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THE deRHAM HOMOTOPY THEORY

OF COMPLEX ALGEBRAIC VARIETIES

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Author's Note

This is a manuscript — christened "Big Red" by Steve Zucker — that I never published, or even submitted, for reasons I explain below. It was written and (and typed) when I was at the Universities of Washington and Utah and was distributed as a University of Utah preprint in the spring of 1984. Most, but not all, of the results in it have been published. These are mostly in the papers:

[1] *The de Rham homotopy theory of complex algebraic varieties I*, Journal of K-Theory 1 (1987), 271–324.

[2] *On a generalization of Hilbert's 21st problem*, Ann. Sci. École Norm. Sup., t. 19 (1986), 609–627.

[3] *On the indecomposable elements of the bar construction*, Proc. Amer. Math . Soc. 98 (1986), 312–316.

By the time I had finished writing *Big Red* I could prove stronger results. So I broke it apart and rewrote it. The papers [2,3] contain material from Section 12 and the appendix. More general versions of the main results were published in [1], where varieties can be singular, the weight filtration is proved to be defined over $\mathbb{Q}$, and MHSs are constructed on a larger zoo of invariants. The Torelli theorem for curves in section 12 was superseded by Mike Pulte's thesis (Duke Math. J. 57, 1988).[1] The holomorphic variation of the Hodge filtration (section 13) was proved with Steve Zucker (Invent. Math. 88, 1987) by a different argument. That said, the exposition in this paper has some merit. The constructions are less technical that those in [1], which makes the basic ideas clearer.

I am posting *Big Red* for historical reasons and also because I need results from Section 11 (Hodge theory and Chen's method of formal connections) in my current work on elliptic motives.

[1] Pulte was unofficially a student of mine.

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C O N T E N T S

In [12], Deligne defined mixed Hodge structures (M.H.S.'s) and showed that the cohomology ring of every quasi-projective algebraic variety over $\mathbb{C}$ has a natural M.H.S. Morgan [38], using Sullivan's minimal models, showed that the rational homotopy groups (and rational minimal model) of every smooth quasi-projective variety has a natural M.H.S. In this paper we show that the real homotopy groups of every quasi-projective variety have a natural M.H.S. and that the real Lie algebra model of every such variety has a M.H.S.

The approach we take is via Kuo-Tsai Chen's iterated integrals [11]. Using iterated integrals, we put a M.H.S. on the homotopy groups of an algebraic variety directly, without passing through a model. Iterated integrals arise naturally in algebraic geometry and provide a natural link between geometry and homotopy theory, in much the same way as differential forms provide a link between the geometry and the cohomology of a manifold.

Several applications of the machine developed are given. We prove that the M.H.S. on the first two stages of $\pi_1(X,*)$, where X is a smooth curve and $*$ is a point on X, determines X and, generically, the base point $*$. This is a generalization of the classical Torelli theorem. Another application is to a natural generalization of Hilbert's 21st problem. We give several applications to the topology of projective varieties. For example, we prove that the rational homotopy groups (and type) of a projective variety that is a rational homology manifold can be computed formally from its rational cohomology ring. This generalizes a result of Deligne-Griffiths-Morgan-Sullivan [13]. We also use iterated integrals to show that the Gauss-Manin connection on the bundle of homotopy groups associated with an algebraic family of pointed, smooth varieties is regular at infinity, satisfies Griffiths' transversality and that the Hodge filtration varies holomorphically, generalizing results of Deligne [56] and Griffiths [20], respectively.

A detailed and leisurely description of the contents of this paper is contained in the introduction, section 1, as well as a discussion of the applications to algebraic geometry.

Acknowledgments. First and foremost, I would like to thank Alan Durfee who rekindled my interest in the project, encouraged me to complete it and critically read the manuscript. Special thanks are also due Jim Carlson, Herb Clemens, Jerzy Dydak and Jim King for helpful discussions and encouragement.

1. Introduction

The fundamental group $\pi_1(U)$ of the punctured Riemann sphere, $U = \mathbb{P}^1 - \{t_1,\ldots,t_N\}$, is a free group on N generators. The fundamental solution of the system of first order ordinary differential equations on $\mathbb{P}^1$, with regular singular points at the t_j,

$$(1) \qquad z_j' = \sum_{i=1}^{m} \sum_{k=1}^{N} \frac{A_{ij}^k}{t-t_k} z_i \quad , \quad j = 1,\ldots,m \; , \; A_{ij}^k \in \mathbb{C}$$

is a $GL(m,\mathbb{C})$-multivalued function on U. Following it around loops in U defines the monodromy representation

$$\rho\colon \pi_1(U) \to GL(m,\mathbb{C}) \quad .$$

As his 21st problem, Hilbert asked if every linear representation of $\pi_1(U)$ arose in this way. Indeed, every linear representation of $\pi_1(U)$ does arise as a monodromy representation, as Hilbert and several others have shown.

Let $\omega = (\omega_{ij})$ be the $gl(m,\mathbb{C})$-valued 1-form on $\mathbb{P}^1$ defined by

$$\omega_{ij} = \sum_{k=1}^{N} \frac{A_{ij}^k}{t-t_k}\, dt \quad .$$

Equation (1) can now be written in the form

$$(2) \qquad d\underset{\sim}{z} = \underset{\sim}{z}\omega \quad ,$$

where $\underset{\sim}{z} = (z_1,\ldots,z_m)$ is a vector valued function on $\mathbb{P}^1$. Because $\omega\wedge\omega = d\omega = 0$, ω defines a flat holomorphic connection on the trivial vector bundle $\mathbb{C}^m \times U \to U$ with regular singular points at the t_j.

