ALGEBRAIC CURVES OVER FUNCTION FIELDS. I

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This paper studies the diophantine geometry of curves of genus greater than unity defined over a one-dimensional function field.

Introduction

Let $X$ be a scheme over $\text{Spec } \mathbb{Z}$ and $K$ an extension of finite type of the field $\mathbb{Q}$ of rational numbers. The fundamental problem in the theory of diophantine equations is the description of the set $X(K)$ of points of the scheme $X$ whose coordinates belong to the field $K$. The solution depends both on $X$ and on $K$. Thus, for example, if $X$ is an algebraic curve, the answer depends on the value of the genus $g$ of the curve. The simplest case is $g = 0$. Either the set $X(K)$ is empty, or the curve is birationally equivalent to a projective line, which gives a rational parameterization of the set $X(K)$ (for an arbitrary field $K$). If $g = 1$, then, since there are rational points on $X$, the structure of a commutative group can be defined on $X(K)$. Mordell proved that the group $X(K)$ has a finite number of generators if $K$ is the field of rational numbers. Finally, when $g > 1$, numerous examples provide a basis for Mordell's conjecture that in this case $X(Q)$ is always finite. The one general result in line with this conjecture is the proof by Siegel that the number of integral points (i.e., points whose affine coordinates belong to the ring $\mathbb{Z}$ of integers) is finite. These results are also true for arbitrary fields of finite type over $Q$. Fundamentally this is because the fields are global, i.e., there is a theory of divisors with a product formula, which makes it possible to construct a theory of the height of quasiprojective schemes of finite type over $K$. Lang's book $^{[13]}$ contains a description of that theory and its application to the proof of the Mordell and Siegel theorems. It appears that further progress in diophantine geometry involves a deeper use of the specific nature of the ground field. This is confirmed by Ju. I. Manin's proof of the functional analog of Mordell's conjecture $^{[2]}$.

Another important problem in diophantine geometry is the classification of algebraic varieties defined over a given global field $K$ (cf. $^{[4]}$). We shall consider only the case of algebraic curves. Every curve $X$ which is nonsingular and geometrically irreducible over $K$ has the following invariants: genus $g$ and type $S$ – the finite set of points of $K$ for which $X$ does not have a good reduction (we assume for simplicity that $K$ is one-dimensional). We shall say that $X$ has a good reduction at the point $p$ of $K$ if there exists a scheme $V$, smooth and proper over $\mathbb{Q}_p$, such that

$$V \otimes_{\mathbb{Q}_p} K_p \simeq X \otimes_K K_p,$$
where $\mathcal{O}_p \subset K_p$ is the local ring of $p$ and $K_p$ its field of quotients. I. R. Šafarevič has made the following conjecture \[^4\].

**Conjecture.** The set of algebraic curves defined over a global field $K$ and having given invariants $g > 1$ and $S$, is finite (in the functional case only non-constant curves are considered).

It can be shown that the problem of classifying algebraic curves is a special case of the problem formulated above of describing the set of rational points of an algebraic variety. More exactly, let $M^g$ be the space of moduli of algebraic curves of genus $g$, itself a scheme over $\text{Spec } \mathbb{Z}$. Then the set of curves in the Šafarevič conjecture is simply the set of $S$-integral points of the scheme $M^g$. Hence this conjecture is most naturally obtained as a corollary of the appropriate generalization of Siegel's theorem to varieties of dimension greater than unity. The only result of this kind to date is the proof of the finiteness conjecture for hyperelliptic curves \[^4\]. It uses Siegel's theorem for curves, which in this case is sufficient because of the simple structure of the space of moduli of hyperelliptic curves.

This paper studies the problems touched on above for curves over function fields. By a function field we mean the field $K = k(B)$ of rational functions on some curve $B$. Every curve over $K$ defines in a natural way a $k$-surface provided with a pencil of curves (cf. Chapter 1). Hence deep results from the theory of surface \[^1, 10\] can be used in this case. The proofs assume that $k = \mathbb{C}$, but in view of the Lefschetz principle the basic results hold for an arbitrary algebraically closed field $k$ of characteristic $0$.

We pass now to a description of our results. Chapters 1 and 2 contain a proof of the Šafarevič conjecture for a certain class of algebraic curves. As usual in diophantine geometry (cf. \[^13\]), the finiteness proof falls into two stages. First it is established that the height of the set of curves in question is bounded. This is done in Chapter 1 for arbitrary curves (§3, Theorem 2). Sections 1 and 2 of this chapter contain the requisite facts concerning fiberings associated with the curves. To pass from the boundedness of the height to the finiteness conjecture one needs a theorem on the rigidity of deformations of fiberings. Such a theorem is obtained for nondegenerate curves in §§1 and 2 of Chapter 2. From this follows the proof of the Šafarevič conjecture for this class of curves (Theorem 3 of Chapter 2). The problem of nondegenerate curves over fields of genus 0 and 1 is also considered in Chapter 2 (§3).

In the last chapter a relation is established between Mordell's conjecture and that of Šafarevič. To do this the properties of branched coverings of curves of genus $g \geq 1$ are studied in §1. From the results obtained it can be established that the Mordell conjecture is a direct corollary of that of Šafarevič (see the end of §2). Moreover, it is shown that the boundedness of the height in the Šafarevič conjecture already implies the boundedness of the height for rational points on curves of genus $g > 1$. This leads to a new proof of the functional analog of Mordell's conjecture (Manin's theorem; see Chapter 3, §2). This proof makes it possible to obtain a bound for the height of rational points on algebraic curves.

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CHAPTER 1. CURVES OVER FUNCTION FIELDS AND FIBERINGS; GENERAL THEORY

§ 1. Preliminary facts about fibering

Terminology. In this paper we use essentially the language of schemes due to A. Grothendieck (cf. [8]). Instead of the term “regular” we use the older term “nonsingular.” The ground field is denoted by \( k \) and all subsequent schemes and morphisms are assumed to be of finite type over \( k \). By an algebraic variety we mean a reduced \( k \)-scheme of finite type. A covering is a finite flat epimorphism. If \( X \) is a scheme, \( X(S) \) denotes the set of points of \( X \) with values in \( S \). If \( Y \subset X \) is a closed subset, \( X - Y \) denotes the uniquely defined reduced subscheme whose topological space is the complement of \( Y \) in \( X \) (cf. [8], I, 5.2.1). Amplification of all notation used can be found in [8] or [1].

Definition 1. A fibering \( f: X \to B \) is an object consisting of a nonsingular irreducible projective surface \( X \), a nonsingular irreducible projective curve \( B \), and a flat projective epimorphism \( f \) whose generic fiber is a nonsingular geometrically irreducible curve.

We shall sometimes denote the fibering by the single letter \( X \) if there is no ambiguity. By a morphism \( g \) of the fibering \( X \) into the fibering \( Y \) is meant a morphism \( g : X \to Y \) of the \( B \)-scheme \( X \) into the \( B \)-scheme \( Y \). The fiberings \( X \) and \( Y \) are considered identical (or isomorphic) if \( g \) is an isomorphism of the surfaces \( X \) and \( Y \).

By the fibers of the fibering \( f: X \to B \) we mean the fibers of the morphism \( f \). All fibers, except for a finite number, are nonsingular curves of the same genus \( g \) (this follows from [8], IV, 12.2.4 and the formulas for the arithmetic genus of a curve on a surface). The remaining fibers are called degenerate. More precisely, these are those fibers that either are not reduced or have a singular point. The points on \( B \) corresponding to degenerate fibers are called the points of degeneracy of the fibering and form a finite set \( S \subset B \). Thus to every fibering corresponds a set \( (B, S, g) \), the type of the fibering.

The types of isomorphic fiberings are clearly identical.

It follows from the smoothness criterion ([8], IV, 17) that the morphism \( f \) is smooth in the open set \( X - f^{-1}(S) \). Note also that all fibers of a fibering \( X \) are geometrically connected, as follows from the Main Theorem of Zariski ([8], III, 4.3, 4.4).

Definition 2. A fibering \( f: X \to B \) is minimal if its fibers contain no exceptional curves of the first kind.

The importance of this concept is that minimal fiberings are uniquely defined by their generic fibers if \( g \geq 1 \). More precisely, we have:

Theorem 0. Let \( X \) be a nonsingular geometrically irreducible curve of genus \( g \geq 1 \), defined over the field \( K \) of rational functions on a nonsingular projective curve \( B \). Then there exists a minimal fibering \( f: V \to B \) whose generic fiber is isomorphic with \( X \).

If \( U \) and \( V \) are minimal fiberings, \( X \) and \( Y \) their generic fibers, and the genuses of \( X \) and \( Y \) are greater than zero, then the canonical mapping

\[
\text{Isom}_B (U, V) \to \text{Isom}_K (X, Y)
\]

is bijective.

The proof of the first assertion follows from the theorem on resolution of singularities of surfaces and the Castelnuovo contractibility criterion. The second assertion is proved in [1] (Chapter VII,
Theorem 1; it is assumed there that $g = 1$, but the proof goes through without change in the case $g > 1$.

Thus, every curve $X$ defined over the function field $k(B)$, nonsingular and geometrically irreducible (absolutely irreducible in the older terminology), determines a uniquely defined minimal fibering. We shall call the latter the nonsingular minimal model of the curve $X$.

Change of base. Let $\phi: B' \rightarrow B$ be a covering of nonsingular curves $B$ and $B'$, and $f: X \rightarrow B$ a fibering. Then we have the surface $X \times_B B'$ and the canonical morphism $f_{(B')}: X \times_B B' \rightarrow B'$. The minimal fibering associated with the generic fiber of the morphism $f_{(B')}$ will be denoted by $f': X_{B'} \rightarrow B'$ and we shall say that it is obtained by a change of base from the fibering $f: X \rightarrow B$. There is a natural projection $\pi: X_{B'} \rightarrow B'$, which is in general a rational mapping. It is obtained from the morphism $X \times_B B' \rightarrow X$ by resolution of the singular points and contraction of the exceptional curves of first kind.

If $Y$ and $Y'$ are the generic fibers of the fiberings $X$ and $X_{B'}$, then

$$Y' \sim Y \otimes K',$$

where $K' = k(B')$.

Definition 3. The fibering $f: X \rightarrow B$ is called trivial if it is isomorphic with a fibering of the form $C \times B \rightarrow B$, and isotrivial if it becomes trivial after suitable change of base.

