2016.11.14

NOTES ON LIMIT MHS AND REGULARIZED PERIODS

§1 Canonical extensions

Suppose that

$$\mathbb{V} \longrightarrow \mathbb{D}^*$$

is a local system of finite dimensional complex vector spaces over a punctured disk. By rescaling the holomorphic coordinate, we may (and will) assume that $1 \in \mathbb{D}^*$.

Denote the fiber of $\mathbb{V}$ over $t \in \mathbb{D}^*$ by V_t. Let

$$h_t : V_t \to V_t$$

be the monodromy operator. The local system is determined by any one

of these. In particular, it is determined by

$$h := h_1 \in \operatorname{Aut} V_1 .$$

Let U be a contractible nbd of $1 \in \mathbb{D}^*$

$$g_t : V_1 \to V_t , \qquad t \in U$$

be the isomorphism, satisfying $g_1 = id$, given by parallel translation. (This can be continued to a multi-valued section of $\operatorname{Aut} \mathbb{V}$.) Then

(1) $h_t = g_t \, h \, g_t^{-1}$

(2) h = result of "analytically continuing" g_t around the unit circle.

Let $\mathcal{V} = \mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{D}^*}$ be the corresponding holomorphic vector bundle. Denote its canonical flat connection by ∇.

<u>Choose</u> a logarithm of h and set

$$N = \frac{1}{2\pi i} \cdot \log h.$$

<u>Remark:</u>

(1) Deligne chooses N such that its eigenvalues λ satisfy

$$0 \le \mathrm{Re}(\lambda) < 1.$$

Such a choice will give rise to Deligne's <u>canonical</u> <u>extension</u> of $\mathcal{V}$ to $\mathbb{D}$.

(2) In our case, h is unipotent. The canonical choice of $\log h$ is

$$\sum_{k=1}^{\infty} (-1)^{k+1} (h-1)^k / k.$$

This sum is <u>finite</u> as $h-1$ is nilpotent, as is N.

Set

$$N_t = g_t N g_t^{-1} \in \mathrm{End}\, V_t.$$

This satisfies

$$e^{2\pi i N_t} = h_t.$$

<u>Deligne</u> <u>trivializations:</u> Suppose $v \in V_1$. Let $v(t)$, $t \in U$, be the flat section of $\mathbb{V}$ over U satisfying $v(1) = v$. Note that $v(t) = g_t v$.

Set

$$\varphi(t) = g_t t^{-N} v \in V_t.$$

Since $N_t = g_t N g_t^{-1}$,

$$\varphi(t) = t^{-N_t} v(t).$$

This can be analytically continued to a possibly multi-valued section of

$\mathcal{V}$. Here

$$t^A := e^{A \log t}.$$

When analytically continued around the unit circle it becomes

$$e^{A(\log t + 2\pi i)} = e^{2\pi i A} t^A.$$

So when $\varphi(t)$ is analytically continued around the unit circle it becomes

$$g_t h\, e^{-2\pi i N} t^{-N} v$$

$$= g_t t^{-N} v$$

$$= \varphi(t)$$

This proves:

<u>PROP</u>: For each $v \in V_1$, the section

$$\varphi(t) = g_t t^{-N} v \qquad t \in \mathbb{D}^*$$

of $\mathcal{V}$ is single valued. □

Trivialize $\mathcal{V}$ over $\mathbb{D}^*$ by

$$\mathbb{D}^* \times V_1 \longrightarrow \mathcal{V}$$
$$(t, v) \longmapsto g_t t^{-N} v$$

<u>PROP</u>: The pullback of the connection on $\mathcal{V}$ to $\mathbb{D}^* \times V_1$ is

$$\nabla v = -Nv \frac{dt}{t},$$

where $v \in V_1$ is identified with the corresponding constant section of $\mathbb{D}^* \times V_1 \to \mathbb{D}^*$.

<u>proof</u>. Since g_t is a <u>flat</u> section of $\mathrm{Hom}(V_1, \mathcal{V})$

$\nabla(g_t v(t)) = g_t \nabla v(t)$

$$\nabla(g_t t^{-N} v) = -g_t t^{-N} N v \frac{dt}{t}$$

This section of $\mathcal{V} \otimes \Omega^1_{\mathbb{D}^*}$

corresponds to $-N v \frac{dt}{t}$ under the trivialization above. $\Box$

This trivialization gives an extension of $\mathcal{V}$ to $\mathbb{D}$. Namely, $\mathbb{D}\times V_1 \to \mathbb{D}$. The embedding

$$\begin{array}{ccc} \mathcal{V} & \hookrightarrow & \mathbb{D}\times V_1 \\ | & & | \\ \mathbb{D}^* & \hookrightarrow & \mathbb{D} \end{array}$$

is given by the identification above.

<u>NB</u>: A priori, this extension depends on the choice of the holomorphic coordinate t in $\mathbb{D}$.

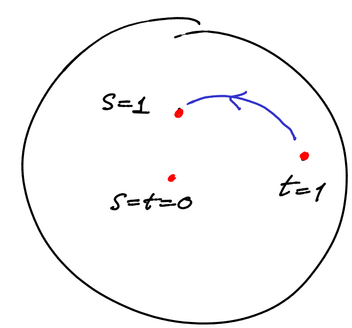
<u>Prop^{n}</u>: This extension does not depend on the choice of holomorphic coordinate t in $\mathbb{D}$.

<u>Proof</u>: Suppose s is another holomorphic

coordinate in $\mathbb{D}$, centered at 0. Write

$$s = t\cdot f(t)$$

where $f(0)\neq 0$. WLOG, $s=1$ is in $\mathbb{D}^*$. (If not, rescale s.)



Let g_0 be the isomorphism

$$g_0 : V_{t=1} \longrightarrow V_{s=1}$$

given by parallel transport. Set

$$h' = g_0 h g_0^{-1} \in \mathrm{Aut}\, V_{s=1}$$
$$N' = g_0 N g_0^{-1} \in \mathrm{End}\, V_{s=1},$$

and $\quad g_s' = g_{t(s)}\, g_0^{-1}$

Then one has the trivialization

$$\begin{tikzcd} V_{s=1} \times \mathbb{D}^* \arrow[r] \arrow[dr] & \mathcal{V} \arrow[d] \\ & \mathbb{D}^* \end{tikzcd}$$

given by $(v, s) \mapsto g_s'\, s^{-N'} v.$

Define $\qquad$ ← t-disk $\qquad$ s-disk ↓

$$\Phi : V_{t=1} \times \mathbb{D}_t \longrightarrow V_{s=1} \times \mathbb{D}_s$$

by

$$(v, t) \longmapsto (g_0\, f(t)^N v,\ s(t))$$

Since is holomorphic at $t=0$ and $f(0) \neq 0$, this is an isom of holomorphic vector bundles.

Restricting to punctured disks, we

have the diagram

$$\begin{tikzcd} V_{t=1} \times \mathbb{D}_t^* \arrow[rr] \arrow[dr] & & V_{s=1} \times \mathbb{D}_s^* \arrow[dl] \\ & \mathcal{V} & \end{tikzcd}$$

$(t, v) \mapsto g_t\, t^{-N} v \qquad (w, s) \mapsto g_s'\, s^{-N'} w.$

This diagram commutes as

$$(v, t) \longmapsto (g_0\, f(t)^N v,\ s(t))$$

$$\downarrow$$

$$(g_t g_0^{-1})\, s(t)^{-N'}\, g_0\, f(t)^N v$$

$$= g_t\, s(t)^{-N} f(t)^N v$$

$$= g_t\, t^{-N} f(t)^{-N} f(t)^N v$$

$$= g_t\, t^{-N} v. \qquad \square$$

So this extension, as a holomorphic vector bundle with connection, depends only on the choice of the logarithm N of h.

