The de Rham Homotopy Theory of Complex Algebraic Varieties II

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Abstract. We show that the local system of homotopy groups, associated with a topologically locally trivial family of smooth pointed varieties, underlies a good variation of mixed Hodge structure. In particular, we show that there is a limit mixed Hodge structure on homotopy associated with a degeneration of such varieties.

Key words. Mixed Hodge structure, homotopy, variation of mixed Hodge structure.

0. Introduction

In this paper we construct a limit mixed Hodge structure (MHS) on the homotopy groups of a degenerating family of pointed smooth algebraic varieties, generalizing the work of Clemens [1], Schmid [10], and Steenbrink [11]. Suppose that $h: Z \to S$ is a proper holomorphic map from a Kähler manifold to a compact Riemann surface. Suppose that $\sigma: S \to Z$ is a section of $h$ and that $\Sigma$ is a closed subvariety of $Z$. Let $X = Z - \Sigma$ and $f: X \to S$ be the restriction of $f$ to $X$. Over some Zariski open subset $S^*$ of $S$, the restriction of $f$ to $f^{-1}(S^*)$ will be a locally trivial $C^\infty$ fiber bundle $g: X^* \to S^*$ and $\sigma(s) \in X_s := f^{-1}(s)$ for each $s \in S^*$.

Thus, over $S^*$ we have the local systems

$$\{g_k(X_s, \sigma(s))\}_{s \in S^*, k \geq 0}$$

(0.1)

of homotopy Lie algebras associated to the pointed family

$$X^* \stackrel{g}{\to} S^*.$$

Our main theorem (1.5.1) asserts that, when the local monodromies about points in

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$S - S^*$ act unipotently on $H^*(X_s, \mathbb{Q})$, the local system (0.1) underlies a good variation of MHS over $S^*$ (cf. [12, 13]). This entails:

(0.2) The Hodge filtration on $g(X_s, \sigma(s))$ varies holomorphically with $s$ and satisfies Griffiths' transversality,

(0.3) the Hodge bundles extend to holomorphic subbundles of the canonical extension to $S$ of the holomorphic vector bundles associated to the local systems (0.1),

(0.4) for each tangent vector of a point $s_0 \in S - S^*$, there is a limit MHS $g_k(X_s, \sigma(s))|_{X_{s_0}}$ on the homotopy of the generic fiber, and

(0.5) the action of the logarithm of the monodromy about $s_0$

$$g_k(X_s, \sigma(s))|_{X_{s_0}} \rightarrow g_k(X_s, \sigma(s))|_{X_{s_0}}$$

is a morphism of MHS of type $(-1, -1)$.

Note that a pointed variety $(X_s, \sigma(s))$ can 'degenerate' for two reasons: either because the topological type of $X_s$ changes or, because the base point $\sigma(s)$ goes to infinity (i.e. $s \rightarrow s_0$ and $\sigma(s_0) \in Z_{s_0} - X_{s_0}$). A natural example of the second situation is provided by the family

$$
\begin{array}{ccc}
X \\ \sigma \downarrow \\
X \\
\end{array}
$$

where $g(x, y) = x$ and $\sigma(x) = (x, x)$ and where $X$ is an incomplete smooth variety. The associated local system of homotopy groups has fiber $g_k(X, x)$ over $x$. By restricting the family (0.6) to a suitable curve, one obtains the following corollary of the main theorem.

(0.7) THEOREM. Suppose that $X = Z - \Sigma$ is a smooth variety where $Z$ is smooth and complete. If $v_x \in T_xZ$ is a tangent vector of a point $x \in \Sigma$, not tangent to $\Sigma$, then there is a natural mixed Hodge structure on $g_k(X, v_x)$, the homotopy Lie algebra of Deligne's fundamental group $\pi_1(X, v_x)$ with basepoint a tangent vector at infinity.

Part of the proof of our main theorem (1.5.1) was worked out in [7] with S. Zucker. There we proved (0.2) [7; (4.17)] and, in [7; §6], that if one can construct a de Rham mixed Hodge complex (MHC) that is quasi-isomorphic to Steenbrink's MHC that computes the limit MHS on cohomology [11], then (0.3), (0.4) and (0.5) hold.

To construct such a de Rham MHC for the limit, we first localize the family about each point $s_0 \in S - S^*$ to obtain a pointed family

$$
\begin{array}{ccc}
X^* \xrightarrow{g} \Delta^* \\
\sigma \downarrow \\
\end{array}
$$

*So, in particular, there is a relative weight filtration on $g_k(X_s, \sigma(s))$ (cf. [12, 13]).
over the punctured disk. We regard $X^*$ as a small deleted neighbourhood of the central fiber $X_0$ and $\Delta^*$ as a small deleted neighbourhood of $s_0$ in $S$. De Rham MHCs for such deleted neighbourhoods were constructed in [4]. As the local monodromy about $s_0$ is unipotent, we can apply I(4.3.1)* to obtain a de Rham MHC for the homotopy fiber $E_f(t)$ of $f$ over $t \in \Delta^*$. Since $f$ is a fibration, the canonical inclusion $X_t \to E_f(t)$ induces an isomorphism on cohomology. We show that the de Rham MHC for $X_t$ obtained as the homotopy fiber complex is quasi-isomorphic with Steenbrink’s MHC for the limit MHS on $H'(X_t)$ associated with the tangent vector $\partial / \partial t$ of $s_0$.

Navarro [8, 9] has obtained results similar to those in this paper.