A natural generalization of Hilbert's 21st problem is the following: Suppose that U is a smooth quasi-projective algebraic variety and that X is a smooth projective completion of U. A $gl(m,\mathbb{C})$-valued 1-form ω

on U defines a connection on $\mathbb{C}^m \times U \to U$: A section corresponds to a function $f: U \to \mathbb{C}^n$. Define $\nabla f = df - f\omega$. This connection is flat if and only if ω is completely integrable. That is,

$$d\omega + \omega \wedge \omega = 0 .$$

A (possibly multivalued) section of $\mathbb{C}^m \times U \to U$ corresponds to a function $f\colon U \to \mathbb{C}^m$. The section is horizontal if and only if

$$df = f\omega . \tag{3}$$

If ω is completely integrable, then one obtains the monodromy (i.e., holonomy) representation

$$\rho\colon \pi_1(U) \to GL(m,\mathbb{C}) . \tag{4}$$

Generalized Hilbert's 21st problem: Characterize the linear representations (4) of $\pi_1(U)$ that arise as the monodromy representation of a flat holomorphic connection on $\mathbb{C}^m \times U \to U$ which has regular singularities along $X-U$ (in the sense of Deligne [56].)

A different generalization of Hilbert's problem has been considered by Deligne [56] (c.f. [31]).

Even in the case of curves, there are non trivial restrictions on the representations of $\pi_1(U)$ that arise in this way. For example, consider the case where $X = U = \mathbb{C}/\Gamma$, an elliptic curve, where $\Gamma = \mathbb{Z} \oplus \mathbb{Z}\tau$ and $\operatorname{Im}\tau > 0$. Denote the loop in U that corresponds to the cut from 0 to 1 in $\mathbb{C}$ by α and the loop in U that corresponds to the cut from 0 to τ by β. Every holomorphic $gl(m,\mathbb{C})$-valued 1-form on U is of the form

$$\omega = A\,dt \quad ,$$

where $A \in gl(m,\mathbb{C})$. All such forms are completely integrable. The associated monodromy representation takes α to $\exp A$ and β to $\exp(\tau A)$. Consequently, the monodromy representation ρ must satisfy

$$\rho(\alpha)^{\tau} = \rho(\beta) \quad .$$

Therefore, only those representations of $\pi_1(U)$ that factor through the representation

$$\theta\colon \pi_1(U) \to \mathbb{C} \quad ,$$

that takes α to 1 and β to τ , arise as the monodromy representation of a completely integrable holomorphic 1-form on U and conversely.

This example suggests that a solution to the generalization of Hilbert's problem may involve the Hodge filtration and Deligne's mixed Hodge structures [12] on $\pi_1(U)$. (Such a mixed Hodge theory has been developed by Morgan [38].)

Another critical observation is that the monodromy representation $\rho\colon \pi_1(U) \to GL(m,\mathbb{C})$ associated with the completely integrable $gl(m,\mathbb{C})$ valued 1-form ω can be computed as follows: If we solve the equation

$$dT = T\omega \quad , \quad T(0) = I \quad , \quad T(t) \in GL(m,\mathbb{C}) \quad ,$$

by Picard iteration along a curve $\alpha(t)$, we find that

$$\rho(\alpha) = I + \int_\alpha \omega + \int_\alpha \omega\omega + \int_\alpha \omega\omega\omega + \ldots \quad , \tag{5}$$

where $\int \omega\ldots\omega$ is an iterated integral as defined by Chen [11]. This observation suggests, at least in the case when ω is nilpotent (so that the series (5) is finite), that iterated integrals should also play a role in the solution of the generalization of Hilbert's problem, providing a link between the geometry of the connection and the homotopy theory.

Let V be a quasi-projective algebraic variety (not necessarily smooth) and $*$ a distinguished point on V. The group ring $\mathbb{C}\pi_1(V,*)$ has a natural augmentation

$$\mathbb{C}\pi_1(V,*) \to \mathbb{C}$$

defined by taking each element of the group to 1. Its kernel, the augmentation ideal, will be denoted by J or $J(V,*)$ and its powers by J^s. In this paper we prove that for each $s \geq 1$, that $\mathbb{C}\pi_1(V,*)/J^s$ has a mixed Hodge structure (M.H.S.) that is natural in the pair $(V,*)$. When V is simply connected we show that the higher homotopy groups $\pi_*(V) \otimes \mathbb{C}$ have a natural M.H.S. that does not depend on the choice of base point. We show that these M.H.S.'s can be obtained directly and naturally using K.-T. Chen's iterated integrals. For a general variety V we could only show that the weight filtration is defined over $\mathbb{R}$. However, for many varieties (e.g., smooth varieties and divisors with normal crossings), the weight filtration is defined over $\mathbb{Q}$.

This work generalizes Deligne's work [12] on the one hand, and Morgan's work [38] on the other. Deligne defined M.H.S.'s and showed that the cohomology ring of every quasi-projective variety has a natural M.H.S. Morgan, using an approach via Sullivan's minimal models [51], then showed that the rational homotopy of a smooth algebraic variety has a natural M.H.S. Actually Morgan proved more. He showed that the complex minimal model of a smooth variety has a natural M.H.S. We also show that the complex Lie algebra model of every quasi-projective variety has a M.H.S. The Lie algebra model (the algebraic cell decomposition of V) contains the same information as Sullivan's model (the algebraic Postnikov tower) (c.f. [23]). But in the case of algebraic varieties, the Lie algebra model is considerably simpler than Sullivan's model, as algebraic varieties tend to have a relatively simple cell structure and often a complicated Postnikov tower.

Armed with this machinery, we return to our generalization of Hilbert's 21st problem and obtain the following result.

Theorem 12.1. If V is a smooth algebraic variety and $m \geq 1$, then there exists a simply connected, nilpotent, complex Lie group $\mathcal{H}_m$ and a representation

$$\theta_m: \pi_1(U) \to \mathcal{H}_m$$

with the property:

If $\rho: \pi_1(V) \to GL(m+1,\mathbb{C})$ is a unipotent representation (i.e. $(\rho(g)-I)^m = 0$ for all g), then ρ is the monodromy representation of a nilpotent, completely integrable, holomorphic 1-form on V that defines a regular connection on $\mathbb{C}^{m+1} \times V \to V$ if and only if ρ factors through θ_m.