Remarks:

① given a choice of N, the extension is determined by a unique isomorphism that is the identity on the subsheaf $\mathbb{V}$ over $\mathbb{D}^*$. The central fiber V_0 is therefore well defined up to a unique isomorphism. Because of this, we can write all trivializations of the extension as

$$\begin{array}{ccc} \mathcal{V} & \hookrightarrow & V_0 \times \mathbb{D} \\ | & & | \\ \mathbb{D}^* & \hookrightarrow & \mathbb{D}. \end{array}$$

So N should be regarded as

an element of $\mathrm{End}(V_0)$.

Any two such trivializations will differ by a map

$$h : \mathbb{D} \to \mathrm{Aut}(V_0)$$

where $h(0) = id_{V_0}$. The corresponding connection on $V_0 \times \mathbb{D}$ will be

$$\nabla v = -hNh^{-1}\frac{dt}{t} + dhh^{-1} \qquad v \in V_0$$

So the connection form is

$$\Omega' = h\Omega h^{-1} - dhh^{-1} = hNh^{-1}\frac{dt}{t} - dhh^{-1}.$$

Since h is analytic and since $h(0) = id$,

$$\mathrm{Res}_0\,\Omega = \mathrm{Res}_0\,\Omega' = N.$$

② For all choices of local parameter t at 0,

$$\lim_{p\to 0} t(p)^{-N_p} v(p) \quad \left\{ \begin{array}{r} p \in \mathbb{D}^* \\ p \mapsto v(p) \\ \text{flat section} \end{array} \right.$$

exists in V_0. This induces the isomorphism $V_p \longrightarrow V_0$ corresponding to t. Perhaps a better way to describe this is to say that the map

$$H^0(\widetilde{\mathbb{D}}^*, \pi^* \mathbb{V}) \longrightarrow V_0$$
$$v \longmapsto \lim_{p\to 0} t(p)^{-N_p} v(p)$$

is an isomorphism for each choice of local parameter t.

③ Suppose that

$$\begin{array}{ccc} \mathcal{V} & = & V_0 \times \mathbb{D} \\ & \searrow & \swarrow \\ & & \mathbb{D} \end{array}$$

is a Deligne trivialization of V.

(ie: for a given t, $\nabla = d - N\frac{dt}{t}$.)

In this trivialization.

$$N_p = N = \mathrm{Res}_0 \nabla$$

for all $p \in \mathbb{D}^*$. Every other trivialization of $\mathcal{V}$ differs from this one by

$$\varphi : \mathbb{D} \longrightarrow GL(V_0)$$

where $\varphi(0) = id$. In this trivialization

$$N_p = \varphi(p) N \varphi(p)^{-1}.$$

So $\lim_{p\to 0} N_p = N.$ indep of trivialization.

④ Now suppose that N is nilpotent.

In this case, the extension is Deligne's canonical extension of $\mathbb{V} \otimes \mathcal{O}_{\mathbb{D}^*}$ to $\mathbb{D}$. It is characterized by the properties:
see Appendix

(i) $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_{\mathbb{D}}(\log 0)$

ie: $(\mathcal{V}, \nabla)$ has a regular singular point at 0.

(2) $\mathrm{Res}_0 \nabla$ is nilpotent.

These conditions imply that the canonical extension of the tensor product of two local systems on $\mathbb{D}^*$ with unipotent monodromy is the tensor product of their canonical extensions. (Similarly with duals.)

Several Variables

This works the same way: Suppose $\mathbb{V}$ is a local system over $(\mathbb{D}^*)^m$. Then

coords $(t_1, \ldots, t_m)$

$$\pi_1(\mathbb{D}^*) = \sigma_1^{\mathbb{Z}} \times \cdots \times \sigma_m^{\mathbb{Z}}$$

where σ_j = positive loop about $t_j = 1$.

Denote the fiber of $\mathbb{V}$ over $\underline{t} = (t_1, \ldots, t_m)$ by $V_{\underline{t}}$. Choose

$$N_1, N_2, \ldots, N_m \in \mathrm{End}\, V_{\underline{1}}$$

such that

(1) $e^{2\pi i N_j}$ = monodromy $\sigma_j : V_{\underline{1}} \to V_{\underline{1}}$

(2) $[N_j, N_k] = 0$.

Rk: When the local monodromy operators $\sigma_j : V_{\underline{1}} \to V_{\underline{1}}$ are unipotent, the canonical (nilpotent) choices of N_j will work as they are polynomials in the $\sigma_j - 1$.

Then on trivializes $\mathbb{V} \otimes \mathcal{O}_{(\mathbb{D}^*)^m}$ by

$$V_{\vec{1}} \times (\mathbb{D}^*)^m \longrightarrow \mathbb{V} \otimes \mathcal{O}_{(\mathbb{D}^*)^m}$$

$$(v, \underline{t}) \longmapsto g_{\underline{t}}\, t_1^{-N_1} \cdots t_m^{-N_m} v$$

where $g_{\underline{t}} : V_{\underline{1}} \longrightarrow V_{\underline{t}}$ is obtained by analytically continuing the id : $V_{\underline{1}} \longrightarrow V_{\underline{1}}$. This gives the extension

$$\mathcal{V} := V_{\underline{1}} \times \mathbb{D}^m$$

of $\mathbb{V} \otimes \mathcal{O}_{(\mathbb{D}^*)^m}$ to $\mathbb{D}^m$. The connection is given by

$$\nabla v = - \sum_{j=1}^{m} N_j v \frac{dt_j}{t_j} .$$

As above, one proves that $(\mathcal{V}, \nabla)$ depends only on the choices of the N_j, and not on t. Details are left as an exercise.

$\mathbb{Q}$ – structures on V_0

Suppose that $\mathbb{V}$ is a $\mathbb{Q}$- (or $\mathbb{R}$-) local system over $\mathbb{D}^*$. As above, the choice of a monodromy logarithm

$$N = \frac{1}{2\pi i} \cdot \log h,$$

where $h : V_1 \to V_1$, determines an extension $\mathcal{V} \to \mathbb{D}$ of $\mathbb{V} \otimes_{\mathbb{Q}} \mathcal{O}_{\mathbb{D}^*}$. Each choice of local coordinate t on $\mathbb{D}$ determines a $\mathbb{Q}$–structure on V_0 as follows: let $v(t)$ be a (multivalued) section of $\mathbb{V}_{\mathbb{Q}}$. Then

$$\lim_{t \to 0} t^{-N_t} v(t) \in V_0$$

is an element of $V_{0,\mathbb{Q}}$. More precisely, the $\mathbb{Q}$–structure on V_0 is given by the map

$$H^0(\widetilde{\mathbb{D}^*}, \pi^* \mathbb{V}_{\mathbb{Q}}) \to V_0$$

$$v(t) \longmapsto \lim_{t \to 0} t^{-N_t} v(t)$$

where

$$\pi : \widetilde{\mathbb{D}^*} \longrightarrow \mathbb{D}^*$$

is a universal covering. This is a $\mathbb{Q}$-structure because

$$H^0(\widetilde{\mathbb{D}^*}, \pi^* \mathbb{V}_{\mathbb{Q}}) \otimes \mathbb{C} \longrightarrow V_0$$

is an isomorphism.

<u>PROP^N</u>: This $\mathbb{Q}$-structure depends only on $\partial/\partial t \in T_0 \mathbb{D}$. If $\partial/\partial s = \lambda \, \partial/\partial t$, then the $\mathbb{Q}$-structure $V_{\partial/\partial s}$ on V_0 associated to $\partial/\partial s$ is

$$V_{\partial/\partial s} = \lambda^{-N} V_{\partial/\partial t}$$

where $\partial/\partial s = \lambda \, \partial/\partial t$ and

$$N = \mathrm{Res}_0 \nabla .$$

<u>proof</u>: Suppose s is another local parameter at 0. Then

$$s = t f(t)$$

where $f(0) \neq 0$. Then

$$ds = f(0)\, dt$$

and

$$\frac{\partial}{\partial s} = f(0)^{-1} \frac{\partial}{\partial t}$$

Set $\lambda = f(0)^{-1}$.

