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RATIONAL POINTS ON ALGEBRAIC CURVES

by Yu. I. MANIN

1. The solution of equations in integers or in rationals is the object of diophantine analysis: the etymology of the name indicates the antiquity of the subject. The apparently elementary nature of these problems has given rise to a flood of literature devoted to particular equations, particular solutions of which can be established by particular means: cf. the 800-page survey volume of Dickson [3] or the booklet of Sierpiński [12]. The history of Fermat's problem since Kummer is the most famous example of exercises of this kind. But all this is irrelevant to the theme of the present survey.

The traditional arguments using divisibility and a consideration of the real "size" of solutions (Fermat's method of descent) acquired the following interpretation after Hensel had introduced p-adic numbers: these arguments reduce to the investigation of the solutions of the problem in the completions of the field of rationals with respect to all its topologies (for by Ostrowski's theorem the real and the p-adic topologies exhaust all possibilities). This interpretation is in accordance with the general "geometric" point of view in which all complex-solutions of a given system of equations are regarded as points of an algebraic variety, and the diophantine problem consists in separating out from them the points whose coordinates lie in a given field or ring. An algebraic variety has a number of structures (of a topological space with respect to all the topologies, of an analytical manifold, and then of a Lie group or transformation space etc.) which must be intensively used in conjunction with arithmetical methods. Besides, algebraic geometry provides a natural language for the expression of such concepts as the number of independent variables, change of variables, the degree of an equation, rational parametrization etc. Finally, it allows us to classify the set of diophantine problems by the invariants of the corresponding algebraic variety (see the survey by Lang [5]).

The simplest invariant of this kind is the dimension of the variety. The following account deals with the one-dimensional case, but an important tool in the investigation will be results about special varieties of higher dimension.

2. A one-dimensional algebraic variety is traditionally called a curve. From the complex point of view, however, an algebraic curve is actually a surface. We shall restrict ourselves to the case of compact connected Riemann surfaces with a tangent plane at every point. Algebraically the problem is reduced to the discussion of irreducible curves without singular points, lying in projective space. This does not restrict the generality if we have in mind only qualitative questions such as the structure of the
set of rational solutions, whether they are finite in number etc.

The topological type of a compact Riemann surface is described by a single integral invariant, its genus, which may take all non-negative values. We call the genus of an algebraic curve the genus of the corresponding Riemann surface. The genus of a plane algebraic curve of degree \( n \) without singular points is \( \frac{1}{2}(n-1)(n-2) \); in particular, curves of degree 1 and 2 have genus 0.

The value of the genus determines to a remarkable degree the structure not merely of all the points on the curve, but also that of its arithmetical subsets, those of all rational or all integral points. Below we give a survey of the known results in this direction.

3. Curves of genus 0. Every curve of genus 0 is isomorphic to a plane curve of the second degree \( ax^2 + bx^2 + cx^2^2 = 0 \). (D. Hilbert and Hurwitz [4]). The isomorphism can be obtained by a birational change of variables with coefficients in the ground field.

If on the curve \( \Gamma: ax^2 + bx^2 + cx^2^2 = 0 \) there is at least one rational point \( P \), then there are infinitely many and they can all be obtained as the points of intersection of the curve with the lines through \( P \) whose equations have rational coefficients. The set of such lines, and with them the set of rational points on \( \Gamma \) can be parametrized by rational functions of a single parameter (in inhomogeneous form) Classical example:

\[
x^2 + y^2 = 1, \quad x = \frac{1-r^2}{1+r^2}, \quad y = \frac{2r}{1+r^2}.
\]

4. Curves of genus 1. A topological surface of genus 1 is a torus, i.e. the direct product of two circles, so that it can be given the structure of a compact commutative Lie group. It is remarkable that if a curve of genus 1 has at least one rational point (which can be taken as the zero of the group law) then it possesses a structure of a Lie group whose law of composition is expressed in terms of rational functions in the coordinates with coefficients in the ground field. Hence, the rational points form a commutative group and the fundamental fact about it, first stated as a conjecture by Poincaré and proved by Mordell [11], is that it has a finite number of generators. The comparatively elementary proof of this fact by the method of descent depends on explicit formulae for the group law on curves of genus 1 in "Weierstrass normal form" \( y^2 = x^3 + ax + b \).