1. A Limit Mixed Hodge Structure on Homotopy

1.1. THE SETTING

Suppose that $Z$ is a complex Kähler manifold, that $h: Z \to T$ is a proper holomorphic map onto the disk and that $D \subseteq Z$ is a divisor in $Z$. View $T$ as having a fixed parameterization by fixing an imbedding $T \to \mathbb{C}$. Denote the components of $D$ by $D_i$ so that $D = \cup D_i$. Set $Y = h^{-1}(0)$. We suppose that $D \cup Y$ is a divisor with normal crossings in $Z$ and that $h$ and its restriction to each $D_{i_0} \cap \ldots \cap D_{i_q}$ is flat over $T$ and smooth over $T^*$, the punctured disk. Set $X = Z - D$ and $X^* = Z - (D \cup Y)$. Denote the restriction of $h$ to $X$ by $f: X \to T$ and its restriction to $X^*$ by $g: X^* \to T^*$. The fiber of $f$ over $t \in T$ will be denoted by $X_t$. Finally, let $\sigma: T \to Z$ be a section of $h$ such that $\sigma(t) \in X_t$ when $t \in T^*$. We shall call this a pointed family over $T$.

Assume, in addition, that the monodromy

$H'(X_t; \mathbb{Q}) \to H'(X_t; \mathbb{Q})$,

is unipotent. This will be the case when $Y$ is reduced. The geometric monodromy $(X_t, \sigma(t)) \to (X_t, \sigma(t))$ induces a well-defined automorphism of $H'(\overline{B(E'X_t)})$. Since $Q_{\mathbb{Q}} H'(\overline{B(E'X_t)})$, is a subquotient of $\otimes H'(X_t)$, it follows that the monodromy $Q_{\mathbb{Q}} H'(\overline{B(E'X_t)}) \to Q_{\mathbb{Q}} H'(\overline{B(E'X_t)})$, is unipotent. Interpreting the cohomology of the bar construction as in [6], Section 2, we obtain

(1.1.1) PROPOSITION. (a) For each $s$, the action of monodromy on $Q_{\pi_1}(X_t, \sigma(t)) / J^{s+1}$, is unipotent.

*The references I(a,b,c) refers to paragraph (a,b,c) of the first part [6] of this paper.
(b) If $X_t$ is a nilpotent space, then the action of monodromy on
\[ a_t(X_t, \sigma(t)), \]
is unipotent.

\[ \Box \]

1.2. A DE RHAM MHC FOR $X^*$

This is a review of the construction in [4] of a Q-de Rham MHC for the punctured neighborhood $N^* = N - V$ of a subvariety $V$ of a smooth subvariety $W$, where $N$ is a regular neighborhood of $V$ in $W$. For each $t \in T^*$ we will obtain Q-de Rham MHCs
\[ P(X^*, \sigma(t)), \quad P(T^*, t), \]
and a morphism
\[ g^*: P(T^*, t) \to P(X^*, \sigma(t)), \]
induced by $g$.

Suppose that $Y = \cup j_0 Y_j$ is the decomposition of $Y$ into irreducible components. We may assume that each $Y_j$ is smooth. Choose tubular neighborhoods $N_j$ of $Y_j$ in $Z$ with each inclusion $Y_j \to N_j$ a homotopy equivalence. Set $N = \cup N_j$ and $N^* = N - (D \cup Y)$. The inclusion $N^* \to X^*$ is a homotopy equivalence.

For each $J = (j_0, \ldots, j_q) \in A^q$, set
\[ N_j = N_{j_0} \cap \ldots \cap N_{j_q}, \quad N^*_j = N_j - (D \cup Y). \]
These fit together to form a (split) simplicial manifold $N^*_j$ whose geometric realization $|N^*_j|$ is homotopy equivalent to $N^*$ and hence to $X^*$.

(1.2.1) One can construct a Q-de Rham MHC $K(N^*_j)$ for $N^*_j$ as follows:

- the rational part $(K_\mathbb{Q}(N^*_j), W)$ is constructed as in I(5.6).
- the complex part $(K_\mathbb{C}(N^*_j), W, F^*)$ is the global sections of the sheaf on $T$ whose sections over the polydisk $U \subseteq N_j$ are the elements of the d.g.a.
\[ E^*(U \cap Y_j \log E_j) \otimes A \left( \frac{dz_1}{z_1}, \ldots, \frac{dz_k}{z_k} \right), \]
where
(a) $z_1 = \cdots = z_k = 0$ is the local defining equation of $Y_j$ in $U$,
(b) $E_j = Y_j \cap (D \cup Y_K)$, where $Y = Y_j \cup Y_K$ and $Y_j$ and $Y_K$ have no common components.

The Hodge and weight filtrations are defined in the usual way. (That is, $F^*$ is defined by counting $dz$'s and $W_i$ by counting $dz/z$'s, cf[4].)

The $K(N^*_j)$ fit together to form a split cosimplicial Q-de Rham MHC $K(N^*_j)$ over the nerve of $Y$. By I(5.5.2) its de Rham complex $P(X^*_j) = DK(N^*_j)$ is a Q-de Rham MHC for $|N^*_j|$. Since $X^*$ is homotopy equivalent to $|N^*_j|$, we have
LEMMA. The pair $\mathcal{P}(X^*) = ((P_0^*, W_0), (P_0^*, W_0, F'))$ is a $Q$-de Rham MHC for $X^*$.

We next construct a de Rham MHC for $(X^*, \sigma(t))$ when $t \neq 0$. There is a unique $j \in A$ such that $\sigma(0) \in Y_j$. We may assume that $j = 0$. Note that $\sigma(0) \notin Y_j$, if $j \neq 0$. Define a d.g.a. homomorphism

$$\sigma^*: K_c(N_0^*) \to \mathbb{C},$$

to be zero if $J \neq \{0\}$ and to be the composite

$$K_c(N_0^*) \to K_c(N_0^*) \overset{\sigma(0)^w}{\longrightarrow} \mathbb{C},$$

when $J = \{0\}$, where the last map is evaluation at $\sigma(0)$. These fit together to define an augmentation

$$\varepsilon: P_c(N) \to \mathbb{C},$$

which is essentially evaluation at $\sigma(0)$. Let

$$\varepsilon_t: E^t(X^*) \to \mathbb{C},$$

be the augmentation induced by the inclusion $\sigma(t) \to X^* (t \in T^*)$.