In the example with V an elliptic curve, $\mathcal{H}_m = \mathbb{C}$ for all m and

$$\theta_m(\gamma) = \int_{\gamma} dt \quad .$$

Suppose now that X is a smooth, projective algebraic curve and that $x \in X$. The M.H.S. on $\mathbb{C}\pi_1(X,x)/J^s$ depends non trivially on the base point x. In modern parlance, the classical Torelli theorem says that the polarized Hodge structure on $H^1(X;\mathbb{C})$ determines X up to isomorphism. Thus a natural question is to what extent does the M.H.S. on $\mathbb{C}\pi_1(X,x)/J^s$ determine the pair (X,x) ? The following theorem provides strong evidence that (X,x) is completely determined by this M.H.S.

Theorem 12.13. The M.H.S. on $\mathbb{C}\pi_1(X,x)/J^3$ determines X and determines x up to a finite number of possibilities. For generic X[1] and generic $x \in X$, the point x is uniquely determined.[2]

This "pointed Torelli theorem" suggests that it might be interesting to study the period mapping

$$\{\text{pointed varieties}\} \to \{\text{M.H.S.'s}\} .$$

In chapter 13 we prove that this map is holomorphic and that the Gauss-Manin connection on the bundle of homotopy groups, associated with an algebraic family of smooth, projective varieties, has regular singularities at infinity and satisfies Griffiths' transversality.

In addition to these geometric applications of our machine, we obtain the following applications to the topology of projective varieties. Recall from [13] that a topological space is formal if its rational homotopy type is a formal consequence of its cohomology ring.

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[1] Here generic means that there is no finite morphism $f: X \to Y$ of degree ≥ 2.

[2] M. Pulte has recently proved the Torelli theorem for all curves X and all but at most two points $x \in X$.

Theorem 12.12. If V is a projective algebraic variety whose rational cohomology ring satisfies Poincaré duality (e.g. smooth varieties and varieties, all of whose singularities are of the form $x_0^3 + x_1^2 + \ldots + x_n^2 = 0$), then V is formal.

This theorem generalizes the result of Deligne-Griffiths-Morgan-Sullivan [13], which asserts that all smooth projective varieties are formal.

The following result shows that there are more restrictions on the homotopy type of a projective variety than those imposed by its cohomology ring.

Theorem 12.10. There exists a simply connected, finite, CW-complex whose integral cohomology ring is isomorphic to that of a projective variety, but which does not have the same (rational) homotopy type as any projective variety.

What were recently called iterated integrals by Kuo-Tsai Chen probably first appeared in algebraic geometry in the work of Lappo-Danilevsky [35] on finding an explicit solution to Hilbert's 21st problem in terms of higher logarithms. Since then, Aomoto [1] and Golubeva [19] have both followed Lappo-Danilevsky and used Chen's methods in their work on the generalization of Hilbert's problem for Zariski open subsets of $\mathbb{P}^n$. Aomoto [2,3,4] has also used the Gauss-Manin connection on holomorphic iterated integrals to study generalized higher logarithms, while Ramakrishnan's result [46] on the higher logarithms can be interpreted in terms of, and proved using, iterated integrals.

Gunning's quadratic abelian integrals [22] are twice iterated integrals of abelian differentials and, as such, have been studied by B. Harris [26,27] and Jablow [30]. Their results are related to our "pointed Torelli theorem."

Hwang-Ma [28] has studied the monodromy maps

$$\theta_m : \pi_1(X) \to H_m$$

of theorem 12.1, where X is a smooth, projective curve, and has discovered that certain arithmetic properties of the curve are reflected in the θ_m. In the case of curves, the representations θ_m were first defined by Parsin [42].

Chen [8] was the first to observe that the existence of a Hodge structure on the cohomology of a compact Kaehler manifold imposed conditions on its fundamental group. This observation was generalized dramatically by Deligne-Griffiths-Morgan-Sullivan [13], who proved that the rational homotopy type of a compact Kaehler manifold can be formally computed from its cohomology ring. Neisendorfer [40] then gave a very explicit description of the rational homotopy types of all smooth complete intersections.

As was noted earlier, Morgan [38] showed that the minimal model of a smooth algebraic variety has a M.H.S. One consequence of this, obtained by Morgan, is that the rational homotopy type of a smooth algebraic variety has negative weights, a non trivial restriction on their homotopy types. Kohno [32,33,34] has used the M.H.S. on the minimal model of the complement of a hypersurface in $\mathbb{P}^n$ in his work on the Alexander module of a plane algebraic curve.

Carlson-Clemens-Morgan [7] initiated the study of the geometry of the M.H.S. on π_3 of a simply connected projective manifold. They gave an example of two simply connected 3-folds with the isomorphic Hodge structures on their cohomology but different M.H.S.'s on π_3, thus showing that the M.H.S. on homotopy is more sensitive to moduli that the M.H.S. on cohomology.

Another paper of interest is that of Neisendorfer and Taylor [41] where the de Rham homotopy analogue of the Dolbeault cohomology groups of a complex manifold is developed.

In broad terms, this paper is in four parts. Part I comprises sections 2 through 5. The main result of this part is that to every quasi-projective variety, we can associate a sub d.g. algebra of its de Rham complex that is a mixed Hodge complex (M.H.C.). Of course, we have to explain what is meant by the de Rham complex of a singular variety. Following Deligne [12], we replace the variety V, if singular, by a hypercovering $\varepsilon\colon X_\bullet \to V$. The hypercovering has a nice de Rham complex. Section 2 is a review of hypercoverings. It is well known that $X_\bullet$ and V have the same cohomology. Since we are concerned with homotopy, we have to show that $X_\bullet$ and V have the same homotopy type. This is done in section 3. In section 4 we prove a suitable version of the de Rham theorem for hypercoverings and in section 5 we describe the "de Rham M.H.C." that computes the cohomology of $X_\bullet$, and hence V. If one is only interested in putting a M.H.S. on the real homotopy type of V, one could, at this point, plug into Morgan's machine [38] to put a M.H.S. on the minimal model of V.