Let us describe those curves that correspond to isotrivial fiberings. A curve $X$ defined over the field $K = k(B)$ is called constant if there exist a finite extension $K' \supseteq K$ and a curve $C$ defined over $k$, such that

$$X \otimes_K K' \sim C \otimes_K K'.$$

To isotrivial fiberings correspond constant curves. The reverse statement follows from the theorem on the minimality of trivial fiberings.

Sections and rational points. Let $f: V \rightarrow B$ be a fibering, $X$ a generic fiber, and $K = k(B)$. There is an important relation between the divisors of the curve $X$ which are rational over $K$ and the divisors of the surface $V$. We consider only the effective divisors. If $S$ is a divisor on $V$, there corresponds to it the divisor $S.X$ on the generic fiber. The divisor $S.X$ is rational over $K$. If $R$ is a divisor consisting of components of fibers, then $R.X = 0$. Given a divisor $\overline{D}$ on the curve $X$ which is rational over $K$, one can single out among the divisors $S$ on $V$ such that $S \cdot X = \overline{D}$ a uniquely defined divisor $\overline{D}$. The condition defining $\overline{D}$ is the flatness of $L$ over the curve $B$ ([8], IV, 2.8.5). In the present case the condition is equivalent to the absence from $D$ of components contained in the fibers (this follows from Proposition 6 of Lecture 6 of [15]). In particular, to every rational point $P \in X(K)$ corresponds a divisor $P$ which is a section of the fibering $V$. The curve $\widetilde{P}$ is nonsingular and irreducible, and the morphism $f$ defines an isomorphism

$$f|_P: \widetilde{P} \sim B.$$

For any closed fiber $V_b$, $b \in B$, the intersection index $V_b \cdot \widetilde{P}$ is equal to unity. It follows that $\widetilde{P}$ cannot intersect the singular points of the fibers and consequently the morphism $f$ is smooth in some neighborhood of the curve $\widetilde{P}$. 
If $\phi : B' \to B$ is a covering and $f : X_{B'} \to B'$ is a change of base for the minimal fibering $f : X \to B$, then to every section $C$ of the fibering $X$ corresponds a section $C'$ of the fibering $X_{B'}$. It is not difficult to verify also that the correspondence constructed above commutes with the operation of changing the base.

§2. The geometry of minimal fiberings

Lemma 1. Let $f : X \to B$ be a minimal fibering whose generic fiber is a curve of genus $g > 1$, while the genus $q$ of the base is also greater than unity. Then:

a) $X$ is a surface of fundamental type;

b) $X$ is a minimal model in the sense of the theory of surfaces.

Proof. By Lemma 1, $X$ is a surface of fundamental type if $X$ has no pencil of rational curves or of elliptic curves, the irregularity $q(X) \neq 0$, and $X$ is not a two-dimensional abelian variety. Let us verify that these conditions hold in the present case. Any curve on $X$ either covers the base, and hence has genus greater than unity, or is contained in a fiber, and so likewise, with the exception of a finite number of possibilities (degenerate fiber), has genus greater than unity. The irregularity satisfies $q(X) \geq q \geq 2$. Finally, if $X$ is an abelian variety, then the canonical class $K_X = 0$, and so

$$2g - 2 = K_X \cdot F + F^2 = 0$$

(where $F$ is a fiber of the morphism $f$, $g$ the genus of $F$), which contradicts the hypothesis. The preceding implies that exceptional curves of the first kind on $X$ must be contained in fibers, and assertion (b) follows from the minimality of the fibering.

Proposition 1. Let $f : X \to B$ be a minimal fibering of type $(B, S, g)$, $g$ the genus of the base $B$, $g > 1$, $q > 1$. Then:

1. $p_g(X) \leq 4g(2g + 2q + s)$,
2. $K^2 \leq 48g(2g + 2q + s)$,
3. $\chi(X) \leq 4g(2g + 2q + s)$,
4. $q(X) \leq 50g(2g + 2q + s)$,

where $s = \text{Card}(S)$, $p_g(X)$ is the geometric genus, $\chi$ the Euler characteristic, $K$ the canonical class and $q(X)$ the irregularity of the surface $X$.

Proof. Denote by $X_S \subset X$ the set of points belonging to degenerate fibers, $X_S \subset f^{-1}(S)$. We have the exact sequence of the pair $(X, X_S)$:

$$H^2(X, X_S) \to H^2(X) \to H^2(X_S)$$

where $H^*$ is cohomology with complex coefficients. The group $H^2(X_S)$ is generated by algebraic cycles (components of fibers). The group $\Omega^2(X)$ of double differentials of the first kind is contained in $H^2(X)$, and since the fibers are one-dimensional it follows that

$$\alpha |_{\Omega^2(\alpha)} = 0.$$ 

Hence $p_g(X) \leq b^2(X, X_S)$. Using the spectral sequence of the differentiable locally trivial fibering

$$X \setminus X_S \to B \setminus S,$$

we find that $\chi(X, X_S) = \chi(B, S)(2g - 2)$ and $b^2(X, X_S) \leq 2g + 2q + s$. From this we obtain a bound for $b^2(X, X_S)$ and so for $p_g(X)$. To obtain inequalities 2, 3, and 4, note that $K^2 > 0$ (Lemma 1 and Lemma 5 of §1, Chapter VII, of [1]) and $\chi(X) > 0$ (which is clear from the formula for
the Euler characteristic and the inequalities \( g > 1 \) and \( q > 1 \) ([1], Chapter IV, §4, Theorems 6 and 7)). Hence the required bounds follow from the Noether formula \( K^2 + \chi = 12p_a = 12(1 - q + p_g) \).

**Proposition 2.** Under the hypothesis of Proposition 1, let \( n \geq 6 \). Then the linear system \( |nK| \) has no fixed component and defines a regular mapping

\[ \varphi: X \to \mathbb{P}^m, \]

where:

1. \( \varphi \) is a biregular imbedding on \( X - f^{-1}(S) \);
2. \( nK = \phi^*(L) \), where \( L \) is a hyperplane in \( \mathbb{P}^n \);
3. \( m = l(nK) - 1 \leq 50n^2(gq + s) \).

**Proof.** It follows from Lemma 1 and Theorems 3 and 5 of [10] that the linear system \( |nK| \) has no fixed component. Theorem 1 of [10] proves that \( \phi \) is regular and biregular on the complement of a set \( E = \bigcup C_i \), where the \( C_i \) are nonsingular rational curves. We assert that \( C_i \subset f^{-1}(S) \). Indeed, if a curve \( C \subset X \) covers the base \( B \), it is irrational, and the nondegenerate fibers of the morphism \( f \) are also irrational \( (g > 1) \); hence \( E \subset f^{-1}(S) \). Assertion 2 of Proposition 2 follows, as is well known, from the regularity of \( \phi \) and the fact that there are no fixed components. Finally, assertion 3 follows from the Riemann-Roch theorem, Proposition 1 and Theorem 5 of [10]. This completes the proof.

**Remark.** For any curve \( C \) on the surface \( X \),

\[ K.C \geq 0. \]

Indeed, if \( C \) is an irreducible fiber, then \( K.C > 0 \); if \( C \) is a component of a reducible fiber, then \( C^2 < 0 \) ([1], Chapter VII, §2) and from the equality \( K.C + C^2 = 2p_a(C) - 2 \) we obtain \( K.C \geq -1 \) and \( K.C = -1 \Leftrightarrow p_a(C) = 0, C^2 = -1 \). The minimality of \( X \) then implies the required result. Now let \( C \) cover the base and let \( C^2 \leq 0 \). Then \( p_a(C) > 1 \) and

\[ K.C = 2p_a(C) - 2 - C^2 > 0. \]

If \( C^2 > 0 \), then \( K.C \geq 0 \) since \( l(K) = p_g > 0 \) (Theorem 5, Chapter IV of [1] and Lemma 1). It can be shown, using Hodge’s index theorem, that \( K.C = 0 \Leftrightarrow p_a(C) = 0 \) and \( C^2 = -2 \).

**Proposition 3.** Let \( f: X \to B \) be a minimal fibering of type \( (B, S, g) \), \( q \) the genus of the base \( B \), \( g > 1 \), \( q > 1 \), \( N > 0 \) an integer. Then there exists a covering \( t: B' \to B \) for which:

1. The fibering \( X_B \) has at least \( N \) different sections \( C_1, \ldots, C_N \).
2. The fibering \( X_B \) is of type \( (B', S', g) \), where \( S' \subset t^{-1}(S) \), and

\[ q' \leq 100 N^2(gq + s), \]

\[ \deg t \leq 200 N^2(gq + s). \]

3. \( f_*(C_i).K_X \leq 100N(gq + s) \),

where \( f \) is the natural projection of \( X_B \) onto \( X \), \( f_*(C_i) \) the proper image of the section \( C_i \), and \( q' \) the genus of \( B' \).

**Proof.** By Proposition 2, there exists in the linear system \( |6nK| \) an irreducible nonsingular curve \( B'' \) not contained in the fibers. Let \( L = k(B), L'' = k(B''), L'/L'' \) be the least normal extension of
$L$ containing $L''$, $B'$ the normalization of $B$ in $L'$, and $t: B' \to B$ the natural covering. Denote by $X' = X_B$, the nonsingular minimal model of the fibering $X \times_B B'$. Then

$$f'(B') = \sum_{i \in \mathbb{N}(2g-2)} C_i + \Theta,$$

where the $C_i$ are sections, $\Theta$ is an effective divisor consisting of fiber components, and $f_*(C_i) = B''$.

The number of sections is equal to the degree of $B''$ over $B$, and since

$$[k(B'') : k(B)] = B''F = 6NK.F = 6N(2g - 2) > N,$$

we obtain assertion 1. Since extensions conjugate to $L''$ are branched wherever $L'$ is, and $\deg t \leq [L'' : L][L'' : L]$, to obtain bounds for $q'$ and $\deg t$ it is sufficient to know the degree $[L'' : L]$ and the number of branch points of the curve $B''$ over $B$. Using the Hurwitz formula, Proposition 1, and the bound

$$2p_a(B'') - 2 = (B'')^2 + B''K \leq 200N^2K^2,$$

we obtain the required result. In a similar manner, assertion 3 follows from the equalities

$$f_*(C_i).K = B''.K = 6NK^2$$

and Proposition 1.

