Elliptic Curves and Multiple Zeta Numbers

Richard Hain

Duke University

November 14, 2008
Suppose that $r \geq 1$ and that $n_1, \ldots, n_r$ are positive integers with $n_r > 1$. The \textit{multiple zeta number} $\zeta(n_1, \ldots, n_r)$ is defined by the convergent series

$$
\zeta(n_1, \ldots, n_r) = \sum_{0<k_1<\cdots<k_r} \frac{1}{k_1^{n_1} k_2^{n_2} \cdots k_r^{n_r}}.
$$

The integer $r$ is called the \textit{depth}, and $n_1 + \cdots + n_r$ is called the \textit{weight} of $\zeta(n_1, \ldots, n_r)$. 

Multiple zeta values of depth 1 are simply values of the Riemann zeta function

$$\zeta(s) = \sum_{k \geq 1} \frac{1}{k^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

at integers $> 1$. When $n$ is even, these are just non-zero rational multiplies of $\pi^n$:

$$\zeta(2m) = -\frac{(2\pi i)^{2m}B_{2m}}{4m(2m-1)!} \in \pi^{2m}\mathbb{Q}^\times$$

where $B_{2m}$ is the $m$th Bernoulli number:

$$\frac{x}{(e^x - 1)} = \sum_{n \geq 0} B_n x^n / n.$$
zeta values (ctd)

Example

1. \[ \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \ldots \]

2. As is well known \( \zeta(3) \) is irrational. It would be interesting to know whether \( \zeta(2m + 1) \) is an irrational multiple of \( \pi^{2m+1} \).
If $n, m > 1$, we have:

$$\zeta(n)\zeta(m) = \left(\sum_{k>0} \frac{1}{k^n}\right)\left(\sum_{\ell>0} \frac{1}{\ell^m}\right)$$

$$= \left(\sum_{0<k<\ell} + \sum_{0<k=\ell} + \sum_{0<\ell<k}\right) \frac{1}{k^n\ell^m}$$

$$= \zeta(n, m) + \zeta(n + m) + \zeta(m, n).$$

Thus we have:

$$\zeta(n)\zeta(m) = \zeta(n, m) + \zeta(n + m) + \zeta(m, n) \quad n, m > 1$$
Let’s try another. If $n_1, n_3 > 1$, then

$$
\zeta(n_1) \zeta(n_2, n_3) = \left( \sum_{k_1 > 0} \frac{1}{k_1^{n_1}} \right) \left( \sum_{0 < k_2 < k_3} \frac{1}{k_2^{n_2} k_3^{n_3}} \right)
$$

$$
= \left( \sum_{0 < k_1 < k_2 < k_3} + \sum_{0 < k_1 = k_2 < k_3} + \sum_{0 < k_2 < k_1 < k_3} + \sum_{0 < k_2 < k_1 = k_3} + \sum_{0 < k_2 < k_3 < k_1} \right) \frac{1}{k_1^{n_1} k_2^{n_2} k_3^{n_3}}
$$

$$
= \zeta(n_1, n_2, n_3) + \zeta(n_1 + n_2, n_3) + \zeta(n_2, n_1, n_3) + \zeta(n_2, n_1 + n_3) + \zeta(n_2, n_3, n_1).
$$
The identity

\[ \zeta(n_1)\zeta(n_2, n_3) = \zeta(n_1, n_2, n_3) + \zeta(n_1 + n_2, n_3) + \zeta(n_2, n_1, n_3) \]

\[ + \zeta(n_2, n_1 + n_3) + \zeta(n_2, n_3, n_1) \]

is an example of a stuffle relation. It implies:

**Theorem**

*The product of two multiple zeta numbers is an integral linear combination of multiple zeta numbers. If the two MZNs have weights \( w_1 \) and \( w_2 \), then their product is a sum of MZNs of weight \( w_1 + w_2 \). If the two MZNs have depths \( r_1 \) and \( r_2 \), then their product is a sum of MZNs of depth \( \leq r_1 + r_2 \).*
The stuffle relations imply that the $\mathbb{Q}$-subspace of $\mathbb{R}$ spanned by the multiple zeta numbers (MZNs) is an algebra. It is better to think of MZNs as spanning a *graded* algebra. For $n \geq 1$ set

$$\text{MZN}_n :=$$

the $\mathbb{Q}$-subspace of $\mathbb{R}$ spanned by MZNs of weight $n$.