In part II (sections 6 through 9), we show that the M.H.S. on the homotopy of a variety may be obained directly and naturally using iterated integrals. Iterated integrals on V compute the cohomology of the loop space of V. In section 6 we review the relationship between the cohomology of the loop space and the homotopy groups of V. Section 7 contains technical results that allow us to show that homotopy and smooth homotopy agree for hypercoverings, facts we need to know when applying Chen's loop space de Rham theorems. These are reviewed in section 8. In section 9 we prove that the bar construction on a M.H.C. is again a M.H.C. and conclude from this that the cohomology of the complex of iterated integrals on a variety has a M.H.S. compatible with its Hopf algebra structure. Applying the results in sections 6 and 8, we show that $\mathbb{C}\pi_1(V)/J^s$ and, when V is simply connected, $\pi_*(V) \otimes \mathbb{C}$, have natural M.H.S.'s.

We prove that the Lie algebra model of V has a M.H.S. in part III, sections 10 and 11. Section 10 is a review of Chen's method of formal connections, the means by which we construct the Lie algebra model. We prove that the Lie algebra model of V has a M.H.S. in section 11.

The geometric applications of the machine developed in parts I, II and III are given in part IV, sections 12 and 13. In section 12 we give applications to the generalization of Hilbert's 21st problem, the pointed Torelli theorem and applications to the rational homotopy theory of projective varieties. In section 13, we study the Gauss-Manin connection associated to an algebraic family of smooth projective varieties. We show that it is regular at infinity and that the Hodge filtration varies holomorphically. We also show that the connection ∇ satisfies Griffiths transversality:

$$\nabla: \mathcal{O}(F^p) \to \mathcal{O}(F^{p-1}) \otimes \Omega^1 .$$

There is also an appendix. In it we prove a dual version of the Poincaré-Birkhoff-Witt theorem for the bar construction on a commutative d.g.a. This technical result is used in section 11.

For topologists, we include a brief description of the algebro-topological significance of iterated integrals. Suppose, for simplicity, that K is a simply connected finite semi simplicial complex with a unique vertex and trivial 1-skeleton. (That is, K is a simplicial set with only a finite number of non degenerate simplices, none of which are 1-dimensional.) The complex of real simplicial cochains, $S^\bullet(K) = S^\bullet(K;\mathbb{R})$, is then a d.g. algebra of finite type. Denote the de Rham complex of K by $\mathcal{E}^\bullet(K)$. Choose the vertex $*$ for a base point. The inclusion of $*$ into K induces augmentations

$$S^\bullet(K) \to \mathbb{R} \quad , \quad \mathcal{E}^\bullet(K) \to \mathbb{R} .$$

Integration induces an augmentation preserving chain map

$$\text{int}: \mathcal{E}^\bullet(K) \to S^\bullet(K) .$$

Except in a few trivial cases, int is <u>not</u> an algebra homomorphism, so that the map

$$\mathcal{B}(\text{int}): \mathcal{B}(\mathcal{E}^\bullet(K)) \to \mathcal{B}(S^\bullet(K)) \quad ,$$

induced on their bar constructions (c.f. section 8), is not a chain map. Algebraic topologists get around this by perturbing $\mathcal{B}(\text{int})$ so that it is a chain map. <u>Iterated integrals do this in a canonical way</u>: Here $\mathcal{B}(S^\bullet(K))$ is the dual of Adam's cobar construction on $S_\bullet(K)$, the real simplicial chains. The map

$$\begin{aligned} \Theta: \mathcal{B}(\mathcal{E}^\bullet(K)) &\to \mathcal{B}(S^\bullet(K)) \\ [w_1|\dots|w_r] &\to \{c \to \langle \textstyle\int w_1 \dots w_r \, , c \rangle\} \quad , \end{aligned}$$

where c is an element of Adam's cobar construction on $S_{\bullet}(K)$ and $\langle\ ,\ \rangle$ denotes the integration pairing, is a d.g. coalgebra map that induces an isomorphism on cohomology. This is the best that we can hope for, as $\mathcal{B}(S^{\bullet}(K))$ is a d.g. coalgebra but not a d.g. Hopf algebra. However,

$$H^{*}(\mathcal{B}(S^{\bullet}(K))) = H^{*}(\Omega_{*}K;\mathbb{R})$$

by Adam's theorem (c.f. [11]), where $\Omega_{*}K$ denotes the space of loops in K based at $*$, so that $H^{*}(\mathcal{B}(S^{\bullet}(K)))$ is a Hopf algebra.

Since $E^{\bullet}(K)$ is commutative, $\mathcal{B}(E^{\bullet}(K))$ is a d.g. Hopf algebra. The second wonderful topological property of iterated integrals is that

$$\Theta_{*}:\ H^{*}(\mathcal{B}(E^{\bullet}(K))) \to H^{*}(\Omega_{*}K;\mathbb{R})$$

is a graded Hopf algebra isomorphism [11,24].

2. Hypercoverings and their Cohomology

Recall that the *simplicial category* Δ is the category whose objects are the finite ordinals

$$[n] = \{0, 1, \ldots, n\}, \qquad n \in \mathbb{N}$$

and whose maps are order preserving functions $f : [m] \to [n]$. The *degeneracy map*

$$s_j : [n] \to [n-1] \qquad 0 \leq j < n$$

is the unique surjective, order preserving function with $s_j(j) = s_j(j+1) = j$. The *face map*

$$d_j : [n] \to [n+1] \qquad 0 \leq j \leq n+1$$

is the unique order preserving, injective function which omits the value j. Every map in Δ, other than the identity maps $[n] \to [n]$, is the composite of face maps and degeneracy maps.