§3. The main theorem

In this section we prove

Theorem 1. Suppose given a nonsingular projective curve $B$ of genus $g \geq 2$, a finite set $S$ of closed points of $B$, and an integer $g \geq 2$. Then there exist nonsingular algebraic varieties $V$ and $W$ and a smooth proper epimorphism $\Theta: V \to (B - S) \times W$ such that:

1. For any closed point $w \in W$ the variety

$$V_w = V \times ((B - S) \times W)$$

is a nonsingular surface, and the morphism $\Theta|_{V_w}: V_w \to B - S$ is smooth.

2. For any minimal fibering $f: X \to B$ of type $(B, S, g)$ there exists a closed point $w \in W$ for which $X - f^{-1}(S) \sim V_w$ over the curve $B - S$.

To prove Theorem 1 we require a number of lemmas and propositions.

First we require the concept of the height $H_K(P)$ of a point $P$ rational over the global field $K$. We use the definition of height in Chapter III of Lang's book. If $f: X \to B$ is a fibering, $K = k(B)$ the field of rational functions on $B$, and $Y$ a generic fiber - an algebraic curve defined over $K$, then there is a one-to-one correspondence between the sections $\tilde{P}$ of the fibering and the points $P$ of $Y$ rational over $K$.

Lemma 2. Let $f: X \to B$ be a fibering with projective base $B$, $Y$ a generic fiber, $K = k(B)$, $P \in Y(K)$. If $\phi: X \to \mathbb{P}^m$ is a morphism of the surface $X$ into projective space that defines an imbedding

$$\phi_K: Y \to \mathbb{P}_K^m,$$
then

\[ H_{φ_K}(P) = e^{\deg i_Q^{-1}(L_\otimes B)} , \]

where \( D = φ^*(L) \) and \( L \) is a hyperplane in \( \mathbb{P}^n \).

**Proof.** The mapping \( φ_K \) is constructed as follows. The morphisms \( φ \) and \( f \) yield the mapping \( Φ: X \to \mathbb{P}^n \), \( Φ = f \times B\phi \), and hence a mapping of generic fibers, which is then \( φ_K \). Let \( Q \in \mathbb{P}^n(K) \); then Propositions 5 and 6 of \([13]\) (Chapter III, §3) show that

\[ H_K(Q) = e^{\deg i_Q^{-1}(L_\otimes B)} , \]

where \( L \) is a hyperplane in \( \mathbb{P}^n \), \( i_Q : B \to \mathbb{P}_B^m \) is the section of the direct product corresponding to the point \( Q \) (in \([13]\) the curve \( B \) is the variety \( W \), \( e = c^{-1}, Q = P \)). Finally we have

\[ H_{φ_K}(P) = H_K(φ_K(P)) = e^{\deg i_{φ_K}^{-1}(L_\otimes B)} \]

\[ = e^{i_{φ_K}^{-1}(L_\otimes B) \cdot φ_K(P)} = e^{(L_\otimes B) \cdot φ_K(P)} \]

\[ = e^{φ_*(L_\otimes B) \cdot P} = e^{φ_*(L_\otimes B) \cdot P}_{φ_K}(P) = e^{P \cdot D} , \]

where

\[ P \in Y(K), \quad i = i_{φ_K}(P) . \]

**Corollary.** In the same situation as in Lemma 2, let \( B' \to B \) be a covering of the curve \( B \), \( K' \supset K \) the corresponding extension of the function field, and \( P \in Y(K') \). Then

\[ H_{φ_{K', K}}(P) = e^{φ_*(P) \cdot D} , \]

where \( \tilde{P} \subset X_{B'} \), \( f : X_{B'} \to X \) is the projection, \( f_\ast \) the proper image, and \( H_{φ_{K, K}} \) is the relative height.

The proof follows from the definition of height, Lemma 2 and standard formulas of intersection theory.

**Lemma 3.** Suppose given a projective space \( \mathbb{P}^m \) and an integer \( d \). Then any two irreducible curves of degree \( d \) in \( \mathbb{P}^m \) coincide if they intersect in more than \( N \) closed points, where \( N = d^2 \).

The proof follows from the inequality \( \deg (X \cap X') \leq \deg X \cdot \deg X' \), where \( X \) and \( X' \) are curves in \( \mathbb{P}^m \), proved in Lang's book ([13]), Chapter 3, §3, Lemma 4).

**Lemma 4.** Let \( X \) be a quasi-projective scheme over \( k \) with a fixed projective imbedding and suppose given an integer \( d > 0 \). Then there exists a quasi-projective scheme \( Y \), called the Hilbert scheme of one-dimensional cycles of degree \( d \) on \( X \), and denoted by \( \text{Hilb}_d(X) \) with the following properties:

1. There exists a scheme \( V \) and a commutative diagram

\[ \begin{array}{ccc}
V & \xrightarrow{q} & \text{Hilb}_d(X) \times X \\
\text{pr}_1 \downarrow & & \downarrow \text{pr}_2 \\
\text{Hilb}_d(X) & \xrightarrow{p} & X
\end{array} \]

where \( p \) is a flat morphism, \( q \) an imbedding, and for any closed point \( y \in \text{Hilb}_d(X) \) the fiber \( V_y \) is a one-dimensional cycle of degree \( d \) in \( X \) with respect to the imbedding \( \text{pr}_2 \circ q \).

2. For any commutative diagram
in the category of schemes satisfying the same conditions as diagram 1, there exists a unique mapping 
\( \phi: S \to \text{Hilb}_d(X) \) such that \( R \cong V \times_\gamma S, \ s = q \times_\gamma S. \)

The scheme \( V \) is called a universal family of one-dimensional cycles of degree \( d \).

The proof can be found in [7]. It is clear from the definition that to each closed point of the 
Hilbert scheme corresponds a one-dimensional cycle in \( X \). Hence if \( a \in \text{Hilb}_d(X) \), we shall speak of 
the cycle (or curve) \( a \). Conversely, if \( L \) is a one-dimensional cycle of degree \( d \) in \( X \), property 2 ap-
p lied to the diagram

\[
\begin{array}{c}
L \to X \\
\downarrow \\
\text{Spec } k
\end{array}
\]

defines a point in the universal family \( V \). This shows that it corresponds in \( \text{Hilb}_d(X) \) to a closed point \( l \) (viz., that for which \( V_l \cong L \)).

Remark. In the scheme \( \text{Hilb}_d(X) \) there exists an open subscheme \( \text{Hilb}_d^0(X) \) defined by the con-
dition that \( l \in \text{Hilb}_d(X) \Leftrightarrow V_l \) is a nonsingular geometrically irreducible curve in \( X \).

Lemma 4 holds for \( \text{Hilb}_d^0(X) \) if we replace the words "one-dimensional cycle of degree \( d \)" by "irreduci-
ble nonsingular curve of degree \( d \)" and the scheme \( V \) by \( V^o = \text{proj}(\text{Hilb}_d^0(X)). \)

We proceed now to the proof of Theorem 1. As we saw in §2, the divisor \( 6K_X \) defines for any 
minimal fibering \( f: X \to B \) of type \( (B, S, g) \) a morphism of the surface \( X \) into projective space \( \mathbb{P}^m \),
where \( m \) depends only on \( g, q \) and \( s \). This mapping, which we denote by \( \phi_X \), is an imbedding on 
\( X - f^{-1}(S) \). For all \( b \in B \) the degree of the divisor \( 6K_X \) on the fiber \( X_b \) is independent of \( b \). Let it 
equal \( d \). By Lemma 4 we have the morphism

\[
\psi^b_X : B \to S \to \text{Hilb}_d(\mathbb{P}^m).
\]

The image of a generic point under this morphism defines a point \( x(X) \) in \( \text{Hilb}_d(\mathbb{P}^m) \otimes K, K = k(B). \)

Since the curve \( B \) is nonsingular and the scheme \( \text{Hilb}_d(\mathbb{P}^m) \) projective (see [7]), there exists a unique 
morphism

\[
\psi_X: B \to \text{Hilb}_d(\mathbb{P}^m),
\]

coinciding with \( \psi^b_X \) on \( B - S \). The graph \( \Gamma_X \) of the morphism \( \psi_X \) is a one-dimensional cycle of the 
scheme \( B \times T \) (here and hereafter \( T = \text{Hilb}_d(\mathbb{P}^m) \)), namely, a section of the morphism \( B \times T \to B \). On 
the generic fiber \( T \otimes K \) there corresponds to it the point \( x(X) \). Let us take as fixed a projective im-
bedding of the scheme \( T \) and with it the height \( H_K \) on \( T \otimes K \).

Lemma 5. There exists a constant \( C \), depending only on \( g, q \) and \( s \), such that

\[
H_K(x(X)) \leq C
\]

for any minimal fibering \( X \) of type \( (B, S, g) \).

Before proving this lemma we shall show how Theorem 1 follows from it. If we choose a projective 
imbedding of the scheme \( B \times T \), Lemma 5 indicates that the degrees of the cycles \( \Gamma_X \) are bounded by
some constant $C'$ when $\Gamma_X$ runs through the set of fiberings of type $(B, S, g)$ ([13], Chapter III, §3, Theorem 2).