(1.2.3) PROPOSITION. There is a natural quasi-isomorphism of augmented d.g.a.s

$$P_c(X^*) \overset{\varepsilon}{\longrightarrow} E^t(X^*) \overset{\varepsilon_t}{\longrightarrow} \mathbb{C}.$$

Proof. By shrinking the $N_j$ if necessary, we may assume that $\sigma(T)$ is disjoint from $N_j$, when $j \neq 0$. Let $\Delta \subseteq T$ be a subdisk containing $0$ such that $\sigma(\Delta) \subseteq N_0$. Define $E^t_\Delta(N_j \log(D \cup Y))$ to be the sub d.g.a. of $E^*(N_j \log(D \cup Y))$ consisting of those $w$ such that

$$\sigma^* w = \begin{cases} 
0 & \text{deg } w > 1, \\
\lambda \frac{dt}{t} & \text{deg } w = 1, \quad \lambda \in \mathbb{C}, \\
\text{constant} & \text{deg } w = 0.
\end{cases}$$

The inclusion

$$E^t_\Delta(N_j \log(D \cup Y)) \to E^*(N_j \log(D \cup Y))$$

is an isomorphism, when $J \neq \{0\}$ and a quasi-isomorphism when $J = \{0\}$, as $\sigma(\Delta)$ is contractible. The $E^t_\Delta(N_j \log(D \cup Y))$ fit together to form a cosimplicial d.g.a. $E^t_\Delta$ and $\sigma^*$ induces an augmentation

$$\varepsilon: DE^t_\Delta \to \mathbb{C}.$$
The natural projection $DE^*_t \to P^*_t(X^*)$ commutes with augmentations.

Similarly, one can define an augmentation $\varepsilon_t : DE^*(N^*_t) \to C$ of the de Rham complex of the cosimplicial d.g.a. $E^*(N^*_t)$ by evaluating at $\sigma(t)$. The natural inclusion $DE^*_t \to DE^*(N^*_t)$ preserves the augmentations. Finally, the d.g.a. homomorphism $E^*(X^*) \to DE^*(N^*_t)$ commutes with the augmentations induced by $\sigma(t)$. \qed

It is routine to use (1.2.3) to show that we have a morphism of MHCs

$$\varepsilon_t : P(X^*) \to Q$$

This completes the construction of a $Q$-de Rham MHC for $(X^*, \sigma(t))$.

To construct a $Q$-de Rham MHC for $(\Delta^*, t)$, apply the above construction when $Z = \Delta$ and $h : Z \to \Delta$ is the identity. Note that

$$P^*_C(\Delta^*) = \Lambda \left( \frac{dt}{t} \right).$$

From the naturality of the construction we have a map

$$g^* : P(\Delta^*, t) \to P(X^*, \sigma(t)),$$

induced by $g$. Thus, we have proved

(1.2.4) LEMMA. For each $t \in \Delta^*$, there are $Q$-de Rham MHCs $P(\Delta^*, t)$ and $P(X^*, \sigma(t))$ and there is a map

$$g^* : P(\Delta^*, t) \to P(X^*, \sigma(t)),$$

induced by $g$. \qed

(1.2.5) REMARKS. (a) The complex parts of $P(\Delta^*, t)$ and $P(X^*, \sigma(t))$ do not depend upon $t \in \Delta^*$. That is, the maps in the diagram

$$\begin{array}{ccc}
P^*_C(\Delta^*) & \xrightarrow{\varepsilon} & C \\
g^* & & \\
P^*_C(X^*) & \xrightarrow{\varepsilon} & C \\
\end{array}$$

are independent of $t$.

(b) The quasi-isomorphism of (1.2.3) depends upon the parameterization $T \subseteq C$ of the disk. Consequently, the MHCs $P(X^*, \sigma(t))$ and $P(\Delta^*, t)$ both depend upon the choice of parameterization of $T$. \qed

Since the monodromy is unipotent and since $g(\Delta^*, t) = H_1(\Delta^*)$ is of type $(-1, -1)$ and since $g(\Delta^*, t)$ is generated by the logarithm of the generator of $\pi_1(\Delta^*, t)$, I(2.5.2), we may apply I(4.3.1) and I(4.3.5) to obtain:

(1.2.6) PROPORTION. For each $t \in \Delta^*$ there is a MHS on the cohomology and homotopy

$$H_1(E_0(t)), Q\pi_1(E_0(t), \sigma(t))/J^{s+1}, \quad g_*(E_0(t), \sigma(t)),$$
of the homotopy fiber $E_g(t)$ of $g: X^* \to T^*$. The logarithm of monodromy acts on each
as a morphism of type $(-1, -1)$. These MHSs are functorial with respect to maps of
pointed families. Furthermore, the sequence of inclusions

$$E_g(t) \cong X_t \to X \cong X_0,$$

induces a morphism of MHS

$$g_*(E_g(t), \sigma(t)) \to g_*(X_0, \sigma(0)),
$$

provided $\sigma(0) \in X_0$.

(1.2.7) REMARKS. (a) The MHS on the cohomology and homotopy of $(E_g(t), \sigma(t))$
depend upon the parameterization of $T$.

(b) Since $g$ is a fibration, the natural inclusion $(X_t, \sigma(t)) \to (E_g(t), \sigma(t))$
duces an isomorphism on cohomology and homotopy. Thus, we have three ways of putting
a MHS on the cohomology (say) of $X_t$: first there is the usual MHS obtained by
viewing $X_t$ as an algebraic variety, next there is the limit MHS corresponding to the
tangent vector $t \partial / \partial t$ of 0 and third there is the MHS constructed in (1.2.6). To
distinguish these, we shall denote them by $H^*(X_t)$, $H^*(X_t)_{\text{lim}}$ and $H^*(E_g(t))$, respectively.
A similar convention applies to homotopy groups. In (1.4) we shall show
that $H^*(X_t)_{\text{lim}}$ and $H^*(E_g(t))$ are isomorphic MHSs, with the isomorphism being
induced by the inclusion of $X_t$ into $E_g(t)$.