Let $\text{MZN}_0 = \mathbb{Q}$. There is a product $\text{MZN}_n \times \text{MZN}_m \to \text{MZN}_{m+n}$, so

$$\text{MZN}_\bullet := \bigoplus_{n \geq 0} \text{MZN}_n$$

is an graded algebra.
Problem

Compute (or estimate) \( \dim_\mathbb{Q} \text{MZ}_n \) for all \( n \).

Approaches

1. find algebraic relations that hold between MZNs;
2. use the techniques of transcendental number theory;
3. other techniques?
If all $\zeta(n_1, \ldots, n_r)$ of weight $n$ were linearly independent, then the dimension of MZN of weight $n \geq 2$ and depth $r \geq 1$ would be

$$= \# \text{ degree } n \text{ monomials } x_1^{n_1} \cdots x_r^{n_r} \quad n_j \geq 1, n_r \geq 2$$

$$= \# \text{ degree } n - r - 1 \text{ monomials } x_1^{m_1} \cdots x_r^{m_r} \quad m_j \geq 0$$

$$= \binom{n-2}{r-1}$$

In this case $\dim_{\mathbb{Q}} MZN_n$ would be

$$\sum_{r=1}^{n-1} \binom{n-2}{r-1} = \sum_{k=0}^{n-2} \binom{n-2}{k} = 2^{n-2}.$$
Consider the algebra $A = \mathbb{Q}[Z_2] \otimes \mathbb{Q}\langle Z_3, Z_5, Z_7, \ldots \rangle$ where $\mathbb{Q}\langle Z_3, Z_5, Z_7, \ldots \rangle$ denotes the free associative algebra generated by indeterminates $Z_3, Z_5, Z_7, \ldots$. If we give $Z_m$ weight $m$, then this algebra can be graded by weight:

$$A = \bigoplus_{n \geq 0} A_n$$

where $A_n$ is the span of all monomials of weight $n$.

**Example**

1. $A_4$ is spanned by $Z_2^2$, so $\dim A_4 = 1$
2. $A_5$ is spanned by $Z_5$ and $Z_2Z_3$, so $\dim A_5 = 2$
Zagier’s Conjecture, Terasoma’s Theorem

Conjecture (Zagier, 1994), Theorem (Terasoma, 2003)
For all $n \geq 0 \dim \text{MZN}_n \leq \dim A_n$ (N.B. $\dim A_n \sim (0.754878)^n$)

All proofs use the theory of \textit{mixed Tate motives} due to Deligne and Goncharov (which uses work of Voevodsky and Levine).

Example
This implies that the $2^{4-2}$ MZN of weight 4:

\[ \zeta(4), \zeta(2, 2), \zeta(1, 3), \zeta(1, 1, 2) \]

are proportional, and thus all rational multiples of $\pi^4$.

The problem is to find enough relations between MZN of weight $n$ to cut the dimension down to $\dim A_n$. 
Definition (K.-T. Chen)

Suppose that $M$ is a smooth manifold. (E.g. $\mathbb{C} - \{0, 1\}$.) For 1-forms $\omega_1, \ldots, \omega_r$ and a piecewise smooth path $\gamma : [0, 1] \to M$, define

$$\int_\gamma \omega_1 \ldots \omega_r = \int \cdots \int f_1(t_1)f_2(t_2) \ldots f_r(t_r) \, dt_1 \, dt_2 \ldots \, dt_r$$

where $\gamma^* \omega_j = f_j(t)dt$.

When $r = 1$, this reduces to the standard line integral $\int_\gamma \omega$. 

Richard Hain

Elliptic Curves and Multiple Zeta Numbers
Take $M = \mathbb{C} - \{0, 1\}$ and let $\omega_0 = \frac{dz}{z}, \omega_1 = \frac{dz}{1-z}$. The iterated integral
\[
\int_0^1 \omega_{e_1} \omega_{e_2} \ldots \omega_{e_r} \quad e_j \in \{0, 1\}
\]
converges if and only if $e_1 = 1$ and $e_r = 0$. 
Example

\[ \int_0^1 \omega_1 \omega_0 = \int_0^1 \int_0^z \frac{dw}{1-w} \frac{dz}{z} = \sum_{n=1}^{\infty} \int_0^1 \left( \int_0^z w^{n-1} dw \right) \frac{dz}{z} \]

\[ = \sum_{n=1}^{\infty} \int_0^1 z^{n-1} \frac{dz}{n} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) \]