Put

$$W_m^1 = \prod_{\varepsilon \in C'} \text{Hilb}_e (B \times T).$$

Denote by $p : U_m \to W_m^1$, $\delta : R \to T$ the universal families over the Hilbert schemes (Lemma 4), by

$$V_m^1 = U_m \times_{(B \times T \times W_m^1)} (B \times R \times W_m^1),$$

where the product is taken with respect to the mappings $y$ and $i_B \times 1_{W_m^1}$. Define the mapping

$$\Theta_m^1 : V_m^1 \to B \times W_m^1$$

as the composition of the natural morphism of the scheme $V_m^1$ into $B \times T \times W_m^1$, and the projection onto the outside factors. Finally, replace $B$ by $B - S$:

$$V_m^2 = V_m^1 \times_B (B - S),$$

$$W_m^2 = W_m^1,$n

$$\Theta_m^2 = \Theta_m^1 \times_B (B - S).$$

The triple $(W_m^2, V_m^2, \Theta_m^2 : V_m^2 \to (B - S) \times W_m^2)$ satisfies condition 2 of Theorem 1. To satisfy condition 1 we must pare down the scheme $W_m^2$. First we replace $W_m^2$ by $(W_m^2)_{\text{red}} = W_m^2 : \Theta_m^2$, $V_m^3$ are obtained by restriction to the subscheme $(B - S) \times (W_m^2)_{\text{red}}$ of the scheme $(B - S) \times W_m^2$. If $W_m^3$ is reducible and $W_m^3 = \bigcup_i (W_m^3)_i$, we replace the union $\bigcup_i (W_m^3)_i$ by the direct sum $\bigoplus_i$, and change $W_m^3$, $\Theta_m^3$ correspondingly. Thus we obtain a triple $(W_m^4, V_m^4, \Theta_m^4)$, in which the scheme $W_m^4$ is a finite direct sum of integral schemes. By [8] (IV, 6.9.1) there exists in each component of $W_m^4$ an open subset over which the morphism $\Theta_m^4 \circ \text{pr}_2$ is flat. Hence ((8), IV, 6.9.3) we can replace each component of $W_m^4$ by a direct sum of subschemes such that in the new triple $(W_m^5, V_m^5, \Theta_m^5)$ the morphism $\Theta_m^5 \circ \text{pr}_2$ is flat. Now retain in $W_m^5$ only those components needed to satisfy condition 2 of the theorem. Then by [8] (IV, 17.3.7 and 17.5.1) and condition 2 of the theorem, there exists in each of these components an open subscheme over which the morphism $\Theta_m^5 \circ \text{pr}_2$ is smooth. Since a flat morphism is open [8], IV, 2.4.6), we can assume that $\Theta_m^5 \circ \text{pr}_2$ is an epimorphism. Finally, by [8] (IV, 6.12.6) there exists in each component of $W_m^5$ an open subscheme consisting of nonsingular points. As above, we can replace each component by a direct sum of integral nonsingular subschemes. Thus, we have obtained a triple $(W_m^6, V_m^6, \Theta_m^6 : V_m^6 \to W_m^6)$ where $\Theta_m^6 \circ \text{pr}_2 : V_m^6 \to W_m^6$ is a smooth epimorphism and the connected components of $W_m^6$ are nonsingular algebraic varieties. Since the set of points of the scheme $V_m^6$ at which the morphism $\Theta_m^6 \circ \text{pr}_2$ is reduced is open ([8], IV, 12.1.7, vi), it follows from condition 2 of the theorem and [8] (IV, 3.3.5, 12.2.1, viii) that we can replace each component of $W_m^6$ by an open subscheme such that after restricting the whole construction we obtain a triple $(W_m^7, V_m^7, \Theta_m^7)$ in which $\Theta_m^7 \circ \text{pr}_2$ is a smooth reduced morphism, $V_m^7$ a reduced scheme, and $W_m^7$ a nonsingular algebraic variety (in general, not connected). By [8] (IV, 17.11.1 and 17.15.2), $V_m^7$ is a nonsingular scheme and hence, by what has been proved above, integral.

Consider now the diagram
By [8] (IV, 1.8.7) and condition 2 of the theorem, there exists in $W_m^7$ a constructive set of points $s \in W_m^7$ such that the mapping $(\Theta_m^7)_s : (V_m^7)_s \to B - S$ is epimorphic. If we replace $W_m^7$ by an appropriate direct sum of subschemes and repeat the argument above, we can ensure that the mapping $\Theta_m^7 \circ \text{pr}_1$ is an epimorphism. The morphism

$$\Theta_m^7 \circ \text{pr}_1 : V_m^7 \to B - S$$

is flat. This follows from Proposition 6 of [15] (Lecture 6), the nonsingularity of $B - S$, the integrity of $V_m^7$, and the fact that $\Theta_m^7 \circ \text{pr}_1$ is an epimorphism. Hence ([18], IV, 2.1.4, 2.1.7) the morphism $\Theta_m^7$ is a flat epimorphism. It follows from the definition of a smooth morphism ([18], IV, 17.3-7) and the fact that $\Theta_m^7 \circ \text{pr}_2$ is open, that there exists in $W_m^7$ an open subscheme $W_m^8$ such that when we restrict $V_m^7$ and $\Theta_m^7$ to it we obtain a smooth epimorphism $\Theta_m^8$. Finally, if we put

$$W = \prod_m W_m^8, \quad V = \prod_m V_m^8, \quad \Theta = \prod_m \Theta_m^8,$$

we obtain a triple satisfying all the conditions of the theorem. This completes the proof.

Proof of Lemma 5. Let $K = k(B)$, let $\tilde{K}$ be the algebraic closure of $K$, and $\tilde{E}$ the set of points of the form $x(X)$ in $\text{Hilb}_d(P_m)$. Choose the integer $N$ so that Lemma 3 holds with $d = 6(2g - 2)$ and $m = \ell(6\ell) - 1$ from Proposition 2 of §2 ($n = 6$). Let $V \to \text{Hilb}_d(P_m)$ be a universal family. Put

$$U = V \times_{\text{Hilb}_d(P_m)} V.$$

We have the canonical imbedding

$$\gamma : U \to (P_m)^N \times \text{Hilb}_d(P_m).$$

Let $\Phi = \text{pr}_1 \circ \gamma : U \to (P_m)^N$. By Lemma 3 we have an imbedding of the sets of $\tilde{K}$-rational points,

$$\Phi(\tilde{K}) : U(\tilde{K}) \to (P_m(\tilde{K}))^N.$$ 

Denote by $\Psi$ the morphism $\text{pr}_2 \circ \gamma$. In $U(\tilde{K})$ we shall construct a set $\tilde{E}'$ with bounded height relative to the imbedding $\Phi(\tilde{K})$ and such that $\Psi(\tilde{K})(\tilde{E}') \supset \tilde{E}$. From this follows the lemma, since the image of a set of bounded height has bounded height ([13], Chapter 4).

If $x \in \tilde{E}$ and $f : X \to B$ is the corresponding fibering, then $\phi_X : X \to P_m$ is a morphism defined by $6K_X$ (Proposition 2 of §2). By Proposition 3 of §2 there exists a covering $B' \to B$ such that the fibering $X_{B'}$ has $N$ different sections $\tilde{P}_1, \ldots, \tilde{P}_N$ and the generic fiber $\tilde{X} \times K'$, $K' = k(B')$, $N$ different $K'$-points $P_1, \ldots, P_N$. Denote by $\gamma \in U(\tilde{K})$ the point $(P_1, \ldots, P_N, x(X))$. Then $\Psi(\tilde{K})(\gamma) = x(X)$ and $\Phi(\tilde{K})(\gamma) = (P_1, \ldots, P_N)$. We now find a bound for the height of the point $\Phi(\tilde{K})(\gamma)$. Since the property of a set of $K$-points of having a bounded height is independent of the choice of the height we can take as the height the function

$$h(P_1, \ldots, P_N) = \sum_i \mu_{\tilde{K}}(P_i).$$
on \((\mathbb{P}^n(\overline{K}))^N\), where \(h_K\) is the absolute height \(^{[13]}\). If \(P \in \mathbb{P}^n(L)\), then

\[ h_K(P) = H_L(P)^{[L:K]} \]

where \(L\) is a finite extension of \(K\). Using Proposition 3 of \(\S\)2 and the corollary to Lemma 2, we obtain a bound for \(h_K(P_i)\) depending only on \(g, q\) and \(s\). This proves the lemma.

**Theorem 2.** Suppose given a nonsingular projective curve \(B\) of genus \(q\), a finite set \(S\) of closed points of the curve \(B\), and an integer \(g \geq 2\). Then there exist integers \(d\) and \(m\) depending only on \(g, q\) and \(s = \text{card}(S)\), and a set \(\mathcal{E}\) of bounded height,

\[ \mathcal{E} \subset \text{Hilb}^d(\mathbb{P}^n)(K), \quad K = k(B), \]

such that for any nonsingular geometrically irreducible curve \(X\) defined over \(K\) and of genus \(g\) and type \(S\), there exists a point \(e \in \mathcal{E}\) such that \(V_e \simeq X\).

**Proof.** If \(q > 1\), the theorem is simply a reformulation of Lemma 5, by virtue of Theorem 0 of \(\S\)1. The general case can be reduced to this one by replacing the ground field by a field of genus \(q > 1\), since, as is easy to see, the type of the minimal fibering associated with the curve is not "increased" by such a replacement.

CHAPTER 2. SMOOTH FIBERINGS AND NONDEGENERATE CURVES

\(\S\)1. The isotriviality criterion

A fibering \(f : V \to B\) is called smooth if the mapping \(f\) has no degenerate fibers, i.e., if \(f\) is a smooth morphism. If \(\phi : B' \to B\) is an arbitrary covering, \(B'\) a nonsingular curve, then the surface \(V' = V \times_B B'\) is nonsingular and the natural morphism \(f' : V' \to B'\) is smooth. This follows from the general properties of smooth morphisms (see \([8]\), IV, 17).

**Lemma 6.** Let \(f : X \to Y\) be a smooth proper morphism of connected algebraic \(k\)-schemes. Put

\[ h(y) = \dim_k H^0(X_y, \Omega^1_{X/y}) \]

for any closed point \(y \in Y\) and assume that \(h(y) = \text{const}\). Then for any morphism \(\phi : Y' \to Y\) we have a smooth morphism \(f_{(Y')/Y}(X' = X \times_Y Y' \to Y', \text{where:})\):

1. The sheaf \(f_{(Y')/(X'/Y)}(\Omega^1_{X'/Y})\) is locally free.
2. \(\phi^* f_{(Y')/Y}(\Omega^1_{X'/Y}) \cong f_{(Y')/Y}(\Omega^1_{X'/Y})\).
3. The mapping \(f_{(Y')/Y}(\Omega^1_{X'/Y}) \otimes k(y') \to H^0(X'_y, \Omega^1_{X'_y} y')\) is bijective for all \(y' \in Y'\).