1.3. NILPOTENT ORBITS

By (1.2.5) the complex parts of the MHCs

$$B(P(X^*), P(\Delta^*), Q) \text{ and } BB(P(X^*), P(\Delta^*), Q),$$

for the homology and homotopy of $E_g(t)$ do not depend upon $t \in \Delta^*$. Thus, given a
parameterization of $\Delta$, we obtain trivializations of the $C^\infty$ complex vector bundles
obtained by tensoring the local systems

$$\{H^k(E_g(t), Q)\}_{i \in \Delta^*}, \quad \{Q \pi_1(E_g(t), \sigma(t))/J^{y+1}\}_{i \in \Delta^*}, \quad \{\mathcal{F}_k(E_g(t), \sigma(t))\}_{i \in \Delta^*}
$$

with $\mathcal{F}$, the sheaf of $C^\infty$ functions on $\Delta^*$. Sections of these bundles which are
constant with respect to these trivializations will be called false constants. Each of
these bundles has a natural flat structure. In this section we compute the connection
form of this connection with respect to the trivialization given by the false
consstants. As a result we shall see that the false constants are holomorphic (i.e.
locally they are an $\mathcal{O}$-linear combination of flat sections.) Thus, the false constants
trivialize the holomorphic vector bundles associated with these local systems. We
shall denote these holomorphic vector bundles by

$$\mathcal{H}^k(E_g/\Delta^*), \quad C\pi_1(E_g/\Delta^*, \sigma)/J^{y+1}, \quad \mathcal{F}_k(E_f/\Delta^*, \sigma).$$

A trivialization of a vector bundle over $\Delta^*$ gives an extension of the bundle to $\Delta$. In
this way the false constants give extensions of these holomorphic vector bundles to
\[ \Delta. \text{ We show that these are the canonical extensions of these bundles to } \Delta \text{ and that each is a nilpotent orbit of MHS.} \]

Recall from I(1.2.4) that since \( \Lambda(dt/t) \) is a commutative d.g.a.,

\[ B(P_{\mathcal{C}}(X^*), \Lambda(dt/t), \mathbb{C}) \]

is a commutative d.g.a. We shall denote it by \( L_0^c \).

(1.3.1) PROPOSITION. There is a natural isomorphism of bifiltered d.g.a.'s

\[ L_0^c \simeq P_{\mathcal{C}}(X^*)[\theta], \]

where \( P_{\mathcal{C}}(X^*)[\theta] \) denotes the polynomial ring over \( P_{\mathcal{C}}(X^*) \) generated by the indeterminate \( \theta \) of degree 0, where

\[ d\theta = g^* \frac{dt}{t}, \text{ and } \theta \in W_2 \cap F^1. \]

Proof. Set \( \theta = -(dt/t) \in L_0^c \) and note that \( d\theta = g^*(dt/t) \) and that

\[ (-1)^p \theta^n = \left( \frac{dt}{t} \right) \wedge \ldots \wedge \left( \frac{dt}{t} \right) = n! \left( \frac{dt}{t} \right) \ldots \left( \frac{dt}{t} \right). \]

Since \( (dt/t) \in W_2 \cap F^1 L_0^c \), the result follows. \qed

Suppose that \( \mathbb{C}^n \times M \to M \) is a vector bundle over the manifold \( M \) with a fixed trivialization. To a connection on this bundle we can associate a parallel transport map \( T: PM \to \text{GL}(m) \) from the space of piecewise smooth paths on \( M \) into \( \text{GL}(m) \). It is defined by taking the path \( \gamma \) to the matrix obtained by parallel transporting a frame along \( \gamma \) and comparing it with the same frame transported along \( \gamma \) via the trivialization (cf. [5]). The map satisfies \( T(\alpha\beta) = T(\alpha)T(\beta) \). Suppose that the connection is given by \( \nabla x = dx - x\omega \), where \( x: M \to \mathbb{C}^n \) is a locally defined function and \( \omega \in E^1 M \otimes \text{gl}(m) \) is the connection form relative to the fixed trivialization. There is an explicit formula for \( T \) due to Chen. For a proof, see [5].

(1.3.2) LEMMA. The transport function \( T: PM \to \text{GL}(n) \) is given by the absolutely convergent power series

\[ \gamma \mapsto I + \int_0^\gamma \omega + \int_0^\gamma \omega \omega + \int_0^\gamma \omega \omega \omega + \ldots. \]

Recall that for each \( t \in \Delta^* \), the false constants give a canonical quasi-isomorphism.

\[ L_0^c \leftrightarrow A_0^c E_0(t). \]

If \( \gamma \) is a path of \( \Delta^* \) from \( t \) to \( b \) then we have a geometric monodromy map

\[ h_\gamma: E_0(t) \to E_0(b) \]

\[ (x, \alpha) \mapsto (x, \alpha \cdot \gamma). \]
(1.3.3) PROPOSITION. The geometric monodromy $h$, induces the d.g.a. homomorphism

$$T_{\gamma}: L_c \to L_c, \quad w\theta^m \mapsto w(\theta + \log t - \log b)^m.$$ 

Proof. Recall that $\theta = -(dt/t)$. According to I(2.8.2)

$$T_{\gamma}(\theta) = -\left( \frac{dt}{t} \right) - \int_{\gamma} \frac{dt}{t} = \theta + \log t - \log b.$$ 

Since $T_{\gamma}$ is an algebra homomorphism, the result follows. \qed

Taking $\gamma$ to be a loop based at $\gamma$ going counterclockwise around $\Delta^*$, we obtain a formula for the action $T: L_c \to L_c$ of the geometric monodromy $h: E_\theta(b) \to E_\theta(b)$.

(1.3.4) COROLLARY. The monodromy is given by the formula

$$T: L_c \mapsto L_c,$$

$$\theta \mapsto \theta + 2\pi i.$$ \qed

Observe that the action of $T$ on $L_c$ is locally unipotent so that we may take its logarithm. Set $N = (1/2\pi i) \log T$. This is a locally nilpotent derivation of $L_c$ of degree 0 that commutes with $d$.