5. Curves of genus \( g > 1 \). Equations of genus \( g > 1 \) cannot be parametrized by rational functions, nor can they be given a Lie group structure (even if one excises from the curve a finite number of points): for the fundamental group of a surface of genus \( g > 1 \) is non-commutative. But one can embed every curve \( \Gamma \) of genus \( g > 1 \) that has a rational point in an algebraic variety \( J \) which is topologically a 2g-dimensional torus and which can be given a Lie group structure in terms of rational functions. The variety \( J \) is called the Jacobian variety of the curve \( \Gamma \). A direct product of a finite number of Jacobian varieties and all possible homomorphic images of such products are called Abelian Varieties. In particular, every curve of genus 1 with a rational point is a 1-dimensional Abelian variety.

From a number-theoretical point of view the importance of the embedding of a curve in its Jacobian variety is brought out by the following theorem of Mordell-Weil [14]: the commutative group of the rational points on any
abelian variety has a finite number of generators. This theorem means that there are comparatively few rational points on \( J \); contrast the affine plane regarded as a group under addition, which does not have a finite number of generators, although it is trivially possible to embed in it a curve without points.

Unfortunately this theorem does not give an immediate method of finding the rational points on the curve itself. Mordell [11] announced the following conjecture: the number of rational points on any curve of genus \( g > 1 \) is finite. Until now Mordell's conjecture remains unproved. The strongest result in this direction is connected with a generalization of the problem. Weil [15] showed that the theorem about the finiteness of the number of generators is valid for abelian varieties over an arbitrary field of algebraic numbers (not merely the rationals), and then Lang and Néron [7] showed that the Mordell-Weil Theorem still remains true if the ground field is taken to be any field generated by a finite number of elements (some of which may be transcendental). Thus, it is natural to assume that Mordell's finiteness conjecture must also be valid for any field of finite type.

Recently the author [9] proved the following result: every curve \( \Gamma \) of genus \( g > 1 \) over a field of finite type either has only a finite number of rational points over that field or can be reduced by a change of variables to a curve whose defining equations have only rational coefficients. The proof depends on regarding the transcendental coefficients as variable parameters. Then one applies the Picard-Fuchs technique in differential equations in which these parameters function as independent variables. For a short exposition of the fundamental ideas see [8].

Thus, the "numerical case" of Mordell's conjecture, which is of fundamental interest for number-theory, remains unsolved. Here only some partial results are known which have the following character: if the rank of the group of rational points on the abelian variety \( A \) is not too large, then on any curve \( \Gamma \subset A \) there lie only a finite number of rational points. In the theorem of Chabauty, [2], for example, it is sufficient that the rank should be less than the dimension of \( A \). Unfortunately, Chabauty's theorem is difficult to apply because of the absence of an effective procedure for computing the rank. The only result known to me is for the curve \( x^k(1-x) = y^l \), where \( 1 < k < l \) and \( l \) is prime. D.K. Faddeev [13] gave an estimate for the rank of the group of rational points on the Jacobian varieties of these curves which permits the application of Chabauty's theorem (or rather a somewhat strengthened variant) for the case when the prime \( l \) is regular in the sense of Kummer.

Another finiteness theorem of similar type was proved by the author [10]. But there is no known argument for tackling the case of large rank.

6. We note that if we consider curves in affine, instead of in projective space, then we can distinguish between points with integral and those with just rational coordinates. The problem of finding the structure of the set of integral points was completely solved by Siegel. His result is as follows: there can be infinitely many points with integral coordinates on a curve only if it has genus 0 and if there exists a rational parametrization by polynomials in \( t \) and \( t^{-1} \). In particular, the number of integral
points on a curve of genus \( g > 1 \) is finite. For \( g > 1 \) this would follow from Mordell's conjecture. Siegel's theorem remains true if the ring of integers is replaced by any ring of finite type (see Lang \cite{6}).

7. In this short survey we do not touch on the extremely interesting problem of the effective determination of the rational points. For curves of genus 0 the solution is known, but definitive results in the remain cases have not been obtained. This question currently attracts great attention: for a survey of the ideas and a bibliography see the Stockholm report of Cassels \cite{1}.