**Proof.** The morphism \(f_{(Y')}\) is smooth and the sheaf \(\Omega^1_{X'/Y}\), locally free (\([8]\), IV, 17.3.3, 17.2.3); hence it is flat on \(Y'.\) Now note that

\[ \Omega^1_{X'/Y} \cong (\phi \times Y X')^* \Omega^1_{X/Y} \quad (1) \]

(\([8]\), 01V, 20.5.5); hence \(\Omega^1_{X/Y}|_Y \cong \Omega^1_{X/Y}\) and Theorem 4 of \([6]\) (\(\S\)7) asserts that \(f^* (\Omega^1_{X/Y})\) is locally free and the canonical morphism

\[ f^* (\Omega^1_{X/Y}) \otimes k(y) \to H^0(X_y, \Omega^1_{X/Y}), \quad y \in Y, \]

is bijective. By \([8]\) (III, 7.7.10b) the theorem on change of base holds in the present situation (\([8]\), III, 7.7.5, II), and assertion (d) of that theorem together with formula (1) imply assertion 2 of the
lemma; whence also 1 and 3 follow, since the inverse image of a locally free sheaf is locally free.

Lemma 7. Let \( f : X \to Y \) be

1. an abelian \( Y \)-scheme

or

2. a smooth fibering.

Then Lemma 6 holds for the morphism \( f : X \to Y \).

Proof. In case 1,

\[
\dim_k H^0(X_Y, \Omega^1_{X_Y}) = \dim X_Y = \text{const}
\]

(equidimensionality of a flat morphism \([\text{I}^8], \text{IV}, 6.1.4\)). In case 2,

\[
\dim_k H^0(X_Y, \Omega^1_{X_Y}) = -\chi(X_Y, \mathcal{O}_{X_Y}) + 1 = \text{const}
\]

(invariance of the Euler characteristic of the fibers of a proper flat morphism \([\text{I}^8], \text{III}, 7.9.4\)).

Proposition 4. Let \( f : V \to B \) be a smooth fibering. Then:

1. The sheaf \( \Omega^1_{V/B} \) is locally free.
2. The sheaves \( R^if_\ast \Omega^1_{V/B} \) and \( R^if_\ast \mathcal{O}_V \) are locally free on \( B \), \( i \geq 0 \).
3. The sheaves \( f_\ast \Omega^1_{V/B} \) and \( R^if_\ast \mathcal{O}_V \) are dual to each other.
4. \( f_\ast \mathcal{O}_V \cong \mathcal{O}_B \).

Proof. Assertion 1 follows from the definition of a smooth morphism and the nonsingularity of \( V \).

To verify assertion 2 we use Theorem 4 of \([6]\) (§7), as in the proof of Lemma 6. Note firstly that \( f \) is flat, and so the sheaves \( \Omega^1_{V/B} \) and \( \mathcal{O}_V \), being locally free, are flat over \( B \). It is easy to see (formula (1)) that for any point \( b \in B \)

\[
\Omega^1_{V/b} |_{V_b} \cong \Omega^1_{V_b},
\]

\[
\mathcal{O}_V |_{V_b} \cong \mathcal{O}_{V_b},
\]

where \( V_b \) is the fiber of \( f \) over \( b \). It follows that \( \dim H^i(V_b, \Omega^1_{V/b} |_{V_b}) \) and \( \dim H^i(V_b, \mathcal{O}_V |_{V_b}) \) are independent of \( b \). The above-mentioned Theorem 4 yields assertion 2. It also follows from that theorem that the fibers of the fiberings over \( B \) associated with the sheaves \( f_\ast \Omega^1_{V/B} \) and \( R^if_\ast \mathcal{O}_V \) are isomorphic respectively with \( H^0(V_b, \Omega^1_{V/b}) \) and \( H^i(V_b, \mathcal{O}_{V/b}) \). There is a natural pairing

\[
f_\ast \Omega^1_{V/B} \times R^if_\ast \mathcal{O}_V \to R^if_\ast \Omega^1_{V/b}.
\]

Since the geometric fibers of a fibering are connected, the sheaf \( R^if_\ast \Omega^1_{V/B} \) has a regular section which nowhere vanishes \([\text{I}^9], \text{Chapter III}, \text{§} 11\). Hence

\[
R^if_\ast \Omega^1_{V/B} \cong \mathcal{O}_B.
\]

Using the duality theorem for the fibers of \( f \), we find that the above pairing is a duality of locally free sheaves.

Finally, assertion 4 is a general fact relating to fiberings with geometrically connected fibers. This completes the proof.
Proposition 5. Let \( f: V \to B \) be a smooth fibering and \( g \) the genus of a generic fiber of the mapping \( f \), with \( g > 1 \). Let

\[
d_{V/B} = -\text{deg} \wedge (\otimes^R f_+ \mathcal{O}_V).
\]

Then:

1. \( d_{V/B} \geq 0 \);
2. \( d_{V/B} = 0 \) if and only if \( V \) is isotrivial.

Proof. The theorem on a flat change of base ([8], III, 1.4.15) shows that for any covering \( B' \to B \) there is a natural isomorphism

\[
\phi^* R^1 f_+ \mathcal{O}_V \cong R^1 f'_+ \mathcal{O}_{V'}.
\]

where \( V' = V \times_B B', f' = f \times_B B' \), and hence

\[
d_{V'/B'} = \text{deg} \varphi \cdot d_{V/B}.
\]

It is therefore clear that we can restrict ourselves to fibrerings with sections, by an appropriate change of base. Thus, let the fibering \( f: V \to B \) have a section. Then ([14], Chapter 6, §1) there exists an abelian scheme \( p: A \to B \) and a \( B \)-morphism \( i: V \to A \) with the following properties. For any point \( b \in B \) the mapping \( i|_{V_b} \) induces an isomorphism \( \text{Pic}^0(V_b) \cong A_b \) and also a sheaf isomorphism

\[
p_* \Omega^1_{A/B} \cong f_* \Omega^1_{V/B}.
\]

This latter mapping is constructed as the composition of the natural mapping

\[
f_* i^* \Omega^1_{A/B} \to f_* \Omega^1_{V/B}
\]

and the change-of-base mapping

\[
p_* \Omega^1_{A/B} \to f_* i^* \Omega^1_{A/B},
\]

and the fact that it is an isomorphism follows from Lemmas 6 (assertion 3) and 7 and the corresponding fact for jacobian varieties. Assertion 3 of Proposition 4 shows that

\[
d_{V/B} = \text{deg} \wedge (\otimes^R f_+ \mathcal{O}_{V,B}) = \text{deg} \wedge (\otimes^R p_* \Omega^1_{A/B}).
\]

The abelian scheme \( A \) has a canonical polarization ([14], Chapter 6, §2). Suppose there exists a covering of the base \( B \) over which \( A \) becomes trivial, as a polarized family. Torelli's theorem shows that all fibers of the fibering \( V \) become isomorphic over some covering of the base. It follows that \( V \) is locally trivial as an analytic fibering, since curves of genus \( g > 1 \) have a local space of moduli [11]. Since the genus of each fiber is greater than unity, for all \( b \in B \) the group \( \text{Aut} V_b \) is finite, and \( V_b \), being a fibering with finite structural group, is trivial over the appropriate covering of the base. Thus the isotriviality of the polarized family \( A \) is equivalent to that of the fibering \( V \).

Let \( \phi: B' \to B \) be a covering of the base such that the abelian scheme \( A' = A \times_B B' \) has jacobian rigidity consisting of the choice of basis in the group of sections of given order \( n \geq 3 \).

Applying Lemma 7, we obtain:

\[
p_* \Omega^1_{A'/B'} \cong \varphi^* p_* \Omega^1_{A/B},
\]
since the sheaf $\Omega^1_{A/B}$ is invariant with respect to change of base (formula (1)). The fundamental theorem in modulus theory ([14], Chapter 7, §3) gives a mapping $\Psi: B' \to M$ into the scheme of moduli of polarized abelian varieties with given rigidity, for which

$$A' \cong \mathcal{M} \times_M B',$$

where $\pi: \mathcal{M} \to M$ is the canonical family of abelian varieties over $M$. Again we have

$$p^*\Omega^1_{A'/B'} \cong \Psi^*p^*\Omega^1_{\mathcal{M}/M}.$$

It follows from [5] (exp. 17) that the sheaf $\mathfrak{X} p^*\Omega^1_{\mathcal{M}/M}$ is ample on $\mathcal{M}^s$ (its sections are modular forms in the analytic interpretation of the scheme $\mathcal{M}$), and we see that always

$$\deg \mathfrak{X} p^*\Omega^1_{A'/B'} \geq 0,$$

where equality

$$\deg \mathfrak{X} p^*\Omega^1_{A'/B'} = 0$$

is equivalent to the mapping $\Psi$ being a point-mapping. Returning to the sheaf $R^1f_*\mathcal{O}_V$, we obtain the two assertions of the proposition.

§2. The finiteness theorem

**Theorem 3.** Let $B$ be a nonsingular proper curve of genus $q$, $q \geq 2$, $g > 1$. Then up to isomorphism there exist only a finite number of smooth non-isotrivial fiberings with base $B$ and generic fiber genus $g$.

**Proposition 6.** Let $f: V \to B$ be a smooth non-isotrivial fibering, $q$ the genus of $B$, $q \geq 2$. Then the sheaf $\Omega^1_{V/B}$ is ample on $V$.

**Corollary.** Under the hypothesis of the proposition,

$$H^1(V, \Omega^1_{V/B}) = 0.$$

The proof follows from Kodaira's theorem on the vanishing of the cohomology of negative fiberings.

**Lemma 8.** Under the hypothesis of Proposition 6, let $K$ be the canonical class of the surface $V$, $K_B$ the canonical class of the curve $B$. Then:

1. $c_1(\Omega^1_{V/B}) = K - f^* (K_B)$,
2. $\Omega^1_{V/B} \cdot \Omega^1_{V/B} = 12d_{V/B}$.

**Proof.** Assertion 1 follows from the usual exact sequence

$$0 \to f^*\Omega^1_B \to \Omega^1_V \to \Omega^1_{V/B} \to 0$$

and assertion 1 of Proposition 4.

We now prove assertion 2. We have:

$$\Omega^1_{V/B} \cdot \Omega^1_{V/B} = (K - f^* (K_B))^2 = K^2 - 2\chi = 12p_a(V) - 3\chi,$$

(2)

*This was established by Grothendieck, as communicated to me by I. R. Šafarevič.*
where \( \chi = (2g - 2)(2q - 2) \) is the Euler characteristic of the surface \( V \). From the spectral sequence of the fibering we obtain:

\[
\chi(V, \mathcal{O}_V) = \sum_i (-1)^i \chi(B, R^i f_* \mathcal{O}_V).
\]

Since \( R^i f_* \mathcal{O}_V \cong \mathcal{O}_B, R^i f_* \mathcal{O}_V = 0, i > 1 \), we have

\[
p_a(V) = \chi(V, \mathcal{O}_V) = d_{V/B} + \frac{\chi}{4},
\]

which with (2) yields the desired result.