If $v \in H^0(\widetilde{\mathbb{D}^*}, \mathbb{V}_{\mathbb{Q}})$, then

$$\lim_{p \to 0} s(p)^{-N_p} v(p)$$

$$= \lim_{p \to 0} f(t(p))^{-N_p}\, t(p)^{-N_p} v(p)$$

$$= f(0)^{-N} \lim_{p \to 0} t(p)^{-N_p} v(p)$$

$$= \lambda^N \lim_{p \to 0} t(p)^{-N_p} v(p). \qquad \square$$

§2 <u>Polarized Variations of Hodge Structure</u>.

This is a summary of Wilfred Schmid's fundamental results about PVHS.

<u>Variations of Hodge structure</u> (VHS)

A $\mathbb{Q}$ (or $\mathbb{Z}$ or $\mathbb{R}$) VHS of weight m over a smooth complex analytic variety T consists of

(1) a $\mathbb{Q}$-local system $\mathbb{V}$ over T of finite rank

(2) Holomorphic sub-bundles $\mathcal{F}^p$ of $\mathcal{V} = \mathbb{V} \otimes_{\mathbb{Q}} \mathcal{O}_T$ such that

$$\cdots \supseteq \mathcal{F}^p \supseteq \mathcal{F}^{p+1} \supseteq \cdots$$

and

$$(V_{t,\mathbb{Q}}, (V_{t,\mathbb{C}}, F^\cdot))$$

is a Hodge structure of weight m for all $t \in T$.

(3) The canonical flat connection ∇ on $\mathcal{V}$ satisfies <u>Griffiths transversality</u>

$$\nabla : \mathcal{F}^p \to \mathcal{F}^{p-1} \otimes \Omega^1_T$$

<u>Example</u>: If $f: X \to T$ is a complex analytic family of compact Kähler manifolds, then

$$\mathbb{V} := R^m f_* \mathbb{Q}$$

is a VHS of weight m over T.

<u>Polarized VHS</u>:

A polarization of a Hodge structure $V = (V_{\mathbb{Q}}, (V_{\mathbb{C}}, F^\cdot))$ of weight m is a $(-1)^m$-symmetric bilinear form

$$\langle , \rangle : V_{\mathbb{Q}} \otimes V_{\mathbb{Q}} \to \mathbb{Q}$$

satisfying:

Riemann - Hodge bilinear relations

(1) $\langle V^{p,m-p}, V^{s,m-s} \rangle = 0$

unless $s = m-p$,

(2) $V^{p,q} \times V^{p,q} \longrightarrow \mathbb{C}$

$(v_1, v_2) \longmapsto i^{p-q} \langle v_1, \bar{v}_2 \rangle$

is a positive definite hermitian form for all (p,q) satisfying $p+q=m$.

<u>Example</u>: X = smooth projective variety of $\dim \geq m$. Let

$$\omega \in H^2(X;\mathbb{Q})$$

be the class of a projective embedding.

Let

primitive part

$$V = PH^m(X)$$

$d = \dim X$

$$:= \ker\{\omega^{d-m+1} : H^m(X) \to H^{2d-m+2}(X)\}$$

and

$$\langle \xi, \eta \rangle = (-1)^{m(m-1)/2} \int_X \xi \wedge \eta \wedge \omega^{d-m}$$

$\xi, \eta \in PH^m(X)$.

This is part of the "Hard Lefschetz Theorem";

<u>Def</u>: A polarized variation of Hodge structure (PVHS) of weight m over a complex manifold T is a VHS $\mathbb{V}$ of weight m over T plus a flat innerproduct

$$\langle\ ,\ \rangle : \mathbb{V} \otimes \mathbb{V} \to \mathbb{Q}_T$$

whose restriction to each fiber V_t is a polarized HS of weight m.

__Example:__ Suppose that $f: X \to T$ is a holomorphic family of smooth projective varieties over a complex manifold T.

(__i.e.__: locally have embedding

$$\begin{tikzcd} X|_U \arrow[r, hook] \arrow[d] & \mathbb{P}^M \times U \arrow[d] \\ U & U \end{tikzcd}$$

s.t. the polarizations match on $U \cap V$.)

Suppose $\dim X_t = d \geq m$. Let $\mathbb{V} \to T$ be the local system whose fiber over $t \in T$ is $PH^m(X_t; \mathbb{Q})$. This is a PVHS of weight m over T.

__Defn:__ A HS is __polarizable__ if it has at least one polarization. A VHS is polarizable if it admits a polarization. (We can regard a PHS as a PVHS over a point.)

__Prop:__ Suppose that $\mathbb{V}$ is a $\mathbb{Q}$-PVHS over T. If $\mathbb{A}$ is a sub VHS of $\mathbb{V}$, then

(1) the restriction of the polarization of $\mathbb{V}$ to $\mathbb{A}$ polarizes $\mathbb{A}$,

(2) $\mathbb{A}^\perp$ is a sub VHS of $\mathbb{V}$

(3) $\mathbb{V} = \mathbb{A} \oplus \mathbb{A}^\perp$ as PVHS.

Pf: Exercise: Use the Riemann–Hodge bilinear relations to show that the restriction of the polarization to $\mathbb{A}$ is non-degenerate, etc. $\square$

__COR:__ The category of polarizable $\mathbb{Q}$-VHS over T is a semi-simple tannakian category.

<u>Remark</u>: If $f: X \to T$ is a family of smooth projective varieties, then $R f_*^m \mathbb{Q}$ is polarizable. Just use the decomposition

$$H^m(X_t;\mathbb{Q}) \cong PH^m(X_t) \oplus \omega \cdot PH^{m-2}(X_t) \oplus \omega^2 PH^{m-4}(X_t) \oplus \cdots$$

when $m \le \dim X_t$, and Poincaré duality when $m > \dim X_t$.

Since every VHS "coming from geometry" is a sub quotient of such a variation, every VHS of "geometric origin" is polarizable.

<u>Schmid's Theorems</u>:

<u>Local monodromy Theorem</u>: If $\mathbb{V}$ is a PVHS over $\mathbb{D}^*$, then

the monodromy operator $h: V_1 \to V_1$ is quasi-unipotent. That is, there are positive integers e and n such that

$$(h^e - id)^{m+1} = 0.$$

This is due (I think) to

- Grothendieck in the algebro-geometric case, where
$$\mathbb{V} = R^m f_* \mathbb{Q}$$
- to Landman in the Kähler case?
- to Griffiths and Borel (and Schmid?) in the PVHS case.

Significance: it implies that, after the base change $\mathbb{D} \longrightarrow \mathbb{D}$, $t \longmapsto t^e$, the local monodromy operator is unipotent.

<u>Nilpotent Orbit Theorem</u> (Schmid).

If $\mathbb{V} \to \mathbb{D}^*$ is a PVHS with unipotent monodromy then the Hodge sub-bundles $\mathcal{F}^p$ of $\mathbb{V} \otimes \mathcal{O}_{\mathbb{D}^*}$ extend to holomorphic sub-bundles of the canonical extension $\mathcal{V} \to \mathbb{D}^*$ of $\mathbb{V}$ to $\mathbb{D}$. In particular, they cut out the "limit Hodge filtration" on V_0, the fiber of $\mathcal{V}$ over $t=0$.

<u>Remark</u>: The nilpotent orbit theorem also holds in the several variable case, provided that one has unipotent monodromy operators. (Cattani–Kaplan–Schmid)

We now have two of the three ingredients we need for the limit MHS on $V_{0/\partial t}$:

(1) The $\mathbb{Q}$-structure

$$H^0(\mathbb{D}^*, \mathbb{V}_{\mathbb{Q}}) \to V_0$$

$$v \longmapsto \lim_{t \to 0} t^{-N_t} v(t)$$

which depends only on $\partial/\partial t$.

(2) The limit Hodge filtration on V_0

We still need a weight filtration.