Iterating this argument yields:

**Formula for \( \zeta(n) \)**

\[ \zeta(n) = \int_0^1 \omega_1 \omega_0 \ldots \omega_0 \]
Kontsevich observed that this generalizes:

\[
\zeta(n_1, \ldots, n_r) = \int_0^1 \omega_1^{n_1-1} \omega_0 \cdots \omega_0 \omega_1^{n_2-1} \omega_0 \cdots \omega_0 \cdots \omega_1^{n_r-1} \omega_0 \cdots \omega_0
\]

For example

\[
\zeta(2, 1, 3) = \int_0^1 \omega_1 \omega_0 \omega_1 \omega_1 \omega_0 \omega_0
\]

Thus every convergent iterated integral of \( \omega_0 \) and \( \omega_1 \) over \([0, 1]\) is an MZN.
Iterated integrals satisfy the shuffle product formula:

\[\int_\gamma \omega_1 \ldots \omega_r \int_\gamma \omega_{r+1} \ldots \omega_{r+s} = \sum_{\sigma \in \text{sh}(r,s)} \int_\gamma \omega_{\sigma(1)} \omega_{\sigma(2)} \cdots \omega_{\sigma(r+s)}\]

For example,

\[\int_\gamma \omega_1 \int_\gamma \omega_2 \omega_3 = \int_\gamma \omega_1 \omega_2 \omega_3 + \int_\gamma \omega_2 \omega_1 \omega_3 + \int_\gamma \omega_2 \omega_3 \omega_1.\]
The shuffle product formula can be used to compute products of MZNs. For example

\[ \zeta(2)^2 = \int_0^1 \omega_1 \omega_0 \int_0^1 \omega_1 \omega_0 \]

\[ = 2 \int_0^1 \omega_1 \omega_0 \omega_1 \omega_0 + 4 \int_0^1 \omega_1 \omega_1 \omega_0 \omega_0 \]

\[ = 2 \zeta(2, 2) + 4 \zeta(1, 3). \]

This is an example of a *shuffle relation*. 
double shuffle relations

New relations can be obtained by equating the stuffle and shuffle formulas for the product of two MZNs. For example, equating the stuffle formula

$$\zeta(2)^2 = 2\zeta(2, 2) + \zeta(4)$$

with the shuffle formula

$$\zeta(2)^2 = 2\zeta(2, 2) + 4\zeta(1, 3)$$

we see that

$$4\zeta(1, 3) = \zeta(4).$$

This is an example of a double shuffle relation — cf. Racinet.
Further properties of iterated integrals

Iterated integrals satisfy other useful relations, such as:

**Antipode and Naturality Properties**

1. \[ \int_{\gamma} - \omega_1 \ldots \omega_r = (-1)^r \int_{\gamma} \omega_r \ldots \omega_1 \]
2. \[ \int_{f \circ \gamma} \omega_1 \ldots \omega_r = \int_{\gamma} f^* \omega_1 \ldots f^* \omega_r \]

where \( \gamma : [0, 1] \to N \), \( f : N \to M \) and \( \omega_1, \ldots, \omega_r \) are 1-forms on \( M \).
Applied to the automorphism $f : z \mapsto 1 - z$ of $\mathbb{C} - \{0, 1\}$, this gives

$$
\int_0^1 \omega_{e_1} \cdots \omega_{e_r} = \int_0^1 \omega_{1-e_r} \cdots \omega_{1-e_1}
$$

where each $e_j \in \{0, 1\}$ as $f^* \omega_e = -\omega_{1-e}$.