(1.3.5) PROPOSITION. The derivation $N: L_c \to L_c$ is $d/d\theta$.

Proof. Since $(T - I)\theta = 2\pi i$,

$$N\theta = \frac{1}{2\pi i} (\log T)\theta = 1 = \frac{d}{d\theta} \theta.$$ 

The derivations $N$ and $d/d\theta$ both annihilate $P_c^\ast(X^\ast)$ and agree on $\theta$. Therefore they are equal. \qed

We can view $L_c \times \Delta^* \to \Delta^*$ as a vector bundle with connection given by the parallel transport map of (1.3.3). This connection can be viewed as a lift of the Gauss-Manin connection to the form level.

(1.3.6) PROPOSITION. This connection is given by

$$\nabla s = ds + \frac{dt}{t} \otimes Ns.$$ 

Proof. The section $s = w\theta^m$ is a false constant so that $ds = 0$. Fix $b \in \Delta^*$ and let $t$ be a variable point and $T_t$ denote the parallel transport $L' \to L'$ from the fiber over $b$ to the fiber over $t$. By (1.3.3)

$$T_t(w\theta^m) = w(\theta + \log t - \log b)^m,$$
so that
\[ \frac{d}{dt} T_i(w^0) = \frac{1}{b} \frac{m w^0}{m^0 - 1} = \frac{1}{b} N(w^0). \]

It follows that
\[ \nabla_S = \frac{dt}{t} \otimes N_S. \]

Since \( N \) is locally nilpotent, \( H(E_g(t), \mathcal{Q}) \) is finite dimensional and since \( V \) has a simple pole at 0, we have proved:

(1.3.7) COROLLARY. The natural extension to \( \Delta \) of \( \mathcal{H}^k(E_g/\Delta^*) \) given by the false constants is its canonical extension in the sense of Deligne [2].

We can do the same for homotopy. There is one subtlety. Namely, the geometric monodromy
\[ h_\gamma : E_g(b) \to E_g(b), \]
does not fix the basepoint \( * = (\sigma(b), \eta, \sigma(b)) \), but takes it to \( (\sigma(b), \gamma) \). However, we can circumvent this by using I(4.3.5).

(1.3.8) PROPOSITION. The natural extensions to \( \Delta \) of \( \mathcal{C} \pi_1(E_g/\Delta^*, \sigma)/\mathcal{P}^{+1} \) and \( \mathcal{G}_k(E_g/\Delta^*, \sigma) \) given by the false constants are their canonical extensions in the sense of Deligne.

Proof. As the proof proceeds similarly to that of (1.3.7), we only sketch the details. The first step is to show that the extension to \( \Delta \) of the local system
\[ \{ g(t, X^*, \sigma(t)) \}_{t \in \Delta^*}, \]
given by the false constants \( QB(P_c(X^*)) \) is the canonical extension of this local system. This follows from the fact that the derivative of the false constant \([w_1, \ldots, w_r]\) is
\[ \sigma^* w_r[w_1, \ldots, w_r - 1] - \sigma^* w_1[w_2, \ldots, w_r] \in \frac{dt}{t} \otimes QB(P_c(X^*)), \]

The second step is to observe that the map
\[ P_c(X^*) \to B(P_c(X^*), \mathcal{O}(\frac{dt}{t}), \mathcal{Q}_t), \]
dual to \( E_g(t) \to X^* \) induces a map of false constants between their respective bar constructions. Since, by I(2.5.6),
\[ g_k(E_g/\Delta^*) \subseteq \{ g_k(X^*, \sigma(t)) \}_{t \in \Delta^*}, \]
and both have the same false constants, the result follows.
1.4. COMPARISON WITH THE LIMIT MHS

In this section we show that the natural isomorphism

\[ H^*(E_y(t)) \cong H^*(X_y)_{\text{lims}}, \quad (1.4.1) \]

induced by the inclusion \( X_y \to E_y(t) \) between the MHS on the homotopy fiber and the usual limit MHS on \( X_y \) (cf. [10]) is an isomorphism of MHS. As (1.4.1) is an isomorphism of \( \pi_1(\Delta^*, t) \)-modules defined over \( \mathbb{Q} \), we need only show that it preserves the Hodge and weight filtrations when \( t = 1 \). We do this as follows. The global MHC \( P^*(X^*) \) is replaced by a cohomological MHC \( \Psi^*(X^*) \). One then obtains a bifiltered sheaf \( \mathcal{L}^*_C \) over \( X \) of d.g.a.s. that computes \( (H^*(E_y(t), C), \mathcal{W}_y, F) \) by adjoining \( \theta \) to \( \mathcal{P}_C^* \). The argument is completed by constructing a filtration preserving quasi-isomorphism \( \mathcal{L}^*_C \leftarrow \mathcal{A}^*_C \) with the complex part of Steenbrink's cohomological MHC for the limit ([11; (4.14)], [12; (5.5)]) which induces (1.4.1) on cohomology.

We begin by constructing a cohomological MHC for the simplicial punctured neighborhood \( N^*_y \). Let \( N \) be a simplical neighborhood of \( Y \) as in (1.2). Define a sheaf over \( Y_{\text{red}} \) as follows: For a multi-index \( J = (j_0, \ldots, j_q) \) set

\[ Y_J = (Y_{j_0} \cap \ldots \cap Y_{j_q})_{\text{red}}. \]

Define

\[ \mathcal{P}_C^*[q] = \bigoplus_{|J| = q+1} \Omega^*_C(\log(D \cup Y)) \otimes e_j \mathcal{G}_{Y_J}. \]

This has Hodge and weight filtrations induced from those of \( \Omega^*_C(\log(D \cup Y)) \). In fact, the weight filtration \( W \) can be written as the convolution

\[ W^*_y = W(D)^* \ast W(Y). \]

of the weight filtration \( W(D) \), coming from the divisor \( D \) at infinity and the weight filtration \( W(Y) \), coming from \( Y \).

