bound \( b \), which satisfies \( a \leq b \).

**Exercise:**
\[
\bigcap_{n=1}^{\infty} I_n = [a, b].
\]

This is non-empty as \( a \leq b \). \( \square \)

This generalizes to the "nested rectangle theorem" in \( \mathbb{C} \) (or \( \mathbb{R}^2 \)) and the "nested box theorem" in \( \mathbb{R}^n \).

**Limit points**

Suppose that \( A \subseteq \mathbb{R} \). A limit point (or point of accumulation) of \( A \) is a real number \( p \) (which may or may not be in \( A \)) such that all neighbourhoods
\[
\{x : \varepsilon < x < \varepsilon + 2\}
\]
contain an infinite number of elements of \( A \).

**Proposition:** \( p \in \mathbb{R} \) is a limit point of \( A \) if and only if there is a sequence \( \{a_n\} \) in \( A \), where \( an \neq p \).
That converges to \( a \).

**Proof:** If \( \{a_n\} \to p \) and no \( a_n = p \), then for all \( \varepsilon > 0 \)

\[
\{a_n\} \cap \{ x : p-\varepsilon \leq x \leq p+\varepsilon \}
\]

is infinite. So \( p \) is an accumulation point of \( A \).

Conversely, if \( p \) is an accumulation point of \( A \), for all \( n \geq 1 \), the set

\[
A \cap \{ p - n \leq x \leq p + n \}
\]

is infinite. Let \( a_n \) be an element, \( a_n \neq p \). Then \( a_n \to p \). \( \square \)

**Boundedness Result**

A subset \( B \) of \( \mathbb{R}^n \) is bounded if there are intervals \( I_1, \ldots, I_n \) in \( \mathbb{R} \):

\[
I_j = [a_j, b_j]
\]

Such that \( B \) is contained in the box

\[
I_1 \times \cdots \times I_n = \{ (x_1, \ldots, x_n) : a_j \leq x_j \leq b_j, 1 \leq j \leq n \}
\]

**Proposition** Every infinite subset of

\[
T = I_1 \times \cdots \times I_n
\]

has a point of accumulation.

**Proof:** Suppose that \( T \) is an infinite subset of \( T \). Cut each \( I_j \) into 2:

\[
I_j = I_j' \cup I_j''
\]

where

\[
I_j' = [a_j, a_j + b_j]
\]

\[
I_j'' = [a_j + b_j, b_j]
\]

This divides \( T \) into \( 2^n \) boxes.

\( \text{E.g.} n = 2 \)

Label the pieces \( T_j^{(1)}, \ldots, T_2^{(n)} \).
At least one of these contains an infinite number of elements of \( A \). Choose one. Call it \( T^{(1)} \).

Divide \( T^{(1)} \) into \( 2^n \) boxes

\[ T^{(2)}, \ldots, T^{(2^n)} \]

At least one of these has to contain an infinite \# of elements of \( A \). Choose one. Call it \( T^{(2)} \).

Continue in this manner to obtain a nested sequence of boxes

\[ T = T^{(2)} \supset T^{(3)} \supset T^{(4)} \supset \ldots \]

where \( T^{(k)} \) is \( \frac{1}{2^k} \times T \) (up to translation)

The nested interval implies that

\[ \bigcap T^{(k)} \]

is non-empty. In fact it is one point as the side lengths of \( T^{(k)} \to 0 \). This point is a point of accumulation of \( A \) as every

\[ T^{(k)} \] contains infinite \# of points of \( A \).

We can now show that continuous functions on closed bounded sets are bounded:

**Proposition**: Suppose that \( K \) is a closed and bounded subset of \( \mathbb{R}^n \). If \( f: K \to \mathbb{R} \) is continuous, then there are \( M_0, M_0 \in \mathbb{R} \) such that

1. \( M_0 \leq f(x) \leq M_0 \) all \( x \in K \)

2. There exist \( x_{\text{max}}, x_{\text{min}} \in K \) with

\[ f(x_{\text{min}}) = M_0, \quad f(x_{\text{max}}) = M_0 \]

**proof**: If \( f \) is not bounded on \( K \), then for each \( n \), we can find \( x_n \in K \) such that \( f(x_n) \geq n \). Set

\[ A = \{ x_n : n \geq 1 \} \]

This is infinite. Since \( K \) is bounded, \( A \in I, x \ldots \cdot I_n \)
It therefore has a point of accumulation $a$. But since $A \in K$ and since $K$ is closed, $a \in K$.

Choose $\epsilon > 0$ such that:

$x \in K \quad |x - a| < \epsilon \Rightarrow |f(x) - f(a)| < \epsilon$

This is possible as $f$ is continuous.

But since $a$ is an accumulation point of $A$

$S := \{ x_n \in A : |x_n - a| < \epsilon \}$

is infinite. This implies that for all $n \geq 1$

$n < f(x_n) < f(a) + \epsilon$.

But this is impossible, so $f$ is bounded above.

It is also bounded below — just apply the above argument to see that $-f$ is bounded above.

Our final task is to show that $f$ has a maximum value. The argument above shows that the set

$\{ f(x) : x \in K \}$

is bounded above. It therefore has a least upper bound, $M_0$.

Claim: There is $x_0 \in K$ such that $f(x_0) = M_0$.

Proof: Since $M_0$ is a least upper bound, for all $n \geq 1$, there is $x_n \in K$ such that

$M_0 - \epsilon < f(x_n) \leq M_0$

If there is an $N$ such that $f(x_N) = M_0$, we are done. If not, the set $\{x_n\}$ is infinite. Since $K$ is closed and bounded, it has a point of accumulation $x_0$. This means that
\(\{x_n\}\) has a subsequence \(\{x_{k_n}\}\) that converges to \(x_0\). So

\[ f(x_0) = \lim_{n \to \infty} f(x_{k_n}) = M_0 \]

as

(i) \(k_n \geq n\)

(ii) \(M_0 - y_n < f(x_{k_n}) \leq M_0\). \(\square\)

Applying the argument above to \(-f\) shows that there is \(x_{\min} \in K\) such that \(f(x) \geq f(x_{\min})\) all \(x \in K\). \(\square\)