<u>Digression</u>: monodromy weight filtration of a nilpotent operator:

Suppose V is a finite dimensional vector space over a field of char 0 and that $N: V \to V$ is nilpotent. Then there is a <u>unique</u> filtration

$$0 = W_{-r-1} \subseteq W_{-r} \subseteq \cdots\cdots \subseteq W_r = V$$

such that

(1) $\quad N\, W_j \subseteq W_{j-2}$

(2) $\quad N^k : Gr_k^W V \to Gr_{-k}^W V$

is an isomorphism for all $k \geq 0$.

<u>Existence:</u> Since N is nilpotent, it can be put in Jordan canonical form over the field. This reduces the existence proof to the case where N has a single Jordan block

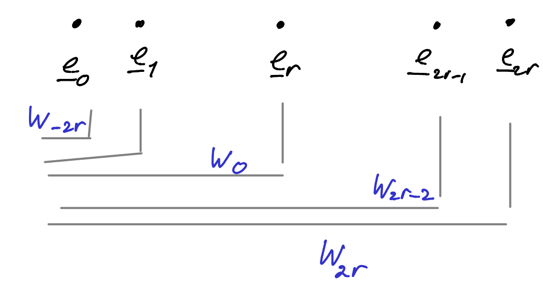
$$\begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & & 0 & 1 \\ 0 & & & & 0 \end{pmatrix}$$

with respect to the basis

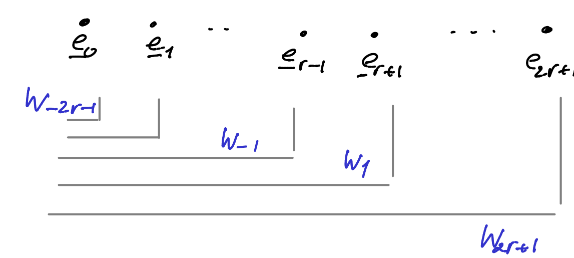
$$\underline{e}_0, \underline{e}_1, \ldots, \underline{e}_d$$

So $\begin{cases} N\underline{e}_j = \underline{e}_{j-1} & j > 0 \\ N\underline{e}_0 = 0 \end{cases}$

$d = 2r: \quad W_{2j+1} = W_{2j} = \mathrm{span}\{\underline{e}_0, \ldots, \underline{e}_{j+r}\}$



$d = 2r+1: \quad W_{2j+2} = W_{2j+1} = \mathrm{span}\{\underline{e}_0, \ldots, \underline{e}_{j+r}\}$



<u>Uniqueness:</u> exercise.

SL_2-orbit Theorem (Schmid)

If $\mathbb{V} \to \mathbb{D}^*$ is a PVHS of weight m with unipotent monodromy, then

$$\left(V_{\partial/\partial t},\ M_\bullet,\ (V_0, F^\bullet) \right)$$

is a MHS, where

(1) $V_{\partial/\partial t}$ = image of

$$H^0(\widetilde{\mathbb{D}^*}, \mathbb{V}_{\mathbb{Q}}) \to V_0$$
$$v \mapsto \lim_{t \to 0} t^{-N_t} v(t)$$

(2) $M_\bullet$ = monodromy weight filtration shifted by m (so that it is symmetric about m).

$$M_j = W(N)_{j-m}$$

(3) $F^\bullet$ is the limit Hodge filtration

Moreover

$$N : V_{\partial/\partial t} \longrightarrow V_{\partial/\partial t}(-1)$$

is a morphism of MHS. Equivalently,

$$N \in \mathrm{Hom}_{MHS}\left(\mathbb{Q}(1), \mathrm{End}(V_{\partial/\partial t})\right).$$

Remark: There is a geometric construction of the canonical extension of $\mathbb{V}$ to $\mathbb{D}$ and of the extended Hodge bundles. It is due to Steenbrink. More on that later, in the section on periods.

Several variables: Schmid's theorems extend to the several variable case. See the papers of Cattani–Kaplan–Schmid.

§3 <u>Limits</u> <u>of</u> <u>mixed</u> <u>Hodge</u> <u>structures</u>

A <u>variation</u> <u>of</u> <u>mixed</u> <u>Hodge</u> <u>structure</u> (VMHS) over a complex manifold T consists of

(1) a $\mathbb{Q}$-local system $\mathbb{V}$ of finite rank over T and a filtration

$$0 \subseteq \mathbb{W}_a \subseteq \cdots \subseteq \mathbb{W}_{r-1} \subseteq \mathbb{W}_r \subseteq \cdots \subseteq \mathbb{W}_b = \mathbb{V}$$

of $\mathbb{V}$ by $\mathbb{Q}$-local systems

(2) holomorphic sub-bundles $\mathcal{F}^p$ $(p \in \mathbb{Z})$ of $\mathcal{V} := \mathbb{V} \otimes \mathcal{O}_T$ satisfying Griffiths transversality

$$\nabla : \mathcal{F}^p \to \mathcal{F}^{p-1} \otimes \Omega^1_T .$$

For each $t \in T$, the fiber V_t of $\mathbb{V}$ should be a MHS with weight filtration cut out by $\mathbb{W}_\bullet$ and Hodge

filtration by $\mathcal{F}^\bullet$.

A VMHS is <u>graded</u> <u>polarizable</u> if each $Gr^W_m \mathbb{V}$ is a polarizable variation of HS.

<u>Examples</u>

(1) (Steenbrink – Zucker, Guillen – Navarro – Puerta). If $X \to T$ is a topologically locally trivial family of algebraic varieties (possibly singular, possibly non-compact) Then each

$$R^m f_* \mathbb{Q}$$

is a VMHS over T. (More on this later.)

(2) (Hain) If

$$\begin{array}{c} X \\ \downarrow \\ T \end{array} \curvearrowleft \sigma$$

is a locally topologically trivial family of smooth varieties, then the local system whose fiber over t is

$$\mathcal{O}(\pi_1^{un}(X_t, \sigma(t)))$$

is a direct limit of VMHS. (More later.)

(3) (Hain) If T is smooth and $\mathbb{H}$ is a PVHS over T, then the local system over T whose fiber over ~~$t \in T$~~ is

$$\mathcal{O}(\pi_1^{rel}(T, t))$$

is a VMHS. Here $\pi_1^{rel}(T, t)$ is the completion of $\pi_1(T, t)$ w.r.t the monodromy rep

$$\pi_1(T, t) \to \operatorname{Aut} V_t.$$

<u>Admissible</u> <u>VMHS</u>:

(1) <u>local</u> <u>behaviour</u>: Suppose that $\mathbb{V} \to \mathbb{D}^*$ is a VMHS. Assume that each $Gr_m^W \mathbb{V}$ is a polarizable VHS. After a base change, we may assume that the monodromy is unipotent. Denote the canonical extension of $\mathbb{V}$ to $\mathbb{D}$ by $\mathcal{V}$. It is filtered by $W_\bullet$

<u>Def</u>: $\mathbb{V}$ is <u>admissible</u> at 0 if

(1) The Hodge bundles $\mathcal{F}^p$ extend to holomorphic sub-bundles of $\mathcal{V}$

(2) there is a filtration $M_\bullet$ of V_0 satisfying

(a) $N M_r \subseteq M_{r-2}$ $\qquad N = \operatorname{Res}_0 \nabla$

(b) the filtration $M_\bullet^{(m)}$ induced on $Gr^W_m V_0$ is the monodromy weight filtration of

$$N : Gr^W_m V_0 \to Gr^W_m V_0$$

shifted by m. (It is called the <u>relative weight filtration</u> of the nilpotent endomorphism of the filtered vectorspace $(V, W_\bullet)$.)

as usual

(3) $\left\{ \begin{array}{l} (V_{o/sq}, M_\bullet, (V_0, F^\bullet)) \\ \quad \| \\ \operatorname{im}\{H^0(\tilde{\mathbb{D}}^*, V_{\mathbb{Q}}) \to V_0\} \\ \qquad v \longmapsto \lim_{t\to 0} t^{-N} \tilde{v}(t) \end{array} \right.$

is a MHS. If it is, so is each $(W_m V_{o/sq}, M_\bullet, (W_m V_0, F^\bullet))$

We say that the MHS on $V_{o/sq}$ is <u>filtered</u> by $W_\bullet$.