There is an analogous construction over \( \mathbb{Q} \): Denote by \( i: N^*_y \to N \) the inclusions. Set

\[ \mathcal{P}_C^*[q] = \bigoplus_{|J| = q+1} \mathcal{P}_C^*[q] \]

This has two weight filtrations \( W(D) \) and \( W(Y) \), coming from the truncation filtrations associated with the inclusions \( N_y - D \to N \) and \( N_y - Y \to N \), respectively. Set \( W^*_y = W(D)^* \ast W(Y) \). Straightforward sheaf theory can be used to establish the following statement.

(1.4.2) PROPOSITION. There is a natural quasi-isomorphism

\[ \mathcal{P}_C^*[q] \otimes C \leftrightarrow \mathcal{P}_C^*[q], \]

which is both \( W(D) \), and \( W^*_y \), filtered.
Denote by $a_q: Y_q \to Y^{\text{red}}$ the natural map. We can form the double complex

$$\mathcal{P}^{p,q} = a_{q*}\mathcal{P}^p [q],$$

of sheaves over $Y^{\text{red}}$. Let $\mathcal{F}_C$ be the total complex of $\mathcal{P}_C$. The Hodge filtration on $\mathcal{F}_C$ is defined in the obvious way. Define the weight filtration $W_\ast$, the filtration $W(D)$, associated with the divisor at infinity by $W(D)$, and a filtration $G$, by

$$W_1 \mathcal{F}_C = a_{q*} W_{1+q} \mathcal{P}^p [q],$$
$$W(D)_1 \mathcal{F}_C = a_{q*} W(D)_{1+q} \mathcal{P}^p [q],$$
$$G_m \mathcal{F}_C = a_{q*} W(Y)_{m+q} \mathcal{P}^p [q].$$

These induce filtrations on $\mathcal{F}_C$ such that $W_\ast = W(D) \ast G_\ast$. Similarly, one can define a complex of sheaves $\mathcal{F}^\ast_C$ on $Y$ with filtrations $W_\ast$, $W(D)$, and $G_\ast$, with $W_\ast = W(D) \ast G_\ast$.

(1.4.3) PROPOSITION. The pair

$$\Psi = (\mathcal{F}_C, W_\ast, (\mathcal{F}_C, W_\ast, F^{\ast}))$$

is a cohomological MHC on $Y^{\text{red}}$ whose cohomology is isomorphic to the MHS on $H^*(X^{*})$ constructed in (1.2).

Proof. We use the notation of section (1.2). Set

$$K_C(N_\ast^q) = \bigoplus_{|j| = q+1} K_C(N_j).$$

The proof follows from the fact that $K_C(N_\ast^q)$ is the global sections of a bifiltered resolution of $\mathcal{F}_C[q]$.

Next define a complex of sheaves $\mathcal{L}_C$ over $Y$ by

$$\mathcal{L}_C = \bigoplus_{m \geq 0} \mathcal{P}_C \theta^m,$$

where $\theta$ is an indeterminate of degree 0 lying in $W_2 \cap F^1$ satisfying

$$d\theta^m = mg^x \frac{dt}{t} \otimes \theta^{m-1}.$$

Once again we have filtrations $W(D)$, and $G$, such that $W_\ast = W(D) \ast G_\ast$, where $\theta \in G_2 \cap W(D)_0$.

Next we recall Steenbrink's cohomological MHC $\Psi'$ for the limit MHS as described in [12; (5.5)]. The complex $\Omega^{\ast}_{\mathcal{L}_C}(\log (D \cup Y))$ carries weight filtrations $W(D)$, and $W(Y)_\ast$, with respect to the divisors $D$ and $Y$, respectively, and $W_\ast = W(D) \ast W(Y)_\ast$. Set

$$\mathcal{A}_{\mathcal{L}_C} = \Omega^{\ast + q+1}_{\mathcal{L}_C}(\log (D \cup Y))/W(Y)_q \quad (p, q \geq 0).$$

The differential $d^\ast: \mathcal{A}_{\mathcal{L}_C} \to \mathcal{A}_{\mathcal{L}_C}^{p+1, q}$ is the de Rham differential, while $d^\ast: \mathcal{A}_{\mathcal{L}_C}^{p+1, q} \to \mathcal{A}_{\mathcal{L}_C}^{p, q+1}$ is cupping with $f^\ast (dt/t)$ on the left. The weight filtrations are defined by

$$W(D)_{1} \mathcal{A}_{\mathcal{L}_C}^p = \text{image of } W(D)_{1, \Omega_{\mathcal{L}_C}^{+ q+1}}(\log (D \cup Y)) \quad \text{in } \mathcal{A}_{\mathcal{L}_C}^p,$$

$$W_k \mathcal{A}_{\mathcal{L}_C}^p = W_2 k+1 \Omega_{\mathcal{L}_C}^{+ q+1}(\log (D \cup Y))/W(Y)_q,$$
and the Hodge filtration is defined by

\[ F^p \mathcal{A}_t^* = \bigoplus_{r \geq p} \mathcal{A}_t^r. \]

The analogous construction for the rational part

\[ (\mathcal{A}_t^0, W_\cdot, (\mathcal{A}_t^0, W_\cdot, F)). \]

is given in [12, §5]. Steenbrink's cohomological MHC \( \mathcal{H} \) for the limit MHS on \( X_t \) associated with the tangent vector \( \partial/\partial t \) of 0 in the disk comprises the pair

\[ ((\mathcal{A}_0, W_\cdot, (\mathcal{A}_0, W_\cdot, F))). \]

It is filtered by \( W_\cdot(D) [12; (5.6)] \).