Remark. The formula \( p_a(V) = d_{V/B} + \chi/4 \) occurs in the paper by Tate \([18]\), from which we have also borrowed the proof.

Proof of Proposition 6. By virtue of the Moishezon-Nakai criterion, it is sufficient to verify that

\[
(K - f^* K_B)^2 > 0,
\]

\[
(K - f^* K_B) \cdot C > 0
\]

for any curve \( C \) on the surface \( V \). The first inequality follows from Proposition 5 and Lemma 8. Note now that \( K^2 > 2\chi > 0 \). Let \( F \) be a fiber of the morphism \( f \). Then if \( C = F \),

\[
(K - f^* K_B) \cdot C = K \cdot F = 2g - 2 > 0.
\]

If \( C \) is mapped by \( f \) onto the curve \( B \), consider the determinant

\[
\begin{vmatrix}
C^2 & C \cdot K & C \cdot F \\
C \cdot K & K^2 & K \cdot F \\
C \cdot F & K \cdot F & F^2
\end{vmatrix} \geq 0.
\]

That it is nonnegative follows from Hodge's index theorem. Noting that

\[
C \cdot F = n = \deg C/B \geq 1,
\]

\[
K \cdot F = 2g - 2,
\]

\[
F^2 = 0,
\]

\[
C^2 + C \cdot K = 2p - 2,
\]

\[
2p - 2 = n(2q - 2) + \alpha, \quad \alpha \geq 0,
\]

we obtain

\[
2(2g - 2)n(2p - 2 - C^2) - n^2 K^2 - (2g - 2)^2 C^2 \geq 0,
\]

or

\[
C^2 \leq \frac{n^2(2\chi - K^3)}{(2g - 2)(2g + 2n - 2)} + \frac{2n\alpha}{2g + 2n - 2}.
\]

Whence

\[
C^2 \leq \alpha.
\]
On the other hand

$$(K - f^*K_B) - C = 2p - 2 - C^2 - (2q - 2)n = a - C^2.$$  

Comparing these relations, we obtain the required inequality. (The idea of using Hodge’s theorem for a bound on $C^2$ was prompted by I. R. Šafarevič.)

Remark 1. It follows from the proof that on a non-isotrivial fibering any section $C$ has a negative square.

Remark 2. It would be interesting to determine whether Proposition 6 is valid for any non-isotrivial fiberings without multiple components. It can be shown that it holds if the dimension of the trace of the jacobian variety of the generic fiber is not less than 2.

Proof of Theorem 3. As follows from Theorem 1 of Chapter 1, it is sufficient to prove the following assertion.

Let \( f: W \rightarrow U \times B \) be a smooth proper epimorphism of non-singular connected algebraic varieties \( W, U, B \), where \( B \) is a proper curve of genus \( q > 1 \), and for any point \( u \in U \), \( f|_W: W_u \rightarrow B \) is a smooth fibering. Then there exist only a finite number of smooth non-isotrivial fiberings of the form \( W_u \).

Denote by

$$p = \text{pr}_2 \circ f: W \rightarrow U$$

a smooth family of surfaces. Let \( u_0 \in U \), \( W_0 = \text{pr}_1^{-1}(u_0) \), and \( f_0 = f|_{W_0}: W_0 \rightarrow B \) be a non-isotrivial fibering. The smoothness conditions imply the existence of families of open sets \( \{ W_{a, \beta} \} \) in \( W \) and \( \{ B_\alpha \} \) in \( B \), and a neighborhood \( U_0 \) of the point \( u_0 \), satisfying the following conditions:

1. \( \bigcup_{a, \beta} W_{a, \beta} \cap W_0 = W_0, \)
2. \( \bigcup_\alpha B_\alpha = B; \)
3. \( (\text{pr}_1 \circ f)(W_{a, \beta}) = B_\alpha; \quad p(W_{a, \beta}) = U_0. \)
4. The fibering \( f|_{W_{a, \beta}}: W_{a, \beta} \rightarrow U_0 \times B_\alpha \) is analytically trivial.

Now let \( t \) be a tangent vector to the variety \( U \) at \( u_0 \). Then there exist tangent vector fields \( \xi_{a, \beta} \) to \( W \) defined in \( W_{a, \beta} \) such that

\[
(\text{pr}_1 \circ f)_* \xi_{a, \beta} = 0,
\]

\[
p_\ast \xi_{a, \beta} = t.
\]

It follows that the Kodaira cocycle \([12]\]

\[
h_t = \{ h_{a, \beta, \gamma, \delta} \}, \quad h_{a, \beta, \gamma, \delta} = \xi_{a, \beta} \mid W_{a, \beta} - \xi_{\gamma, \delta} \mid W_{a, \beta},
\]

which is an element of the group \( H^1(W_0, \mathcal{O}_{w_0}(T_{w_0})) \), belongs to the image of the mapping

\[
H^1(W_0, \mathcal{O}_{w_0}(T_{w_0/B})) \rightarrow H^1(W_0, \mathcal{O}_{w_0}(T_{w_0})),
\]

where \( T_{w_0/B} \) is the relative tangent bundle and \( T_{w_0} \) the tangent bundle. By definition, \( \mathcal{O}_{w_0}(T_{w_0/B}) = \mathcal{O}_{w_0/B} \) and the corollary to Proposition 6 shows that \( h_t = 0 \) for any vector \( t \). Hence the family \( p: W \rightarrow U \) is analytically trivial in the neighborhood of \( u_0 \) \([12]\). Note now that for all \( u \in U \) the fibering \( W_u \) is non-isotrivial. Indeed, applying Proposition 5 and the formula \( d_{V/B} = p_\ast (V) - \chi/4 \), we
see that it is sufficient to show that $p_a(W_u)$ and $\chi(W_u)$ are independent of the choice of $u \in U$. For the arithmetic genus this is a general property of a smooth (or even flat) proper morphism ([8], III, 7.9.4). The constancy of the Euler characteristic follows from the equation $\chi = (2g - 2)(2g - 2)$, valid for any smooth fibering ([1], Chapter IV, §4).

It now remains to use the connectedness of $U$ and the theorem of de Franchis ([13], p. 139) to complete the proof of the theorem.

Remark. As shown by Kodaira, for sufficiently large $q$ there exist smooth non-isotrivial fiberings for which the genus of the base is $q$ (see the Appendix).

§3. Smooth fiberings over algebraic curves of genus 0 or 1

Theorem 4. Let $f: V \to B$ be a smooth fibering and $g$ the genus of its generic fiber, $g > 1$. If the genus of the curve $B$ is 0 or 1, then the fibering $V$ is isotrivial.

Proof. The group $\text{Aut} B$ acts transitively on $B$. If $\sigma: B \to B$ is an automorphism of the curve, denote by $V_\sigma$ the smooth fibering $f_\sigma: V \times_B B \to B$, $f_\sigma = f \times_B B$. If $b \in B$, the fiber of $V_\sigma$ at the point $b$ is $V_\sigma(b)$.

Now note that Theorem 3 remains true when the genus of the base is $q \leq 1$. Indeed, if the genus of $B$ is at most unity, there exists a normal covering $B' \to B$ such that the genus of $B'$ is greater than unity. The change-of-base functor gives a mapping of $B$-fiberings into the set of $B'$-fiberings, and the standard constructions of Galois theory in conjunction with the fact that the group $\text{Aut}_{g_\sigma}V$ is always finite ($g > 1$ and Theorem 0) imply that this mapping is finite. Hence our assertion follows from Theorem 3 for $q \geq 2$.

The set $H = \{ \sigma \in \text{Aut} B, V_\sigma \cong V \text{ over } B \}$ forms a subgroup having, by what has just been said, a finite index. Taking into account the structure of the group $\text{Aut} B$, we find that $H$ is also transitive on $B$. Using the relation between the fibers of $V_\sigma$ and $V$, we find that $V_b \cong V_{\sigma(b)}$ for all $\sigma \in H$. It follows that all the fibers of $V$ are isomorphic. Hence $V$ is analytically locally trivial [11]. Since for all $b \in B$ the group $\text{Aut} V_b$ is finite, we find that $V$ is an analytic fiber space with finite structural group and hence becomes trivial over some (even unbranched) covering of the base. This proves the theorem.

Remark 1. In this chapter we have used the definition of isotriviality in §1 of Chapter 1. But for smooth fiberings a more natural definition is Serre's, i.e., the requirement of triviality over some unbranched covering of the base. It is easy to see that for smooth fiberings the definitions coincide. Indeed, smoothness and isotriviality imply the isomorphism of all the fibers, and the argument with which we have just concluded the proof of Theorem 4 shows that a fibering isotrivial in our sense is isotrivial in Serre's.

Remark 2. For curves $B$ of genus 0 the theorem has been proved by other means by Šafarevič [3].

Remark 3. Arguing as in the proof of Theorem 4, we can obtain from the conjecture mentioned in the Introduction the result that any fibering over a base of genus 0 has at least three points of degeneracy, and over an elliptic base at least two.

We note further that the results obtained in this chapter can be reformulated in the language of
curves. Call the curve $X$ nondegenerate if its associated minimal fibering is smooth.

**Theorem 3.** The set of nondegenerate nonconstant $K$-curves of given genus $g > 1$ is finite.

**Theorem 4.** Every nondegenerate curve of genus $g > 1$ defined over a field of genus 0 or 1 is constant.

**CHAPTER 3. RATIONAL POINTS OF CURVES OF GENUS $g > 1$ DEFINED OVER A FUNCTION FIELD**

§1. Branched coverings of curves of genus $g > 1$

Let $K$ be, as usual, the field of rational functions on a nonsingular projective curve $B$. Consider a complete nonsingular curve $X$ of genus $g > 1$, defined over $K$ and geometrically irreducible. Denote by $f: V \to B$ the corresponding minimal fibering (§1, Chapter 1).