<u>Remarks:</u> Unlike the case of PVHS

(1) the existence of the limit MHS is not automatic,

(2) for the generic nilpotent endomorphism $N : (V, W_\bullet) \circlearrowleft$ of a filtered vector space, there is no relative weight filtration.

<u>Example</u> of nilpotent $N : (V, W_\bullet) \circlearrowleft$ with no relative weight filtration.

$$V = H^1(E - \{p, q\}, \mathbb{Q}) \qquad p \neq q$$

E = elliptic curve

Have ses

$$0 \to H^1(E) \to V \xrightarrow{\text{Res}} \underbrace{\mathbb{Q}}_{\tilde{H}_0(\{p,q\})} \to$$

Define $W_\bullet$ as usual:

$$\begin{array}{ccccc} 0 & \subseteq & H^1(E) & \subseteq & V \\ \| & & \| & & \| \\ W_0 & & W_1 & & W_2 \end{array}$$

Define N by:

(a) $N|_{H^1(E)} = 0$

(b) $N\gamma \in H^1(E)$, $N\gamma \neq 0$

where $\gamma \notin H^1(E)$.

Then the shifted monodromy filtration on the $Gr^W_m V$ are

$$0 = M_1 \subseteq M_2 = Gr^W_2 V$$

$$\& \quad 0 = M_0 \subseteq M_1 = Gr^W_1 V$$

as $Gr^W_\bullet N = 0$. If there were a relative weight filtration $M_\bullet$ on V, it would satisfy

$$V = M_2 V \text{ and } M_0 V = 0.$$

This would imply that $N \equiv 0$ as

$$NV = NM_0 \subseteq M_{-2}.$$

Since $N \neq 0$, there is no relative weight filtration.

<u>Definition</u>: Suppose that T is a smooth projective curve and that Σ is a finite subset. A variation of MHS $\mathbb{V}$ over $T' = T - \Sigma$ is <u>admissible</u> if

(1) it is graded polarizable;

(2) it is admissible at each $t \in \Sigma$.

A VMHS $\mathbb{V}$ over a smooth variety $\mathcal{U}$ is <u>admissible</u> if its restriction to each curve $T' \to \mathcal{U}$ is admissible.

<u>NB</u>: If the local monodromy is not locally unipotent in the curve case, kill the eigenvalues of the local monod operators by passing to a finite branched covering of T.

<u>Remark</u>: Steenbrink & Zucker defined admissible VMHS over a curve; Kashiwara extended this to higher dimensional bases.

Examples:

(1) Steenbrink–Zucker:

$$\mathbb{V} = R^m f_* \mathbb{Q}$$

where $f : X \to T'$ is a topologically locally trivial family of <u>smooth</u> varieties over an algebraic curve T'

(2) Guillen–Navarro–Puerta:

$$\mathbb{V} = R^m f_* \mathbb{Q}$$

where $X \to T'$ is a topologically loc trivial family of varieties – not necessarily smooth or complete.

(3) Hain:

$$\begin{array}{c} X \\ \downarrow \uparrow \sigma \\ T' \end{array} \qquad V_t = W_m \mathcal{O}\left(\pi_1^{un}(X_t, \sigma(t))\right)$$

where $X \to T'$ is a topologically locally trivial family of smooth varieties over a curve.

§ 4 Asymptotics of periods

Suppose we have dual local systems

$$\begin{array}{ccc} \mathbb{V} & & \check{\mathbb{V}} \\ | & & | \\ \mathbb{D}^* & & \mathbb{D}^* \end{array}$$

with unipotent monodromy. Suppose

$$\begin{array}{ccc} \mathcal{V} & & \check{\mathcal{V}} \\ | & & | \\ \mathbb{D} & & \mathbb{D} \end{array}$$

be their canonical extensions to $\mathbb{D}$. These have nilpotent residue at 0:

$$\mathrm{Res}_0 \nabla = -N \in \mathrm{End}\, V_0$$

$$\mathrm{Res}_0 \check{\nabla} = \check{N} \in \mathrm{End}\, \check{V}_0$$

Fix a parameter t in $\mathbb{D}$. Suppose that $\gamma(t) \in H^0(\widetilde{\mathbb{D}}^*, \pi^* \mathbb{V})$ is a flat multivalued section of $\mathbb{V}$ and that

$\omega(t)$ is a holomorphic section of $\check{\mathcal{V}}$.

<u>PROP$^{\underline{N}}$</u>: The period

$$\int_{\gamma(t)} \omega(t) := \langle \gamma(t), \omega(t) \rangle$$

is a polynomial $\sum_{j=0}^{d} a_j(t)(\log t)^j$

in $\log t$ with coefficients in $\mathcal{O}(\mathbb{D})$.

Moreover "canonical regularization" "naive regularization"

$$\lim_{t\to 0} \langle t^{-N_t}\gamma(t), \omega(t)\rangle = a_0(0).$$

<u>Proof</u>. Suppose that $\{\gamma_1(t), \dots, \gamma_m(t)\}$ is a basis of $H^0(\tilde{\mathbb{D}}^*, \pi^*\mathbb{V})$ and that

$$\{\omega_1(t), \dots, \omega_m(t)\}$$

is a framing of $\check{\mathcal{V}}$ over $\mathbb{D}$. Let

$$\varphi_j(t) = t^{-N_t}\gamma(t) \qquad j=1,\dots,m$$

be a Deligne framing of $\mathcal{V}$ over $\mathbb{D}$ and $\{\check{\varphi}_1, \dots, \check{\varphi}_m\}$ be the dual

framing of $\check{\mathcal{V}}$. That is

$$\langle \varphi_j, \check{\varphi}_k \rangle = \delta_{jk}.$$

Identify V_0 with $\bigoplus_{j=1}^{m} \mathbb{C}\varphi_j$. Recall that $N = \mathrm{Res}_0 \nabla$ acts on V_0. The flat sections of $\mathcal{V}$, with respect to this framing, are $\mathbb{C}$-linear combinations of

$$t^{N}\varphi_j, \qquad j=1,\dots,m.$$

Let $A \in M_m(\mathbb{C})$ be the matrix of N w.r.t the basis $\{\varphi_1, \dots, \varphi_m\}$:

$$(N\varphi_1, \dots, N\varphi_m) = (\varphi_1, \dots, \varphi_m)A$$

It is nilpotent. We can write

$$(\omega_1(t), \dots, \omega_m(t)) = (\check{\varphi}_1, \dots, \check{\varphi}_m)\,\Phi(t)$$

where $\Phi \in GL_m(\mathcal{O}(\mathbb{D}))$. So

$$\left\langle \begin{pmatrix} \gamma_1(t) \\ \vdots \\ \gamma_m(t) \end{pmatrix}, (\omega_1(t), \dots, \omega_m(t)) \right\rangle$$

$$= \left\langle \begin{pmatrix} t^{N}\varphi_1 \\ \vdots \\ t^{+N}\varphi_m \end{pmatrix}, (\check{\varphi}_1, \dots, \check{\varphi}_m)\, \Phi(t) \right\rangle$$

$$= t^{+A^T} \underbrace{\left\langle \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_m \end{pmatrix}, (\check{\varphi}_1, \dots, \check{\varphi}_m) \right\rangle}^{\text{identity matrix}} \Phi(t)$$

$$= t^{+A^T}\, \Phi(t).$$

$$= e^{A^T \log t}\, \Phi(t)$$

$$\in M_m(\mathcal{O}(\mathbb{D})[\log t]).$$

Finally, write $\gamma = c_1\gamma_1(t) + \dots + c_m\gamma_m(t)$ and $w(t) = f_1(t)\,\omega_1(t) + \dots + f_m(t)\,\omega_m(t)$ to deduce the first statement.