For example,

$$
\zeta(3) = \int_0^1 \omega_1 \omega_0 \omega_0 = \int_0^1 \omega_1 \omega_1 \omega_0 = \zeta(1, 2).
$$

and

$$
\zeta(4) = \int_0^1 \omega_1 \omega_0 \omega_0 \omega_0 = \int_0^1 \omega_1 \omega_1 \omega_1 \omega_0 = \zeta(1, 1, 2).
$$
Zagier’s Conjecture predicts that \( \dim \text{MZN}_3 = 1 \) and \( \dim \text{MZN}_4 = 1 \). We have just seen that the inversion formula implies that \( \zeta(3) = \zeta(1, 2) \), which proves Zagier’s conjecture in weight 3. In weight 4, we have

\[
\zeta(2)^2 = \left(\frac{\pi^2}{6}\right)^2 = \frac{5 \pi^4}{290} = \frac{5}{2} \zeta(4) \quad \text{classical}
\]

\[
\zeta(2, 2) = \frac{1}{2} \left( \zeta(2)^2 - \zeta(4) \right) = \frac{3}{4} \zeta(4) \quad \text{stuffle}
\]

\[
\zeta(1, 3) = \frac{1}{4} \zeta(4) \quad \text{double shuffle}
\]

\[
\zeta(1, 1, 2) = \zeta(4) \quad \text{inversion}
\]

So our identities imply Zagier’s Conjecture in weight 4 as well.
Elementary relations, such as Euler, stuffle, shuffle and inversion, do not always imply Zagier’s bound. The first interesting cases occur in weights 12 and 16.

**Theorem (Gangl-Kaneko-Zagier, 2006)**

\[
28 \zeta(3, 9) + 150 \zeta(5, 7) + 168 \zeta(7, 5) = \frac{5197}{691} \zeta(12) \\
66 \zeta(3, 13) + 375 \zeta(5, 11) + 686 \zeta(7, 9) + 396 \zeta(11, 5) = \frac{78967}{3617} \zeta(16)
\]

These relation comes from the *cusp forms of weight* 12 and 16 of the *modular group* $SL_2(\mathbb{Z})$. This relation generalizes to all cusp forms of $SL_2(\mathbb{Z})$. 
The modular group

\[ \text{SL}_2(\mathbb{Z}) = \{2 \times 2 \text{ integer matrices of determinant 1}\} \]

acts on the upper half plane \( \mathfrak{h} := \{\tau \in \mathbb{C} : \text{im} \tau > 0\} \) by fractional linear transformations.

The quotient \( \text{SL}_2(\mathbb{Z}) \backslash \mathfrak{h} \) is the “moduli space \( \mathcal{M}_{1,1} \) of elliptic curves”.
Definition

A *modular form of weight* $n$ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ that satisfies

$$f((a\tau + b)/(c\tau + d)) = (c\tau + d)^n f(\tau) \quad A \in \text{SL}_2(\mathbb{Z})$$

(in particular $f(\tau + 1) = f(\tau)$) and its Fourier expansion

$$f(\tau) = \sum_{n \in \mathbb{Z}} a_n q^n \quad q = e^{2\pi i \tau}$$

satisfies $a_n = 0$ when $n < 0$. It is a *cusp form* if, in addition, $a_0 = 0$. 

Richard Hain
Elliptic Curves and Multiple Zeta Numbers
examples

There are no non-zero modular forms of odd weight.

Examples

1. The *Eisenstein series* $G_{2k}$ of weight $2k \geq 4$:

   $$G_{2k}(\tau) = \sum_{(m,n)\neq(0,0)} \frac{1}{(m\tau + n)^{2k}}.$$

2. The *Ramanujan tau function*

   $$\Delta(\tau) = \left(\frac{G_2}{60}\right)^3 - \left(\frac{G_3}{140}\right)^2 = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

   is a cusp form of weight 12.
The coefficients of the relation between MZNs that comes from the cusp form $f$ of weight $2n$ are derived from the modular symbol of $f$, which is defined to be the homogeneous polynomial

\[ r_f(X, Y) := \int_0^{i\infty} f(\tau)(\tau Y - X)^{2n-2} d\tau \in \mathbb{C}[X, Y] \]

of degree $2n - 2$.

Why modular symbols give relations between MZN remains mysterious.
cusp forms and relations between MZN

Why should cusp forms give relations between MZN?

- MZN are “periods” of *mixed Tate motives*;
- there is a theory of *mixed elliptic motives* — Hain-Matsumoto (Goncharov);
- mixed elliptic motives become mixed Tate motives when restricted to the nodal cubic \( y^2 = x^2(x - 1) \);
- cusp forms give relations between mixed elliptic motives — H-M.
- this is explicit in certain cases (Pollack) — and Pollack’s relations imply the Galois analogue (Ihara) of these relations between MZN.