1. Suppose that $f : S \to \mathbb{C}$ is analytic and that the closed disk $|z - z_0| \leq r$ is contained in $S$, where $r > 0$.

(i) Suppose that $|f(z)| > 0$ for all $z$ on the circle $|z - z_0| = r$. Use the identity theorem to show that the set
$$A = \{z \in \mathbb{C} : |z - z_0| \leq r \text{ and } f(z) = 0\}$$
is finite.

(ii) Suppose that $A = \{a_1, \ldots, a_N\}$. For each $j$ satisfying $1 \leq j \leq N$, let $n_j$ be the positive integer such that $f(z)/(z - a_j)^{n_j}$ is analytic and non-vanishing at $z = a_j$. Show that
$$g(z) := \frac{f(z)}{(z - a_1)^{n_1} \cdots (z - a_N)^{n_N}}$$
is analytic in $S$ and has no zeros in $|z - z_0| \leq r$.

(iii) Show that
$$\frac{df}{f} = \frac{dg}{g} + \sum_{j=1}^{N} \frac{n_j dz}{z - a_j}.$$

(iv) Show that
$$\oint_{|z - z_0|=r} \frac{df}{f} = 2\pi i(n_1 + n_2 + \cdots + n_N).$$

2. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is an analytic function that satisfies $f(z) = f(z + 1)$ and $f(z + i) = f(z)$. Show that $f$ is constant. Hints:

- Let $R$ be the rectangle $0 \leq \text{Re}(z) \leq 1$, $0 \leq \text{Im}(z) \leq 1$. Show that for all $z \in \mathbb{C}$, there is a point $w \in R$ such that $f(z) = f(w)$.
- Show that $f$ is bounded. (You may assume that a continuous function $\phi : S \to \mathbb{C}$ is bounded on each closed bounded subset $K$ of $S$.)