For the last statement, note that

$$t^{-N_t}\gamma(t) = \sum_{j=1}^{m} c_j\, t^{-N_t}\gamma(t) = \sum_{j=1}^{m} c_j \varphi_j.$$

The limit is then $\Phi(0)\begin{pmatrix} f_1(0) \\ \vdots \\ f_m(0) \end{pmatrix}$. $\square$

This is the const term

In practice, one is given a trivialization $\mathcal{V} = V_0 \times \mathbb{D}$ where it is easy to compute $N = \mathrm{Res}_0 \nabla \in \mathrm{End}\, V_0$, but not so easy to compute the N_t.

The following provided an alternative method for regularizing periods.

PROP$^{\underline{N}}$: If $v : \mathbb{D}^* \to V_0$ is a flat section of $\mathcal{V} = V_0 \times \mathbb{D}$, then

"canonical regularization" "mortar board regulariz-n"

$$\lim_{t \to 0} t^{-N_t} v(t) = \lim_{t \to 0} t^{-N} v(t).$$

for t in any angular sector.

Remark. The local sections $t^{-N} v(t)$ of $\mathcal{V} = V_0 \times \mathbb{D}$ are single-valued on $\mathbb{D}^*$ if & only if this is a "Deligne trivialization".

proof. If $\mathcal{V} = V_0 \times \mathbb{D}$ is a Deligne trivialization of $\mathcal{V}$, then every flat section is of the form $t^N v$, where $v \in V_0$. In this case, $N_t = N$ for all $t \in \mathbb{D}^*$ so that $t^{-N_t} v(t) = v$ and

$$\lim_{t \to 0} t^{-N} v(t) = \lim_{t \to 0} t^{-N_t} v(t) = v.$$

In general, the trivialization differs from a Deligne trivialization by a holomorphic map $\varphi : \mathbb{D} \to GL(V_0)$ with $\varphi(0) = id$. In this case

$$N_t = \varphi(t) N \varphi(t)^{-1}.$$

So, in any angular sector $\theta_0 < \arg t < \theta_1$,

$$\| t^{N_t} - t^N \| = \| \varphi(t) e^{N \log t} \varphi(t)^{-1} - e^{N \log t} \|$$

$$\overset{\text{as } N \text{ nilpotent}}{\leq} \sum_{k=0}^{K} \overbrace{\| \varphi(t) N^k \varphi(t)^{-1} - N^k \|}^{\text{holo on } \mathbb{D}, \text{ van at } 0} \frac{|\log t|^k}{k!}$$

$$\leq C \, |t| \, |\log t|^K$$

$$\to 0 \quad \text{as} \quad |t| \to 0. \quad \square$$

<u>Remark:</u> As we shall see, this gives Brown's "mortar board" regularization.

§ 5 <u>Regularizing periods</u>

Suppose that K is a number field. Fix an embedding $K \overset{\sigma}{\hookrightarrow} \mathbb{C}$. Suppose that

$$\begin{array}{c} \overline{X} \\ f \big\downarrow \\ T \end{array}$$

a curve

is a proper family of smooth projective varieties defined $/K$. Suppose that $D \subseteq \overline{X}$ is a DNC, each of whose components is transverse to the fibers of f. Suppose that $\Sigma \subseteq T(K)$ is finite. Set

$$E = f^{-1}(\Sigma).$$

assume for simplicity that $E = E^{red}$.

Suppose that $D \cup E$ is a DNC and that the restriction $f' : X' \to T'$, where

$$X' := \overline{X} - (D \cup E) \ \& \ T' = T - \Sigma,$$

is locally topologically trivial.

51.

Results of Steenbrink and Steenbrink Zucker imply that the connection

$$\begin{array}{c} \mathcal{V}_K^{DR} := \left(Rf_* \Omega^\bullet_{\bar{X}/T} \log(D \cup E),\ \nabla \right) \\ | \\ T \end{array}$$

satisfies

(1) it is a vector bundle defined $/K$

(2) ∇ has regular singular points at each $t \in \Sigma$ with nilpotent residue

(3) after tensoring with $\mathbb{C}$, it is isomorphic to the canonical extension of

$$\begin{array}{ccc} H^m(X_t') & \subseteq & R^m f'_* \mathbb{C} \\ | & & | \\ t & \in & T' = T - \Sigma \end{array}$$

to T.

52.

So we have

$$\begin{array}{ccc} (\mathcal{V}_K^{DR}, \nabla) & & \mathbb{V}_{\mathbb{Q}}^B \\ | & \text{and} & | \\ T_{/K} & & T'^{\,an} \\ \text{DeRham} & & \text{Betti} \end{array}$$

and a comparison isomorphism

$$\mathbb{V}_{\mathbb{Q}}^B \otimes \mathcal{O}_{T'} \longrightarrow \mathcal{V}_K^{DR} \otimes \mathbb{C}\big|_{T'}$$

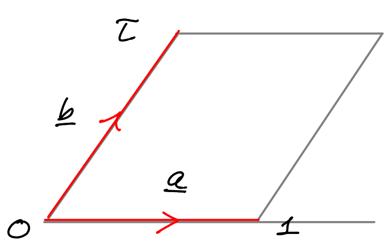
The formulas in the previous section tell us how to regularized this comparison map over $\vec{v} \in T_p T$, where $p \in \Sigma$ and $\vec{v} \neq \vec{0}$.

§6 Example.

Let $\begin{array}{c}\mathcal{E}\\ f\downarrow\\ \mathcal{M}_{1,1}\end{array}$ be the universal elliptic curve and $\mathbb{H}_A$ be the local system $R^1 f_* A$, where $A = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \ldots$ We can pull this back the upper half-plane $\mathfrak{h}$ and the q disk $\mathbb{D}^*$:

$$\begin{array}{ccccc} \mathbb{H}_{\mathfrak{h}} & — & \mathbb{H}_{\mathbb{D}^*} & — & \mathbb{H} \\ | & & | & & | \\ \mathfrak{h} & \longrightarrow & \mathbb{D}^* & \longrightarrow & \mathcal{M}_{1,1} \\ \tau & \longmapsto & q = e^{2\pi i \tau} & & \end{array}$$

Let $\underline{a}$ and $\underline{b}$ be the standard basis of $H_1(E_\tau)$, where $E_\tau = \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau$.



polarization: $\langle \underline{a}, \underline{b} \rangle = 1$ intersection pairing

Let $\check{\underline{a}}, \check{\underline{b}}$ be the dual basis of $H^1(E_\tau)$. The Poincaré duality isom

$$H_1(E_\tau) \longrightarrow H^1(E_\tau)$$

takes $\underline{a}$ to $\check{\underline{b}}$ and $\underline{b}$ to $-\check{\underline{a}}$.

The pullback of $\mathbb{H}$ to $\mathfrak{h}$ is

$$\mathbb{H}_{\mathfrak{h}} = \mathfrak{h} \times (\mathbb{Z}\underline{a} \oplus \mathbb{Z}\underline{b})$$

The left $SL_2(\mathbb{Z})$-action is

$$\gamma : (\underline{a}, -\underline{b}) \longmapsto (\underline{a}, -\underline{b})\gamma$$

Set $\mathcal{H} = \mathbb{H} \otimes \mathcal{O}_{\mathcal{M}_{1,1}}$ etc.

Then

$$\underline{\omega}(\tau) = 2\pi i (\check{\underline{a}} + \tau \check{\underline{b}})$$
$$= -2\pi i \underline{b} + \log q \, \underline{a},$$

the class of $2\pi i \, dz$ in $H^1(E_\tau; \mathbb{C})$, is a section of $\mathcal{H}$ over $\mathfrak{h}$ and $\mathbb{D}^*$.