A *simplicial object* $K_\bullet$ in a category C is a contravariant functor $K_\bullet : \Delta \to C$. The set K_n of n-simplices of $K_\bullet$ is the object $K_\bullet[n]$ of C. A map $K_\bullet \to L_\bullet$ between simplicial objects in C is a natural transformation. The category of simplicial objects in C will be dentoed by C^Δ. A simplicial object in *Set*, the category of sets, is called a *simplicial set*, while a simplicial object in *Top*, the category of topological spaces, will be called a *simplicial space*. Of special interest to us is the category *Alg*, whose objects are quasi-projective varieties over $\mathbb{C}$ (i.e. Zariski open subsets of complex projective varieties) and whose morphisms are regular maps between varieties. A simplicial object in Alg will be called a simplical variety.

We shall denote by $\Delta(k)$ the full subcategory of Δ that contains the objects [0], [1], ..., [k]. A contravariant functor $\Delta(k) \to C$ is called a k-truncated simplicial object in C. The category of k-truncated simplicial objects in C will be dentoed by $C^{\Delta(k)}$. The inclusion functor $\Delta(k) \to \Delta$ induces a covariant functor

$$r_k : C^{\Delta} \to C^{\Delta(k)}.$$

If C is closed under finite colimits, then the restriction functor, r_k, has a left adjoint

$$sk_k : C^{\Delta(k)} \to C^{\Delta}$$

that is called the k-skeleton functor. That is, there is a natural, one to one correspondence between maps $sk_k\ X \to Y$ and maps $X \to r_k\ Y$, where $X \in C^{\Delta(k)}$ and $Y \in C^{\Delta}$. If C is closed under finite limits, then r_k has a right adjoint

$$cosk_k : C^{\Delta(k)} \to C^{\Delta}$$

which is called the k-coskeleton functor. That is, there is a natural, one to one correspondence between maps $r_k X \to Y$ and $X \to cosk_k Y$, where $X \in C^{\Delta}$ and $Y \in C^{\Delta(k)}$. For a discussion of the of the skeleton and coskeleton functor, see Artin and Mazur [5], p. 7.

Especially important to us are the split simplicial objects in a subcategory C of Set, which are the simplicial objects whose degeneracies contain no essential geometric information. For $n, m \in \mathbb{N}$, let $S(n,m)$ denote the set of degeneracies $[n] \to [m]$: i.e., the set of surjective maps $[n] \to [m]$. For a simplicial object $X_\bullet$ in C, we denote by $N(X_n)$ the non-degenerate part of X_n, which is defined by

$$N(X_n) = X_n - \bigcup_{s \in S(n,n-1)} s(X_{n-1}).$$

For all $n \geq 0$, the natural map

$$\sigma_n : \coprod_{m \leq n} \coprod_{s \in S(n,m)} N(X_m) \to X_n$$

is a bijection. The simplicial object $X_\bullet$ is said to be <u>split</u> if σ_n is an isomorphism in C for all n.

Standard references for simplicial objects are Artin and Mazur [5] and Gabriel and Zisman [17].

<u>Hypercoverings</u>. To compute the cohomology of a topological space using Cech cohomology, one covers it by nice open sets and then computes the cohomology of the nerve of the covering. From the point of view of Hodge theory, the correct way to compute the cohomology of a projective algebraic variety V is to "cover" it by smooth projective varieties and then compute the cohomology of the "cover". Of course, in general, we cannot cover V by smooth projective varieties in the usual sense. The appropriate generalization of coverings that enable us to compute the cohomology of V in this way is called a <u>hypercovering</u>. Roughly speaking, these are simplical varieties, all of whose simplices are smooth, and that are "homotopy equivalent" to V.

We now recall the definition of a hypercovering. Standard references for hypercoverings are Artin and Mazur [5], B. Saint-Donat [47], Deligne [12] and Friedlander [16].

A map $f : X \to Y$ in Alg will be called a covering if it is a proper surjective map and if X is smooth. A map $f : X \to Y$ in Top will be called a covering if it is surjective.

For an object V in a category C, denote by C_V the category whose objects are maps $Y \to V$ in C and whose maps $(Z \to V) \to (Y \to V)$ are commutative triangles

$$\begin{tikzcd}[column sep=small] Z \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & V & \end{tikzcd}.$$

The category $C^{\Delta(0)}$ is naturally isomorphic to C. Thus we can associate to each object V of C the simplicial object $\operatorname{cosk}_0 V$ which is characterized by:

(a) $(\operatorname{cosk}_0 V)_n = V$ all $n \geq 0$

(b) every face and degeneracy map is the identity.

Every simplicial object of C_V can then be regarded as a simplicial object $X_\bullet$ in C together with a map $\varepsilon : X_\bullet \to \operatorname{cosk}_0 V$ which we shall call the augmentation. There is an obvious inclusion of C_V into C. Now suppose that there is a notion of covering map in C. A map in C_V is called a covering if its image in C is a covering.

A hypercovering of an object V of C (a category in which there is a notion of a covering) is a simplicial object $Y_\bullet$ in C_V that satisfies:

(1) The natural map $\varepsilon : Y_0 \to V$ is a covering.

(2) For each $n \geq 0$, the natural map

$$Y_{n+1} \to (\operatorname{cosk}_n r_n Y_\bullet)_{n+1}$$

is a covering.

We call $Y_\bullet$ a <u>split hypercovering</u> if, in addition, it is a split simplicial object in $\mathcal{C}_V$. (For enlightenment, see Artin and Mazur [5], p. 96.)

<u>Theorem</u> 2.1 (Deligne [12], Saint-Donat [47]). (1) Every quasi-projective variety V has a split hypercovering $X_\bullet$ in <u>Alg</u>. Moreover, if V is not projective, then we can choose $X_\bullet$ such that there is a smooth projective simplicial variety $\overline{X}_\bullet$ in <u>Alg</u> and a map

$$\begin{array}{ccc} X_\bullet & \hookrightarrow & \overline{X}_\bullet \\ \downarrow & & \\ S & & \end{array}$$

such that $\overline{X}_\bullet - X_\bullet$ is a divisor in $\overline{X}_\bullet$ with normal crossings.