To compare \( \mathcal{L}_t \) and \( \mathcal{A}_t \) we need a simplicial version of \( \mathcal{H}^* \). Define, for each multi-index \( I = (i_0, \ldots, i_q) \),

\[ \mathcal{A}_{t,I}^* = \mathcal{A}_t^r \otimes \mathcal{O}_{Y_I}. \]

This has filtrations \( F^r, W_\cdot, W_\cdot(D) \) induced by those on \( \mathcal{A}_t^* \). By applying the construction of [12; (5.12)] to \( N^\delta_t \), we obtain a double complex of sheaves \( \mathcal{A}_{t,I}^* \) on \( Y_I \) with filtrations \( W_\cdot, W_\cdot(D) \).

When \( t \neq 0 \) is small enough, \( \{X_t \cap N_I\}_I \) is an open covering of \( X_t \). Set \( X_{t,I} = X_t \cap N_I \). Steenbrink's arguments [11; (4.19)] (cf. [4; (4.2.2)]) can be applied to prove the following proposition.

(1.4.4) PROPOSITION. The pair of complexes of sheaves on \( Y_I \)

\[ \mathcal{H}_I = ((\mathcal{A}_{t,I}^0, W_\cdot, (\mathcal{A}_{t,I}^0, W_\cdot, F))). \]

is a cohomological MHC for \( X_{t,I} \).

We can now glue these together as in I(5.5) to obtain a cohomological MHC for \( X_t \). Set

\[ \mathcal{A}^q = \bigoplus_{|I| = q+1} \mathcal{A}_I. \]

These form a cosimplicial cohomological MHC \( \mathcal{H} \) [\( ]\).

(1.4.5) PROPOSITION. The cohomological MHC \( \tilde{\mathcal{H}} \) associated to \( \mathcal{H} \) via [3; (8.1.10.1)] is a cohomological MHC for \( X_t \) and the natural map \( \mathcal{H} \to \tilde{\mathcal{H}} \) is a morphism of cohomological MHCs.

(1.4.6) COROLLARY. The cohomological MHC \( \tilde{\mathcal{H}} \) computes the limit MHS on \( H^*(X_t) \) associated to the tangent vector \( \partial/\partial t \) of 0.

We can now prove the main result of this section.

(1.4.7) THEOREM. The isomorphism

\[ H^*(E_q(t)) \to H^*(X_t)_{\text{lim}} \]

induced by the inclusion \( X_t \to E_q(t) \) is an isomorphism of MHS.
Proof. As remarked previously, it suffices to construct a filtration preserving chain map \( \mathcal{L}_* \to \mathcal{J}_* \) corresponding to the inclusion \( X_1 \to E_q(1) \), as we already know that the induced map

\[
H'(E_q(1)) \to H'(X_1),
\]

is a \( \pi_1(\Delta^*, 1) \) module homomorphism, defined over \( \mathcal{O} \). Such a chain map is defined by

\[
\mathcal{L}^p[q] = \bigoplus_{m > 0} \mathcal{P}^p[q] \theta^m \to \mathcal{P}^p[q] \overset{\Delta^* \theta^m}{\longrightarrow} \mathcal{A}^p \theta^m[q] \tag{1.4.8}
\]

Because of the large number of quasi-isomorphisms that occur, we only sketch the proof that the map on cohomology induced by (1.4.8) is the natural isomorphism \( H'(E_q(1)) \to H'(X_1) \). To simplify notation, we suppose that \( Y \) is reduced and we consider only the proper case (i.e., \( D = \emptyset \)). Both of these simplifications are purely notational. Neither affects the proof.

The first reduction is to replace the sheaves in (1.4.8) by simpler ones. Let \( \mathcal{P}^* \) be the complex of sheaves

\[
\Omega_X(\log Y) \otimes_{\mathcal{O}_M} \mathcal{O}_Y[\theta]
\]

on \( Y \), obtained by adjoining the polynomial generator \( \theta \) of degree zero to the restriction of the log complex to \( Y \). The differential of \( \theta \) is defined to be \( g^\theta (dt/t) \). The morphism \( Y_\to Y \) induces quasi-isomorphisms

\[
\mathcal{P}^* \to \mathcal{L}^* \quad \text{and} \quad \mathcal{A}^{**} \to \mathcal{J}^{**}.
\]

Recall that \( \mathcal{A}^{**} \) is a resolution of the complex of sheaves

\[
\Omega_{X/A}(\log Y) \otimes_{\mathcal{O}_M} \mathcal{O}_Y,
\]

on \( Y \) [11; (4.15)]. Define a chain map

\[
\mathcal{L}^* \to \Omega_{X/A}(\log Y) \otimes_{\mathcal{O}_M} \mathcal{O}_Y,
\]

as the composite

\[
\mathcal{L}^p = \bigoplus_{m > 0} \Omega_X^p(\log Y) \otimes_{\mathcal{O}_M} \mathcal{O}_Y \cdot \theta^m \overset{\text{proj}}{\longrightarrow} \Omega_X^p(\log Y) \otimes_{\mathcal{O}_M} \mathcal{O}_Y \to \Omega_{X/A}(\log Y) \otimes_{\mathcal{O}_M} \mathcal{O}_Y.
\]

Composing with the natural map

\[
\Omega_{X/A}(\log Y) \otimes_{\mathcal{O}_M} \mathcal{O}_Y \to \mathcal{A}^{**} \to \mathcal{J}^{**}
\]

[11;(4.15)], we obtain a commutative square of complexes of sheaves on \( Y \):

\[
\begin{array}{ccc}
\mathcal{L}^* & \xrightarrow{\phi} & \Omega_{X/A}(\log Y) \otimes_{\mathcal{O}_M} \mathcal{O}_Y \\
\downarrow & & \downarrow \\
\mathcal{L}^* & \to & \mathcal{A}^{**}
\end{array}
\]
Thus, to prove the theorem it suffices to show that $\psi$ induces the isomorphism $H^*(X_1) \to H^*(E_g(1))$ on homology.