Note that

$$\mathcal{H}_{\mathbb{D}^*} = \mathcal{O}_{\mathbb{D}^*}\underline{a} \oplus \mathcal{O}_{\mathbb{D}^*}\underline{\omega}$$

Extend it to $\mathbb{D}$ as

$$\mathcal{H}_{\mathbb{D}} = \mathcal{O}_{\mathbb{D}}\,\underline{a} \oplus \mathcal{O}_{\mathbb{D}}\,\underline{w}$$

I claim that this is the canonical extension of $\mathcal{H}$ to $\mathbb{D}^*$. To see this, note that the connection satisfies

$$\nabla \underline{a} = \nabla \underline{b} = 0.$$

so that

$$\nabla \underline{a} = 0 \quad \text{and} \quad \nabla \underline{w} = \underline{a}\,\frac{dq}{q}.$$

Thus, in this extension,

$$\nabla = d + \underline{a}\,\partial/\partial\underline{w}\;\frac{dq}{q}.$$

So the extended connection has a regular singular point at $q=0$. The residue at $q=0$ is the nilpotent endomorphism

$$N = -\underline{a}\,\partial/\partial\underline{w}$$

It follows that

(1) $\mathcal{O}_{\mathbb{D}}\,\underline{a} \oplus \mathcal{O}_{\mathbb{D}}\,\underline{w}$ is the canonical extension of $\mathcal{H}_{\mathbb{D}}$ to $\mathbb{D}$.

(2) $\{\underline{a}, \underline{w}\}$ is a Deligne frame.

So $N_q = N$ all $q \in \mathbb{D}$

The Hodge bundle $\mathcal{F}^1$ is $\mathcal{O}_{\mathbb{D}}\,\underline{w}$. It clearly extends to

$$V_0 = \mathbb{C}\underline{a} \oplus \mathbb{C}\underline{w}$$

The flat $\mathbb{Q}$ (even $\mathbb{Z}$) sections are spanned $\underline{a}$ and

$$\underline{b}(q) = -\frac{1}{2\pi i}\,\underline{w}(q) + \frac{\log q}{2\pi i}\,\underline{a}$$

Since $N^2 = 0$,

$$q^{-N} = e^{-N\log q} = 1 + \log q\;\underline{a}\,\partial/\partial\underline{w}$$

and $q^{-N}\underline{a} = \underline{a}$

$$2\pi i\, q^{-N}\underline{b}(q) = -\underline{w}(q) + \log q\,\underline{a} - \log q\,\underline{a} = -\underline{w}(q).$$

So

$$V_{\partial/\partial q} = \mathbb{Z}\underline{a} \oplus \frac{1}{2\pi i}\mathbb{Z}\underline{w}$$

It follows that

$$V^B_{\partial/\partial q} = \mathbb{Z}\,\underline{a} \oplus \mathbb{Z}\,\underline{b}$$

$$V^{DR}_{\partial/\partial q} = \mathbb{C}\underline{a} \oplus \mathbb{C}\underline{w}, \quad F^1 V^{DR}_{\partial/\partial q} = \mathbb{C}\underline{w}$$

and $-2\pi i\,\underline{b} \longmapsto \underline{w}$

$$\therefore \quad V_{\partial/\partial q} = \mathbb{Z}(0) \oplus \mathbb{Z}(-1).$$

This extends (by linear algebra) to $S^m\mathbb{H}$: over $\mathbb{D}$:

$$S^m\mathcal{H} = \mathcal{O}_{\mathbb{D}}\,\underline{a}^m \oplus \mathcal{O}_{\mathbb{D}}\,\underline{a}^{m-1}\underline{w} \oplus \cdots \oplus \mathcal{O}_{\mathbb{D}}\,\underline{w}^m.$$

$$F^p\mathcal{H} = \bigoplus_{s \geq p} \mathcal{O}_{\mathbb{D}}\,\underline{a}^{m-s}\,\underline{w}^s.$$

$$\nabla = d + \underline{a}\,\partial/\partial\underline{w}\,\frac{dq}{q}$$

This satisfies Griffiths transversality

$$\nabla \mathcal{F}^p \subseteq \mathcal{F}^{p-1} \otimes \Omega^1_{\mathbb{D}}(\log 0)$$

It is polarized by $\langle\,,\,\rangle$, extended

to $S^m\mathcal{H}$ by linear algebra. The fiber over $\partial/\partial q$ is

$$V^B_{\partial/\partial q} = \mathbb{Z}\,\underline{a}^m \oplus \mathbb{Z}\,\underline{a}^{m-1}\underline{b} \oplus \cdots \oplus \mathbb{Z}\,\underline{b}^m$$

$$V^{DR} = \mathbb{C}\,\underline{a}^m \oplus \mathbb{C}\,\underline{a}^{m-1}\underline{w} \oplus \cdots \oplus \mathbb{C}\,\underline{w}^m.$$

The limit MHS is

$$V_{\partial/\partial q} = \mathbb{Z}(0) \oplus \mathbb{Z}(-1) \oplus \cdots \oplus \mathbb{Z}(-m).$$

Fiber of $\mathbb{H}$ over $\lambda\,\partial/\partial q$: use the coordinate $t = q/\lambda$. Then $\partial/\partial t = \lambda\,\partial/\partial q$.

$$(q/\lambda)^{-N} = 1 + \log q\ \underline{a}\,\partial/\partial\underline{w} - \log\lambda\ \underline{a}\,\partial/\partial\underline{w}.$$

So $(q/\lambda)^{-N}\,\underline{a} = \underline{a}$ and

$$2\pi i\,(q/\lambda)^{-N}\,\underline{b} = -\underline{w} + \log\lambda\ \underline{a}$$

So $V_{\lambda\,\partial/\partial q}$ is the element λ of $\mathrm{Ext}^1_{MHS}(\mathbb{Z}(-1), \mathbb{Z}) \cong \mathbb{C}^*$.

§7 Regularizing Iterated Integrals

I'll illustrate this with a few examples. Suppose that X is a smooth projective curve and that D is an effective reduced divisor in X. Suppose that

(1) $\omega_1, \dots, \omega_r \in H^0(X, \Omega^1_X(\log D))$

(2) $P \in D$. $\quad Q \in X - D$.

(3) $\vec{v} \in T_P X, \quad \vec{v} \neq 0$.

We want to compute the regularized integral

$$\int_{\vec{v}}^{Q} (\omega_1 | \dots | \omega_r)$$

To this end, let

$I = (X_1, \dots, X_n)$

$$A = \mathbb{C}\langle X_1, \dots, X_n \rangle / I^{r+1}$$

Imbed A in $\mathrm{End}(A)$ by left mult^n.

Set

$$\Omega = \omega_1 X_1 + \dots + \omega_r X_r \in H^0(\Omega^1_X(\log D)) \otimes A.$$

This defines a flat connection on

$$\begin{array}{c} A \times X \\ | \\ X \end{array} \qquad \nabla = d + \Omega$$

so $N = -\mathrm{Res}_P \Omega$

with regular singular points along D. Its residue at P is left multiplic^n by

$$\mathrm{Res}_P \Omega = \sum_{j=1}^{n} \mathrm{Res}_P \omega_j \cdot X_j.$$

This is nilpotent in A. Set

$$T(x) = \left\langle 1 + \int(\Omega) + \int(\Omega|\Omega) + \dots + \int(\Omega|\dots|\Omega), \; {}_{x}^{Q} \right\rangle$$

Then

$$dT = -\Omega T \quad \Big\}\ \text{by formula of } d \text{ it int.}$$

Choose a holomorphic coordinate t at P such that $\vec{v} = \partial/\partial t$. Set

$$\Omega_0 = \sum_{j=1}^{n} \mathrm{Res}_P \omega_j \frac{dt}{t} X_j.$$

You can think of this as a log form on $T_p X$ with $\mathrm{Res}_0 \Omega_0 = \mathrm{Res}_p \Omega$.