(2) If V is a quasi-projective variety and $X_\bullet$ and $Y_\bullet$ are two hypercoverings of V in <u>Alg</u>, then there is a third hypercovering $Z_\bullet$ of V and maps $Z_\bullet \to X_\bullet$ and $Z_\bullet \to Y_\bullet$. If V is not projective, we may assume that these maps extend to maps $\overline{Z}_\bullet \to \overline{X}_\bullet$ and $\overline{Z}_\bullet \to \overline{Y}_\bullet$. If $X_\bullet$ and $Y_\bullet$ are split, we may assume that $Z_\bullet$ is also split.

(3) If $V \to W$ is a regular map between quasi projective varieties, then we can find split hypercoverings of $X_\bullet$ of V and $Y_\bullet$ of W and a map $X_\bullet \to Y_\bullet$ which can, if necessary, be assumed to extend to a map $\overline{X}_\bullet \to \overline{Y}_\bullet$. $\square$

Geometric Realization. View the standard n-simplex Δ^n as the convex hull of the standard basis vectors $e_0, e_1, \ldots, e_n$ of $\mathbb{R}^{n+1}$. Each map $f : [n] \to [m]$ induces an affine map $|f| : \Delta^n \to \Delta^m$ by taking e_j to $e_{f(j)}$. Suppose that $X_\bullet$ is a simplicial space. Let $\sim$ denote the equivalence relation on $\coprod_{n\geq 0} X_n \times \Delta^n$ generated by

$$(x, |f|(\xi)) \sim (f^*(x), \xi),$$

where $|f|: \Delta^n \to \Delta^m$ and $f^* : X_m \to X_n$ are the maps induced by $f : [n] \to [m]$. The geometric realization $|X_\bullet|$ of X is the quotient space

$$\coprod_{n\geq 0} X_n \times \Delta^n/\sim$$

Suppose that V is a topological space and that $X_\bullet$ is a simplicial object in $\underline{\underline{\mathrm{Top}}}_V$. Define the geometric realization $|X_\bullet|$ of $X_\bullet$ to be the geometric realization of the simplicial space obtained by composing $X_\bullet$ with the inclusion functor $\underline{\underline{\mathrm{Top}}}_V \to \underline{\underline{\mathrm{Top}}}$. The augmentation $\varepsilon : X_\bullet \to \mathrm{cosk}_0 V$ induces a continuous map

$$|\varepsilon| : |X_\bullet| \to V,$$

as it is easily seen that $|\mathrm{cosk}_0 V| = V$.

Examples. (a) Suppose that $V = \bigcup_{\alpha\in A} V_\alpha$ is a variety which is the union of smooth varieties with normal crossings. If we set

$$X_0 = \coprod_{\alpha\in A} V_\alpha \to V,$$

then $X_\bullet = \mathrm{cosk}_0\, X_0$ is a hypercovering of V. In fact, it is not hard to see that

$$X_n = X_0 \times_V X_0 \times_V \cdots \times_V X_0 = \coprod_{A^{n+1}} V_{\alpha_0} \cap V_{\alpha_1} \cap \cdots \cap V_{\alpha_n} .$$

The face map $d_j : X_n \to X_{n-1}$ is induced by the natural inclusions

$$V_{\alpha_0} \cap \ldots \cap V_{\alpha_n} \to V_{\alpha_0} \cap \ldots \cap \hat{V}_{\alpha_j} \cap \ldots \cap V_{\alpha_n} ,$$

and the degeneracy $s_j : X_n \to X_{n+1}$ is induced by the natural isomorphisms

$$V_{\alpha_0} \cap \ldots \cap V_{\alpha_n} \to V_{\alpha_0} \cap \ldots \cap V_{\alpha_{j-1}} \cap V_{\alpha_j} \cap V_{\alpha_j} \cap \ldots \cap V_{\alpha_n} .$$

It follows that $X_\bullet$ is a split hypercovering.

To understand the geometric realization of $X_\bullet$, we consider the special case where V is the union of three transversally intersecting hyperplanes in $\mathbb{C}^3$ which we represent schematically in figure 2.1(a).

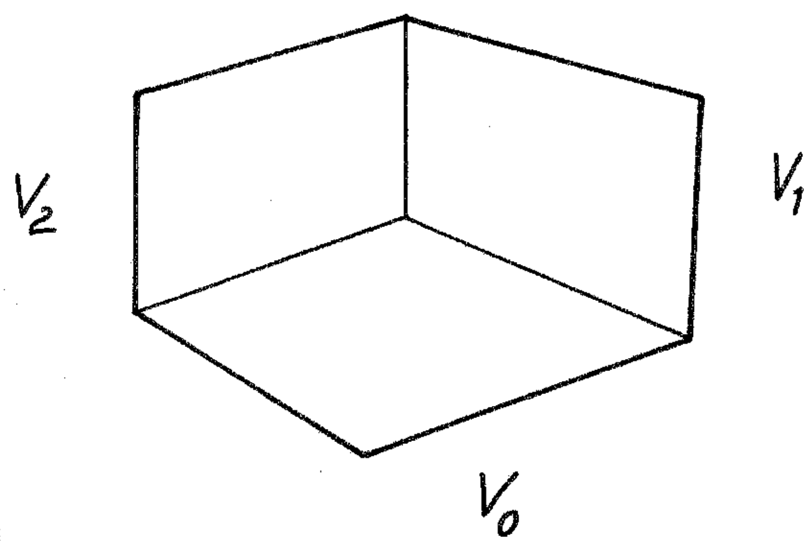

Figure 2.1(a).