Next, let $\tilde{X}^*$ be the covering of $X^* = X - Y$ defined such that the diagram

\[
\begin{array}{ccc}
\tilde{X}^* & \xrightarrow{\tilde{f}} & X \\
\downarrow \quad \quad \downarrow & & \downarrow \\
U & \xrightarrow{f} & \Delta \\
\quad \downarrow \quad & & \quad \downarrow \\
\sigma & \xrightarrow{e^{2\pi i u}} & \\
\end{array}
\]

commutes, where

$$\Delta = \{ z \in \mathbb{C} : |z| < e^{2\pi \varepsilon} \} \quad U = \{ u \in \mathbb{C} : \text{im } z > -\varepsilon \}$$

and $\varepsilon > 0, \sigma \in \mathbb{R}$.

By elementary covering space theory, there is a continuous map $\varphi : E_g(1) \to \tilde{X}^*$, unique up to a deck transformation of $\tilde{X^*}/X^*$, such that the diagram

\[
\begin{array}{ccc}
\tilde{X}^* & \xrightarrow{\varphi} & X^* \\
\downarrow \quad \quad \downarrow & & \downarrow \\
E_g(1) & \xrightarrow{j} & X^* \\
\downarrow \quad \quad \downarrow & & \downarrow \quad \quad \downarrow \\
(x, \gamma) & \xrightarrow{\varphi} & x
\end{array}
\]

commutes. The composite

$$X_1 \to E_g(1) \xrightarrow{\varphi} \tilde{X}^*$$

imbeds $X_1$ as a fiber of $\tilde{X}^* \to U$. By adjusting $\varphi$ by a deck transformation if necessary, we may assume that $X_1$ is imbedded as the fiber over 0. That is, the diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{j} & \tilde{X}^* \\
\downarrow \quad \quad \downarrow & & \downarrow \\
E_g(1) & \xrightarrow{\varphi} & \\
\end{array}
\]

(1.4.9)

commutes, where $\tilde{f} \circ j(x) = 0$.

Recall that Steenbrink [11] constructs the isomorphism

$$H^*(X_1, \mathbb{C}) \simeq H^*(Y, \Omega_{X/\mathbb{A}^2} \otimes \partial_{\psi} \partial_{\varphi})$$

as a composition of isomorphisms

$$H^*(X_1, \mathbb{C}) \simeq H^*(\tilde{X}^*, \mathbb{C})$$

$$\simeq H^*(Y, i^* k_\star \Omega_{X^*})$$
\begin{equation}
\simeq \mathcal{H}^*(Y, \mathcal{D}^*)
\end{equation}

\begin{equation}
\pi_* \mathcal{H}^*(Y, \Omega^*_{X/\alpha} (\log Y) \otimes \theta_X, \theta_Y).
\end{equation}

In view of this and the commutativity of (1.4.9), we only need to show that the isomorphism

\[ H'(E_\varphi(1), \mathbb{C}) \simeq \mathcal{H}^*(Y, \mathcal{D}^*) \]

constructed via (1.4.3) equals the composition

\[ H'(E_\varphi(1), \mathbb{C}) \cong H'(\hat{X}^*, \mathbb{C}) \simeq \mathcal{H}^*(Y, i^* k_\varphi \Omega^*_X) \cong \mathcal{H}^*(Y, \mathcal{D}^*). \]

This can be proved by resolving \( \mathcal{D}^* \) and \( \Omega^*_X \), by \( C^\infty \) forms and noting that the function \( u \circ \hat{h} \) on \( \hat{X}^* \) pulls back to \( \theta \) along \( \varphi \).

### 1.5 THE LIMIT MHS ON HOMOTOpy

We now combine the results of (1.3), (1.4), [7; §6] and [12; §5] to deduce the existence of a limit MHS on homotopy.

Suppose that \( h: Z \to S \) is a proper map of a smooth variety onto a smooth projective curve and that \( D \) is a divisor in \( Z \). Denote the components of \( D \) by \( D_i \), so that \( D = \bigcup D_i \). Denote the set of critical values of \( h \) by \( \Sigma \). Set \( Y = h^{-1}(\Sigma) \) and \( S^* = S - \Sigma \). Suppose that \( D \cap Y \) is a divisor with normal crossings in \( Z \) and that \( h \) and its restriction to each \( D_i \cap D_j \) is flat over \( S \) and smooth over \( S^* \). Set \( X = Z - D \) and \( X^* = Z - (D \cap Y) \). Denote the restriction of \( h \) to \( X \) by \( f: X \to S \) and its restriction to \( X^* \) by \( g: X^* \to S^* \). Let \( \sigma: S \to Z \) be a section of \( h \) such that \( \sigma(t) \in X_i \) when \( t \in S^* \).

#### (1.5.1) THEOREM. If each of the local monodromy transformations \( H'(X_i; \mathbb{C}) \to H'(X_i; \mathbb{C}) \) about \( \Sigma \) is unipotent, then the local system

\[ \mathcal{G}_i(X^*/S^*, \sigma) = \{ g_i(X_i, \sigma(t)) \}_{i \in S} \]

underlies a good unipotent variation of mixed Hodge structure in the sense of [12; (3.4), (3.13)]. Moreover, if we fix a parameterization of a disk \( T \) centered at a point of \( \Sigma \), then the limit MHS on \( g_i(X_i, \sigma(t)) \) associated to the tangent vector \( t \partial \sigma / \partial t \) at 0 corresponds to the MHS on \( g_i(E_\varphi(t), \sigma(t)) \) under the canonical isomorphism. Finally, the natural map

\[ g_i(X_i, \sigma(t))_{\text{lin}} \to g_i(X_0, \sigma(0)) \]

is a morphism of MHS, provided \( \sigma(0) \in X_0 \).

The analogous statements with \( g_i(X_i, \sigma(t)) \) replaced by \( Z \pi_1(X_i, \sigma(t)) / J^* \) are also true.
References


Correction to Part I: C. Ogle kindly pointed out that the assertion that the free loop space of $X$ is the fiber of the diagonal map $X \rightarrow X \times X$ is false. Nonetheless, Theorem I (6.3.5) remains true. It follows directly from I(3.4.1) and I(6.2.1).