We shall say that the curve $X$ is nondegenerate at the point $b \in B$ if the fiber $V_b$ of the fibering $V$ is nondegenerate.

The set $S$ of points $b \in B$ at which the fibering $V$ is degenerate is finite and is called the type of the curve $X$.

**Lemma 9.** Let $\phi: B' \to B$ be a covering of nonsingular curves, $K' \supset K$ the corresponding extension of the field of functions. Then for the type $S'$ of the curve $X' = X \otimes K'$ we have:

$$S' \subset \phi^{-1}(S).$$

The proof follows easily from the definitions.

We proceed now to the fundamental construction of this section. Let $P_i \in X(K)$, $i = 1, \ldots, n$, be distinct $K$-rational points of $X$. They correspond to sections $\tilde{P_i}$ of the fibering $V$. Put $D = \Sigma P_i$ and consider the set $S_D$ of points of the curve $B$:

$$S_D = \{b \in B: A_{i,j} \tilde{P_i} \cap \tilde{P_j} \cap V_b \neq \emptyset \}.$$

If $P \in X(K)$ and $P \neq P_i$, $i = 1, \ldots, n$, denote by $D \cap P$ the following set of points of $B$:

$$D \cap P = \{b \in B: \exists i \tilde{P_i} \cap P \cap V_b \neq \emptyset \}.$$

It is not difficult to see that $S_D$ and $D \cap P$ are finite sets of closed points of $B$.

We now construct a certain covering of $X$ and study its type. To do this we require generalized jacobian varieties, whose definitions and properties are given in Serre's book [17]. In the present situation there is determined the algebraic variety $I^{(1)}_D(X)$. A point $P \in X(K) \setminus \text{Supp} D$ determines a point in $I^{(1)}_D(X)$ and turns it into a commutative algebraic group defined over $K$ and denoted below by $I_{D, P}(X)$. We have the canonical mapping

$$\Psi: X \setminus \text{Supp} D \to I_{D, P}(X).$$

If $m \delta$, $m > 1$, is the mapping representing multiplication by $m$,

$$m \delta: I_{D, P}(X) \to I_{D, P}(X),$$

its restriction to $X \setminus \text{Supp} D$ determines a covering

$$\Theta': X' \to X \setminus \text{Supp} D,$$
defined over $K$. Denoting the normalization of the curve $X'$ by $X_{D, p, m}$, we obtain a geometrically irreducible nonsingular complete curve and a covering

$$\Theta: X_{D, p, m} \to X,$$

unbranched outside $\text{Supp} D$. The covering $\Theta$ is normal and its Galois group is $(\mathbb{Z}/m\mathbb{Z})^{2g + n - 1}$. If $n > 1$, then $\Theta$ is branched in at least one point, since $H_1(X, \mathbb{Z}) = (\mathbb{Z})^{2g}$.

Lemma 10. Put

$$R = S \cup S_D \cup (D \cap P).$$

Then the curve $X_{D, p, m}$ is nondegenerate outside $R$.

Proof. We recall firstly some facts about generalized jacobian varieties. Let $Y$ be a complete non-singular algebraic curve defined over an algebraically closed field $k$ of characteristic 0, $A$ its jacobian variety, and $G_m$ the multiplicative group. If $P_1, \ldots, P_n, P$ are distinct points of $Y$, denote by $I_D, P, \ldots, P$ as above, the generalized jacobian variety corresponding to the "modulus" $D = \Sigma P_i$. Consider also the group $L_D$

$$1 \leftarrow L_D \leftarrow \prod_{i=1}^{n} (G_m)_l \leftarrow G_m \leftarrow 1,$$

where $\alpha$ is the diagonal injection. Then $I_D, P$ is an extension of $L_D$ by $A$ (the ordinary jacobian variety) and hence determines a point $d$ in $\text{Ext}(A, L_D)$. We have the commutative diagram

$$
\begin{array}{ccc}
0 & \leftarrow & \text{Ext}(A, L_D) \\
\downarrow \cong & & \downarrow \cong \\
\prod_{i=1}^{n} \text{Pic}^0 A & \cong & \text{Pic}^0 A \\
\downarrow \cong & & \downarrow \cong \\
\prod_{i=1}^{n} \text{Pic}^0 Y & \cong & \text{Pic}^0 Y \\
\end{array}
$$

where $\epsilon$ is the composite of the morphisms (cf. [17], Chapter VII, no. 16,23). The set of divisors

$$(P_1 - P, \ldots, P_n - P)$$

determines a point in $\prod_{i=1}^{n} \text{Pic}^0 Y$ and, as follows from [17] (Chapter VII, no. 20),

$$\epsilon(P_1 - P, \ldots, P_n - P) = d.$$

The fiberings over $Y$ corresponding to the divisors $P_i - P$ have rational sections $s_i$ whose divisors are respectively equal to $P_i - P$. The set of these sections defines a rational mapping of $Y$ into an algebraic group $I_D$, which is an extension of $\prod_{i=1}^{n} (G_m)_i$ by $A$. This extension is determined by the element $d$, as is clear from the commutative diagram above. We have an exact sequence

$$0 \leftarrow I_{D, p} \leftarrow I_D' \leftarrow G_m \leftarrow 1.$$
and the composition \( \alpha \circ (s_1, \ldots, s_n) \) gives a rational mapping \( \phi_D: Y \to I_D, p \). It is regular in \( P \) and is the canonical mapping of the curve \( Y \) into the generalized jacobian variety ([17], Chapter VII, no. 20).

We turn now to the proof of the lemma. Put

\[
V' = V - f^{-1}(R), \quad B' = B - R, \quad f' = f|_{V'}: V' \to B'.
\]

Then we have a smooth fibering \( f': V' \to B' \), a group scheme \( I \) over \( B' \), and a morphism \( \phi: V' \to I \), where for any point \( b \in B' \) the fiber \( I_b \) is the jacobian variety of the fiber \( V_b = \text{Pic}^0 V_b \), while \( \phi_b = \phi|_{V_b}: V_b \to I_b \) is the canonical morphism of the curve into the jacobian ([14], Chapter 6). The divisors \( \bar{D}_i = \bar{P}_i - \bar{P} \) determine sections of the scheme \( \text{Pic}^0 (V'/B') \), and hence of \( \text{Pic}^0 (I/B') \) (since \( \phi \) induces an isomorphism \( \text{Pic}^0 V'/B' \cong \text{Pic}^0 I/B' \)). By the generalized Weil-Barsotti formula ([16], Theorem 18.1) we obtain a set of points of the group \( \text{Ext}_{B'}(I, (G_m)_B) \) and, using the exact sequence

\[
0 \to \text{Ext}_{B'}(I, (L_D)_{B'}) \to \prod_{i=1}^n \text{Ext}_{B'}(I, G_{mB'}) \to \text{Ext}_{B'}(I, G_{mB'}) \to 0,
\]

a point of the group \( \text{Ext}_{B'}(I, (L_D)_{B'}) \). Thus we have constructed an extension of the group scheme \( (L_D)_{B'} = L_D \times B' \) by \( I \). Denote the group scheme thus obtained by \( I' \). We have the exact sequence

\[
0 \to I' \to \prod_{i=1}^n I_i \to (G_m)_{B'} \to 0,
\]

where \( I_i \) is the extension of \( (G_m)_{B'} \) corresponding to the divisor \( \Delta_i \). Restricting \( I_i \) to \( V' \) via the morphism \( \phi \), we obtain a one-dimensional fibering over \( V' \) determined by the divisor \( \Delta_i \). It has a rational sections \( s_i \), of which \( \Delta_i \) is the divisor. The set of sections \( \alpha' \circ (s_1, \ldots, s_n) \) gives a rational mapping of the surface \( V' \) into \( \prod_{i=1}^n I_i \) and the composition \( \alpha' \circ (s_1, \ldots, s_n) \) defines, as above, a morphism

\( \phi_D: V' \setminus \text{Supp} \Delta \to I' \). The scheme \( I' \) and morphism \( \phi_D \) have the following properties, as follows from the facts above concerning jacobian varieties:

1. The generic fiber of \( I' \) is \( I_D, p (X) \).
2. The morphism \( \phi_D \) on the generic fiber of \( V' \) is \( \Psi \).
3. For all \( b \in B' \), \( I_b = I_{D_b}, p_b (X) \), where \( D_b, p_b \) are the restrictions of \( \Delta \) and \( \Delta \) to the fiber \( V_b \).
4. For all \( b \in B' \) the morphism \( \phi_D|_{V_b}: V_b \to I_b \) is the canonical mapping of the curve into the generalized jacobian.

The restriction of the morphism

\[
m \delta: I' \to I'
\]

defines covering

\[
u: V'' \to V' - \Delta.
\]

The generic fiber of \( V'' \) is \( X' \), and on the generic fiber the morphism \( \nu \) coincides with \( \Theta \). If \( b \in B' \), then

\[
V_b = I'_b \times P_b (V_b - D_b)
\]
and we see that the genus of the normalization of the curve $V_b^g$ is independent of $b$ (since it is equal to $1 + m^{2g+n-1}(g - 1) + \frac{1}{2} (m - 1) nm^{2g+n-2}$, as follows from the Hurwitz formula).

Thus, we have constructed a model of the curve $X_D, P, m$, in general incomplete. Its completion may have singularities. If we resolve these and contract the exceptional curves of the first kind lying in the fibers, we obtain the nonsingular minimal model $\mathcal{W}$ of the curve $X_D, P, m$. For each point $b \in B'$ the fiber $\mathcal{W}_b$ contains as a component a completion of the curve $V_b^g$, since the genus of $V_b^g$ is greater than zero. Moreover, the genus of this component is equal to the genus $g'$ of the generic fiber of the fibering $\mathcal{W}$. We shall show that $\mathcal{W}$ is nondegenerate at the point $b \in B'$. Let

$$W_b = pX + Y, \ p \gg 1,$$

where $X$ is birationally equivalent to $V_b^g$. We have

$$K \cdot W_b = 2g' - 2 = pK \cdot X + K \cdot Y = p(2g' - 2 + 6) - pX^2 + K \cdot Y;$$

here $K$ is the canonical class of the surface $\mathcal{W}$, $K \cdot Y \geq 0$ (as we have seen in §2 of Chapter 1), and $\delta = 2p_0(X) - 2g' \gg 0, g' \gg 1$. Hence we find:

$$pX^2 \geq K \cdot Y + p\delta + (p - 1)(2g' - 2) \gg 0$$

and since $X^2 \leq 0$ always ([1], Chapter VII), we have

$$X^2 = 0.$$

This means, firstly, that $p = 1, \delta = 0$ (otherwise $X^2 > 0$); and secondly, that $Y = \emptyset$ ([1], Chapter VII, Theorem 2). This proves our assertion, and thereby the lemma.