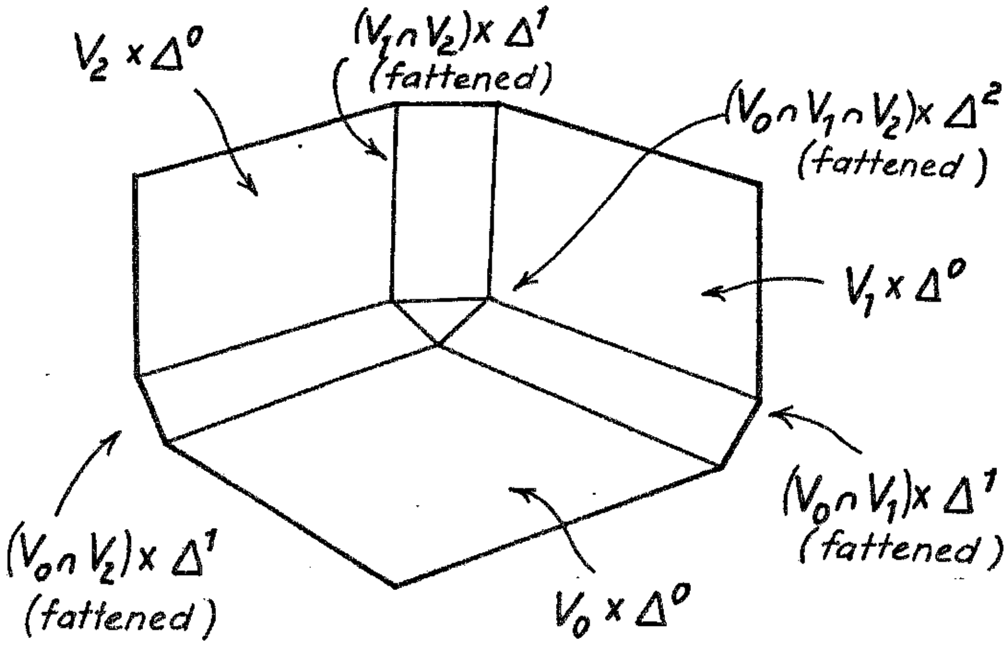

Figure 2.1(b).

The geometric realization of $X_\circ$ is illustrated in figure 2.1(b).

(b) A split hypercovering of the reducible curve V illustrated in figure 2.2(a) is $\mathrm{cosk}_0(V' \coprod V'' \to V)$, where $V' \coprod V''$ is the normalization of V. The first two stages of this hypercovering are illustrated in figure 2.2(b) and its geometric realization in figure 2.2(c).

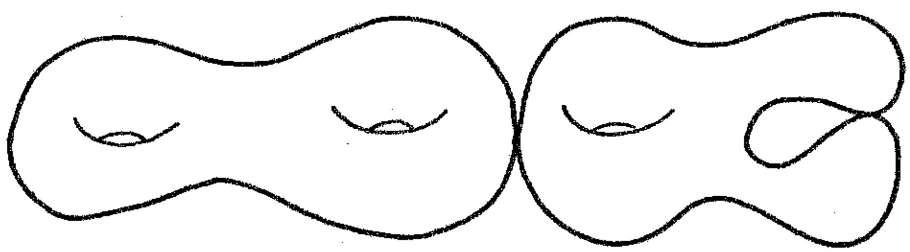

Figure 2.2(a).

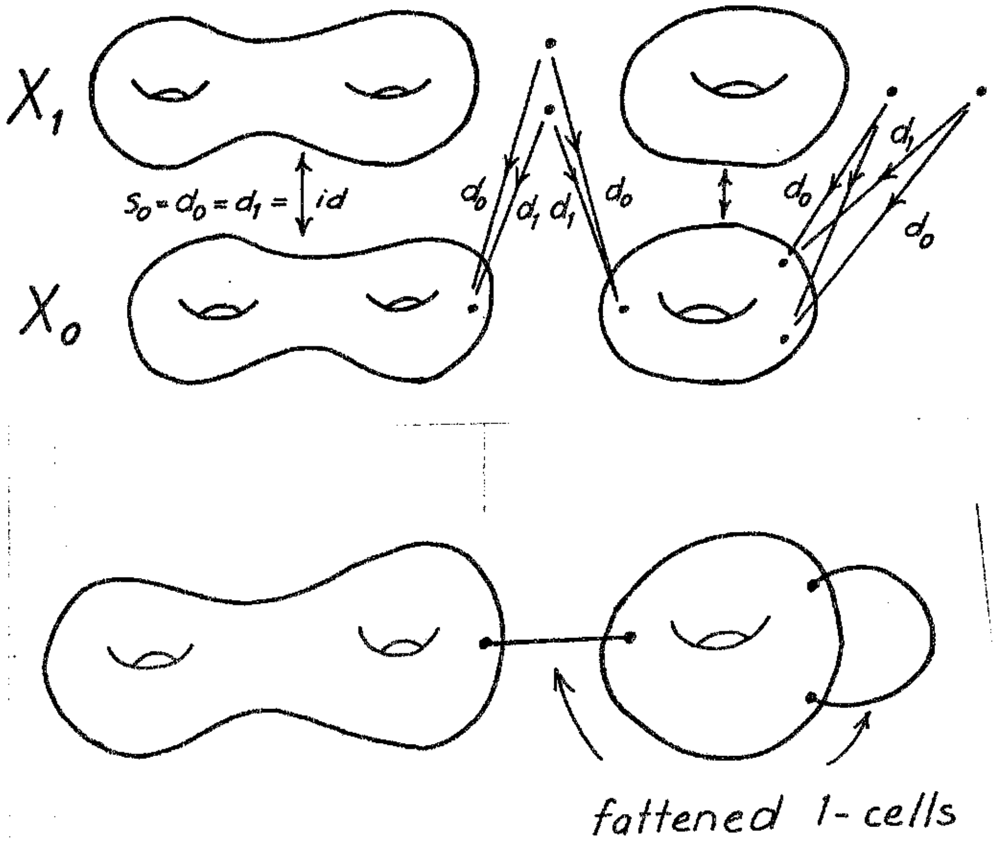

Figure 2.2(c).