Francis Brown denotes

$$\mathrm{Res}_p\, \omega_j \cdot \frac{dt}{t} \quad \text{by} \quad \overline{\omega}_j .$$

Since
$$\int_1^t (\underbrace{\tfrac{dt}{t} | \cdots | \tfrac{dt}{t}}_{m}) = \frac{1}{m!}(\log t)^m$$

$$\left\langle 1 + \int(\Omega_0) + \int(\Omega_0 | \Omega_0) + \cdots \; , \; {}_1\!\curvearrowright^{t} \right\rangle = t^{\mathrm{Res}_0 \Omega}.$$

So
$$t^{-N} T(t) = t^{\mathrm{Res}_0 \Omega}\, T(t)$$

The regularized iterated integrals are the coefficients of

$$\lim_{t \to 0} t^{\mathrm{Res}_0 \Omega}\, T(t)$$

and
$$\int_{\partial/\partial t}^{\varphi} (\omega_1 | \cdots | \omega_r) = \text{coefficient of } X_1 \cdots X_r .$$

This gives

This is the "mortar board" regularization

$$\int_{\partial/\partial t}^{\varphi} (\omega_1 | \cdots | \omega_r) = \sum_{j=0}^{r} \int_1^t (\overline{\omega}_1 | \cdots | \overline{\omega}_j) \int_t^{\varphi} (\omega_{j+1} | \cdots | \omega_r)$$

<u>Example</u>: (1) $X = \mathbb{P}^1$, $D = \{0, \infty\}$, $\varphi \in \mathbb{C}^*$

$\vec{v} = \partial/\partial z$.

$$\int_{\partial/\partial z}^{\varphi} \frac{dz}{z} = \lim_{t \to 0} \left(\int_1^t \frac{dz}{z} + \int_t^{\varphi} \frac{dz}{z} \right)$$

$$= \lim_{t \to 0} \int_1^{\varphi} \frac{dz}{z} = \log \varphi$$

(2) If $\vec{v} = \lambda\, \partial/\partial z$, take $t = z/\lambda$,

Then
$$\int_{\vec{v}}^{\varphi} \frac{dz}{z} = \lim_{t \to 0} \left(\int_{z=\lambda}^{\rho} \frac{dz}{z} + \int_{\rho}^{\varphi} \frac{dz}{z} \right)$$

$t=1$

$$= \log \varphi - \log \lambda .$$

<u>The Drinfeld Associator</u>. This is the regularization of

$$\langle T, \underline{dch} \rangle$$

where $\underline{dch}$ is the path in $\mathbb{P}^1 - \{0, 1, \infty\}$ from $\vec{v}_0 := \partial/\partial x \in T_0 \mathbb{P}^1$ to $\vec{v}_1 := -\partial/\partial x \in T_1 \mathbb{P}^1$

$$0 \longrightarrow 1$$

and

$$T = 1 + \int(\Omega) + \int(\Omega|\Omega) + \cdots$$

$$\Omega = \frac{dx}{x} X_0 + \frac{dx}{1-x} X_1 \quad \left\{ \begin{array}{l} \text{KZ} \\ \text{connection} \end{array} \right.$$

$$\Phi = \lim_{\substack{t \to 0 \\ s \to 0}} t^{X_0} \langle T, [t, 1-s] \rangle s^{X_1}.$$

Here the limit is taken with $s, t > 0$.

<u>APPENDIX</u>

Here we show that if

$$\begin{array}{c} \mathcal{V} \\ | \\ \mathbb{D} \end{array}$$

is a holomorphic vector bundle with flat connection

$$\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_{\mathbb{D}}(\log 0)$$

with residue $N \in \operatorname{End} V_0$ satisfying

(*) no two eigenvalues of N differ by a non-zero integer | eg: N is nilpotent.

then

(1) each h_t is conjugate to $h_\cdot = e^{2\pi i N}$

(2) $\mathcal{V}$ is the Deligne extension

of $\begin{array}{c}\mathbb{V}\\ |\\ \mathbb{D}^*\end{array}$ to $\mathbb{D}$ associated to this choice of $\log h_t$.

<u>Lemma:</u> If $B \in M_m(\mathbb{C})$, then the eigenvalues of

$$ad_B : M_m(\mathbb{C}) \to M_m(\mathbb{C})$$

are $\{\lambda_j - \lambda_k : \lambda_j, \lambda_k \text{ eigenvalues of } B\}$.

<u>proof:</u> Jordan canonical form implies that

$$B = D + E$$

where E is nilpotent, D is diagonal and $[D, E] = 0$. The eigenvlaves of ad_D are $\lambda_j - \lambda_k$. Since

semi-simple $\quad\leftarrow$ nilpotent

$$ad_B = \underbrace{ad_D + ad_E}_{\text{commute}}$$

The eigenvalues of ad_B and ad_D are the same. $\square$

<u>COR:</u> If $c \neq 0$ and

$$c \notin \{\lambda_j - \lambda_k : \lambda_j, \lambda_k \text{ eigenvalues of } B\}$$

then the equation

$$cX = -[B, X] + C$$

has a unique solution for all C in $M_m(\mathbb{C})$.

<u>proof</u>: The lemma implies that

$$ad_B - cI_d : M_m(\mathbb{C}) \to M_m(\mathbb{C})$$

is invertible. $\square$

<u>proof of the main assertion</u>. Trivialize $\mathbb{V}$ over a nbd of 0. (Since $\mathbb{D}$ is Stein, $\mathbb{V}$ is, infact, trivial over $\mathbb{D}$.) The connection becomes

$$\nabla v = dv - Av\frac{dt}{t}.$$

where $v : \mathbb{D}^* \to V_o$. Note that

$A(0) = N$. We need to show that

There is a change of gauge

$$P : \mathbb{D} \longrightarrow \mathrm{Aut}(V_0)$$

such that $P(0) = id_{V_0}$ and the

transformed connection

(*) $$PAP^{-1}\frac{dt}{t} - dP\,P^{-1} = N\frac{dt}{t}.$$

That is, we have to solve the equation

$$PA - NP = tP' \quad , \quad P(0) = I$$

for P. Expand P and A:

$$P = I + \sum_{k=1}^{\infty} P_n t^n \qquad P_n \in \mathrm{End}\, V_0$$

$$A = \underbrace{N}_{A_0} + \sum_{k=1}^{\infty} A_n t^n. \qquad A_n \in \mathrm{End}\, V_0$$

The equation (*) becomes

$$K P_K = \sum_{k=0}^{K-1} P_k A_{K-k} + [P_K, N]$$

for all $K \geq 1$. The Corollary above

and the assumption that no two

eigenvalues of N differ by a non-zero

integer, imply that we can inductively

solve these equations to find a formal

solution $P(t)$ of

$$PAP^{-1}\frac{dt}{t} - dP\,P^{-1} = N\frac{dt}{t}.$$

Convergence: recall

$$\|A\|^2 = \text{largest eigenvalue of } A^T A.$$

So $$\|A + K id\| = K\,\|id + \tfrac{1}{K}A\| \geq K/2$$

and $$\|A + K id\|^{-1} \leq \frac{2}{K}.$$ $\Big\}\ K \gg 0$

We have

$$(ad_N + K id)\,P_K = \underbrace{P_0}_{Id} A_K + \cdots + P_{K-1}A_1$$

$$\therefore \|ad_N + K id\|\,\|P_K\| \leq \underbrace{\|P_0\|}_{1}\|A_K\| + \cdots + \|P_{K-1}\|\,\|A_1\|$$

So, for K sufficiently large,

$$\rho_K \leq \frac{2}{K}\left(a_K + a_{K-1}\rho_1 + \cdots + \rho_{K-1}a_1\right)$$

where $\rho_j = \|P_j\|$ and $a_j = \|A_j\|$.

By rescaling t, we may assume that

$$\sum_0^\infty A_n t^n$$

converges absolutely on $|t| \leq 2$. This implies that $a_n = \|A_n\| \leq \frac{1}{2}$ when $n \gg 0$.

So

$$\rho_K \leq \frac{1}{K}\left(1 + \rho_1 + \cdots + \rho_{K-1}\right) \qquad K \gg 0.$$

$$\leq \max\{\rho_k : 0 \leq k < K\}$$

$\therefore$ $\{\rho_k : k \geq 0\}$ is bounded above

and

$$\sum_{n=0}^\infty \rho_n t^n$$

converges when $|t| < 1$.