Proposition 7. Let $X$ be a nonsingular complete geometrically irreducible curve of genus $g \geq 1$ defined over the function field $K = k(B)$, $P \in X(K)$ a rational point, and $S$ the type of $X$. Then there exist a finite extension $K' \supset K$ and a covering $\phi: X' \to X$, defined over $K'$ with the following properties:

1. $K'$ is unbranched outside $S$.
2. The degree of the morphism $\phi$ and $[K' : k]$ depend only on $g$.
3. The morphism $\phi$ is branched at $P$ and unbranched outside $P$.
4. $\phi^*(P) = \sum e_i P_i, \ P_i \in X'(K').$
5. The type of the curve $X'$ is at most that of the curve $X \otimes K'$.

Proof. Let $\Phi: Y \to X$ be the covering induced through multiplication by 2 in the jacobian of the curve $X$. The type of $Y$ is at most that of $X$. The cycle $\Phi^*(P)$ is rational over $K$. Let $K' \supset K$ be the extension over which $\Phi^*(P)$ is rational over $K$. Then $K'$ is unbranched outside $S$ and the degree $[K' : K]$ is at most $2^{2g}$. On the curve $Y \otimes K'$ we have the divisor $D = \sum_i P_i$ and the point $P \in \text{Supp} D$. Put
Then assertion 5 follows from Lemmas 9 and 10. We have already verified assertions 1 and 4. Finally, assertions 2 and 3 follow from the general properties of generalized jacobian.

§2. Manin's theorem

Theorem 5 (Manin \([2]\)). Let \(X\) be a complete nonsingular geometrically irreducible nonconstant curve of genus \(g > 1\) defined over the function field \(K = k(B)\). Then the set \(X(K)\) of \(K\)-rational points is finite.

Proof. Let \(f: V \to B\) be the minimal fibering associated with the curve \(X\). We know \([1, 3]\) that the finiteness of the set \(X(K)\) will follow from the fact that the numbers \(C \cdot K_V\), where \(C\) runs through all sections of the fibering \(V\) and \(K_V\) is a canonical divisor of \(V\), are bounded above.

Let \(P \in X(K)\); let \(X_P\) denote the covering of \(X\) constructed in Proposition 7, and let \(C\) be the section of the fibering \(V\) corresponding to \(P\). Let \(V_P\) and \(V'\) be the nonsingular minimal models of the curves \(X_P, X \otimes K'\); and \(\psi: V_P \to V'\) the rational mapping equal on the generic fiber to the morphism \(\phi\) of Proposition 7. By applying \(\sigma\)-processes to the surface \(V_P\) we can make the mapping \(\psi\) regular (\([1]\), Chapter 1). Let \(V''\) be the surface thus obtained, \(s\) the number of \(\sigma\)-processes needed. We have the diagram

\[
\begin{array}{ccc}
V'' & \xrightarrow{\psi} & V' \\
\downarrow \tau & & \downarrow \psi \\
V_P & & V'
\end{array}
\]

where \(\tau\) is the product of the \(s\) \(\sigma\)-processes and \(u\) is a morphism. We can assume that the genus of the base \(B\) is greater than 1. Hence the geometric genus of the surface \(V'\) is greater than 0 (see Chapter 1, §2), and there exists a regular differential \(\omega \neq 0\) of degree 2 on \(V'\). Let us find the canonical divisor \(K_{V''}\) of the surface \(V''\). We have

\[
K_{V''} = (u^*(\omega))_{V''}.
\]

It follows that

\[
K_{V''} = u^*(K_{V'}) + M,
\]

where \(M\) is an effective divisor. On the other hand, on the generic fiber \(X_P\) of the fibering \(V''\) we have the equations

\[
\psi^*(P) = \Sigma e_i P_i, \quad P_i \in X_P(K'),
\]

\[
K_{X_P} = \psi^*(K_X) + \Sigma (e_i - 1) P_i = X_P. u^*(K_{V'}) + \Sigma (e_i - 1) P_i.
\]

If \(C_i\) denotes the section of \(V''\) corresponding to the point \(P_i\), we find that the divisor \(M\) is

\[
\Sigma (e_i - 1) C_i + \theta,
\]

where \(\theta\) is an effective divisor consisting of fiber components. Finally, note that the exceptional curves of the first kind occur in \(M\) with multiplicity 1. Indeed,

\[
K_{V''} = (\tau^*\psi^*(\omega))_{V''},
\]
and we need only observe that \( \psi^*(\omega) \) is a regular differential and apply the formula for the behavior of a canonical class under a \( \sigma \)-process. Thus we have

\[
K_{V''} = u^*(K_V) + \Sigma (e_i - 1)C_i + \Theta,
\]

where \( K_{V''} \cdot C_i \geq 0 \), \( K_{V''} \cdot \Theta \geq -s \) (as we have seen in §2, Chapter 1, it is always the case that \( K \cdot C \geq -1 \) and \( K \cdot C = -1 \) means that \( C \) is an exceptional curve of the first kind).

We come now to the fundamental bound:

\[
K_{V''} \cdot K_{V''} \geq K_{V''} \cdot u^*(K_V) - s \geq u^*(K_V) \cdot u^*(K_V) + \Sigma (e_i - 1)C_i \cdot u^*(K_V) + \Theta \cdot u^*(K_V) - s
\]

\[
\geq (\deg u)K_{V''} \cdot K_{V''} + \Sigma (e_i - 1)u_*(C_i) \cdot K_{V''} + u_*(\Theta) \cdot K_{V''} - s \geq \Sigma (e_i - 1)u_*(C_i) \cdot K_{V''} - s.
\]

There exists an \( i_0 \) for which \( e_{i_0} > 1 \), and we obtain

\[
C' \cdot K_{V''} \leq K_{V''} \cdot K_{V''} + s \leq K_{V''} \cdot K_{V''},
\]

where \( C' = u_*(C_{i_0}) \) is the section of \( V' \) obtained from the section \( C \) of \( V \).

By conditions 1 and 2 of Proposition 7, there are only finitely many fields \( K' \). Hence if the set \( X(K) \) is infinite, there exist an infinite number of points \( P \in X(K) \) such that \( K_P \cong K' \). Each of these points determines a point \( P' \in (X \otimes K')(K') \) and a section \( C' \) of \( V' \). But as follows from the above bound for \( C' \cdot K_{V''} \), there are only finitely many such sections (and points). This contradiction proves the theorem.

Remark 1. It is easy to show that the proof of Theorem 5 is entirely effective. It leads to the following bound for the height:

\[
C \cdot K \leq (q_3 + 1) \exp(\exp 4g)!
\]

Further considerations can be used to obtain a more exact bound.

Remark 2. We can show that Mordell's conjecture follows from that of Šafarevič, formulated in the Introduction, and Proposition 7. If \( P \in X(K) \), we have the covering \( X_P \) defined over the field \( K_P \) (Proposition 7). It is easy to see that all the curves \( X_P \), as well as \( X \), are nonconstant. Moreover, as already remarked, there are only a finite number of fields \( K' \). Hence from Proposition 7 (assertions 2 and 5) and the Šafarevič conjecture it follows that the set of curves \( X_P \) is finite up to isomorphism. Since any curve has only a finite number of morphisms onto a curve of genus \( g > 1 \), we conclude that the set of coverings \( X_P \to X \) is finite. From this and Proposition 7 (assertion 3) it follows that the set \( X(K) \) is finite.

Appendix

1. Kodaira's examples. The proof in Chapter 2 that the number of smooth non-isotrivial fiberings is finite left open the question of the existence of such fiberings. The first examples of such fiberings were constructed by Kodaira (cf. note on Russian p. 174 of [4]). We now give a construction of these examples, essentially coinciding with Kodaira's.

Let \( Y \) be an arbitrary nonsingular proper curve defined over \( k \). Assume the genus \( g(Y) \) is greater than 1, and choose a covering \( \phi: B \to Y \). The graph of the morphism \( \phi \) defines a section \( C \) of the
trivial fibering \( f: V = Y \times B \to B \). This section is nonconstant, i.e., it does not have the form \( Y \times B \), \( y \in Y \). Consider the curve \( X = Y \otimes K, K = k(B) \). It is defined over \( K \), and is nonsingular, proper and geometrically irreducible. The section \( C \) determines a rational point on \( X \), which we denote by \( P \).

Now apply Proposition 7. We obtain a covering \( \phi: X' \to X \) of \( X \), which is branched at \( P \). The minimal fibering associated with \( A'' \) is smooth (assertion 5 of Proposition 7). We show that it is not isotrivial. Denoting it by \( f': V' \to B' \), we find from the proof of Proposition 7 that for each point \( b \in B' \) the fiber \( V'_{b} \) is a covering of a fiber of \( V \), i.e., of the curve \( Y \), branched at the point \( \phi(b) \).

If the fibering \( V' \) were isotrivial, then, as we have seen in §3, Chapter 2, all its fibers would be isomorphic. Our construction would then give a continuous family of subfields of the function field \( K = k(V_{b}) \). Since all these subfields are distinct (this follows from a comparison of the branch points and the choice of the section \( C \)), the contradiction proves the required result.

2. Spaces of moduli. The results obtained in Chapters 1 and 2 give some information on the spaces of moduli of algebraic curves of given genus \( g > 1 \). The definition of the space of moduli, denoted below by \( M_{g} \), is given in [14]. Theorem 2 of Chapter 1 leads to the following fact.

Take a fixed hyperplane section on \( M_{g} \). Then curves of given genus \( g \) contained in \( M_{g} \) have bounded degree.

If we consider curves on \( M_{g} \) isomorphic with a given curve \( B \), we find from Theorem 3 of Chapter 2 that the set

\[ \{X \subset M_{g}: X \cong B\} \]

is always finite. Moreover, \( M_{g} \) contains no curves of genus 0 or 1. It goes without saying that the above is true only for proper curves.

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