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SMOOTH FOLIATIONS ON THREE-DIMENSIONAL MANIFOLDS

S.P. NOVIKOV

We first consider a smooth (for example, closed and analytic) $n$-manifold $M$, on which there is given a non-singular 1-form $\omega$ of class $C^\infty$. The equation

$$\omega = 0$$

defines a field of $(n-1)$-dimensional directions on $M$, smoothly dependent on the point. As usual, we call hypersurfaces touching our field everywhere solutions of $\omega = 0$. These hypersurfaces are called leaves and the family of them is called a foliation. Locally, in a coordinate system $(x_1, \ldots, x_n)$ near a point, the equation takes the form

$$\sum P_i(x_1, \ldots, x_n) \, dx_i = 0;$$

it is called Pfaff's equation (without singularities) and is soluble under the integrability condition of Frobenius

$$\omega \wedge d\omega = 0.$$ 

Foliations have been studied by a number of authors (Ehresmann, Reeb, Haefliger and others) in a much more general situation than that given here, they have obtained a number of results on the properties of foliations, on the behaviour of closed leaves of special kinds, on certain properties of one-dimensional curves on leaves, on the existence of analytic leaves, and have also constructed several interesting examples.

However, not a single result was known that establishes the existence of closed leaves. The simplest problem of this type is Kneser's conjecture on the existence of closed leaves of any smooth foliation on the usual 3-sphere $S^3$ (the leaf has dimension 2); in all known examples Kneser's existence hypothesis is satisfied. My aim is to prove a somewhat stranger statement than this conjecture.

**Theorem 1.** If the universal covering of a closed manifold $M^3$ is non-contractible, then any smooth orientable foliation on it has a closed leaf, and either this is a torus $T^2$, homologous to zero in $M^3$ and bounding in $M^3$ a full torus $D^2 \times S^1$ with a special foliation, or all the leaves are spheres $S^2$ or projective planes $P^2$ and the universal covering is the product of a 2-sphere and a line.

In connection with the method of proof I remark that a sufficient (not necessary) characterization of a closed leaf is the property that no closed transversal passes through it; this trivial property serves as a peg to hang the proof on. To use it, a domain is constructed engulfing transversals...
not inclined at too small an angle to the leaf, and the closed leaf is approximated by such transversals (by decreasing this angle).

The construction of the domain engulfing the transversal is the central part of the proof from both the technical and conceptual point of view, and closed curves on the leaves figure in various ways in it. Such curves are often, for example, "limit cycles" round which the nearby leaves spiral indefinitely; they may be curves that are not limit cycles relative to nearby leaves. The latter case interests us under the condition that the curve is not homotopic to zero in its leaf, but is homotopic to zero after a displacement to an arbitrarily close leaf (reconstruction of the topology of the leaf). A leaf with a curve of this type turns out to be closed. The domain engulfing the transversals to the leaves is glued together from films stretched over this curve after all possible displacements to nearby leaves. These films "capture" each other with some period and are glued together to form a domain with the necessary properties. The boundary of this domain is the required closed leaf.

We indicate some results obtained in passing immediately following from Theorem 1.

a) A dynamical system which is a transversal to a foliation on $M^3$ always has a periodic orbit. Further it has a system of periodic orbits knotted in the topological sense.

A dynamical system of this sort cannot be conservative, since it enters the domain bounded by the closed leaf. The latter fact is of a global character, since a transverse foliation can always be constructed locally.

b) A pair of transverse foliations with everywhere dense leaves is connected with one interesting class of dynamical systems ($U$-systems and $U$-cascades in the sense of Anosov).

If the manifold $M^3$ with the $U$-system is three-dimensional, it is easy to show that its fundamental group is infinitive (Anosov). It also turns out that

$$\pi_2(M^3) = 0.$$  

If we are dealing with $U$-cascades, then $M^3$ has the topological type of the torus $T^3$. Some facts can also be obtained here for $n > 3$.

c) The rank of $M^n$ is the maximum number of linearly independent vector fields on $M^n$ whose commutators $[\ ]$ are pairwise zero (Milnor). What, for example is the rank of $S^3$?

V.I. Arnol'd has shown me a proof that the rank of $S^3$ is 1, which is a consequence of the results on foliations given above. This was conjectured by Milnor.

In addition, the rank

$$M^n \leq n - 2,$$

if $\pi_1(M^n)$ is finite, or

$$\pi_2(M^n) \neq 0$$

($M^n$ is compact and closed).

d) On simply-connected manifolds $M^n$, $n \geq 3$, there are no analytic foliations, but there may be infinitely differentiable ones (Haefliger). For $n = 3$, it turns out that there cannot be orientable analytic foliations.
if \( \pi_2(M^3) \neq 0 \) (or \( \pi_1 \) is finite); this follows from Theorem 1. Foliations of class \( C^0 \) always exist if \( M^3 \) is closed and orientable (Zieschang).

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