<u>Cohomology</u>. Suppose that $X_\bullet$ is a simplicial object in the category $\mathcal{C}$. A contravariant functor $A : \mathcal{C} \to \underline{\mathrm{Ab}}$, into the category of abelian groups, associates to $X_\bullet$ a cochain complex $(A^\bullet, \delta)$ as follows: $A^p = A(X_p)$ and $\delta : A^p \to A^{p+1}$ is defined by

$$\delta = \sum_{i=0}^{p} (-1)^i \, d_j^*,$$

where $d_i : X_{p+1} \to X_p$ denotes the i-th face map. We shall call the δ the <u>combinatorial differential</u> associated to A.

More generally, given a contravariant functor $K^\bullet: C \to \underline{\underline{Ab}}^\bullet$ into the category of cochain complexes of abelian groups, one can associate to a simplicaial object $X_\bullet$ in C the double complex $(K^{\bullet\bullet},\delta,d)$, where $K^{p,q}=K^q(X_p)$, where $\delta : K^{p,q} \to K^{p+1,q}$ is the combinatorial differential associated to $K^q : C \to \underline{\underline{Ab}}$, and where $d : K^{p,q} \to K^{p,q+1}$ is the internal differential $(-1)^p K(d)$. The skeleton filtration of $K^{\bullet\bullet}$ is defined by

$$G^s K^{\bullet\bullet} = \bigoplus_{t \geq s} K^{t\bullet}.$$

The E_1 term of the associated spectral sequence is

$$E_1^{s,t} = H^t(K^\bullet(X_s))$$

and $d_1 : E_1^{s,t} \to E_1^{s+1,t}$ is the associated combinatorial differential.

For example, if $S^\bullet : \underline{Top} \to \underline{Ab}^\bullet$ is the singular cochain complex functor, then $S^\bullet$ associates to each simplicial space $X_\bullet$ a double complex $S^{\bullet\bullet}$. If $X_\bullet$ is split, then there is a natural isomorphism

$$H^*(|X_\bullet|; \mathbb{Z}) \approx H^*(S^{\bullet\bullet}, \delta + d).$$

as may be seen using the above spectral sequence and Mayer-Vietoris (c.f. [48].).

A cosimplicial object $F^\bullet$ in a category C is a covariant functor from the simplicial category Δ into C. A sheaf $F^\bullet$ over the simplicial space $X_\bullet$ is a cosimplicial object in the category of sheaves such that F^n is a sheaf over X_n. Such a sheaf $F^\bullet$ yields a double complex $(C^{\bullet\bullet}, \delta, d)$, where $0 \to F^p \to C^{p,0} \xrightarrow{d} C^{p,1} \xrightarrow{d} C^{p,2} \to \ldots$ is the canonical flabby resolution of

F^p (c.f. [18], p. 167) and where $\delta : C^{p,q} \to C^{p+1,q}$ is the combinatorial differential defined by the face maps of $F^\bullet$. The cohomology of $X_\bullet$ with coefficients in $F^\bullet$ is defined to be $H^*(C^{\bullet\bullet}, \delta+d)$.

Suppose that V is a topological space and that F is a sheaf over V. If $X_\bullet$ is a simplicial object in $\underline{\mathrm{Top}_V}$, then F can be pulled back to a sheaf ε^*F over X along the augmentation $\varepsilon : X_\bullet \to \operatorname{cosk}_0 V$. A basic result from the theory of cohomological descent ([47], [12] Section 5.3.1) is the following.

Theorem 2.2 (Deligne [12], B. Saint-Donat [47]). Suppose that V is an algebraic variety and that F is a sheaf over V. If $\varepsilon : X_\bullet \to V$ is a hypercovering of V, then the natural map

$$H^*(V,F) \to H^*(X_\bullet, \varepsilon^*F)$$

is an isomorphism. □

3. The Homotopy Type of a Hypercovering

To compute the mixed Hodge structure on the homotopy of an algebraic variety V, we need a commutative differential graded algebra that computes both the de Rham homotopy type of V and the Hodge theory of V. As we shall see in Section 5, the de Rham complex of a split hypercovering $X_\bullet$ of V will do the job. Our first task is to show that $|X_\bullet|$ and V have the same homotopy type.

Theorem 3.1. If $X_\bullet$ is a hypercovering in Alg of the connected algebraic variety V, then the natural map

$$|\varepsilon| : |X_\bullet| \to V$$

is a homotopy equivalence.[1]

Proof. It suffices to prove that $|X_\bullet|$ is connected and that the induced map $\pi_1(|X_\bullet|) \to \pi_1(V)$ is an isomorphism. For once we have done this, it follows from obstruction theory and the fact that

$$H^*(V, F) \to H^*(|X_\bullet|, \varepsilon^* F)$$

is an isomorphism, for all locally constant sheaves of abelian groups F on V (c.f. 2.2, see also Theorem (4.3) and Corollary (10.8) [5]), that $|\varepsilon|$ is a homotopy equivalence.

Choose a base point $*$ of V and a base point $*' \in X_0$ of $|X_\bullet|$ such that $\varepsilon(*') = *$.

We first show that $\pi_1(|X_\bullet|, *') \to \pi_1(V,*)$ is surjective. Let

$$X_0 = Y_1 \coprod Y_2 \coprod \cdots \coprod Y_k$$

[1] According to Deligne, a shorter proof of 3.1 can be given using the theory of descent for non-abelian cohomology. However, in section 7 we need this proof to show that $\pi_1(|X_\bullet|)$ can be computed as smooth loops modulo smooth homotopy.

be the decomposition of X_0 into connected components. Since ε is proper, the image $\varepsilon(Y_j)$ of each Y_j is a subvariety of V. By Hironaka [29] there is a semi-algebraic triangulation of V such that the $\varepsilon(Y_j)$'s and $*$ are subcomplexes. Each element of $\pi_1(V,*)$ may be represented by an edge path γ. Since ε is surjective and γ is a piecewise real algebraic curve, each of whose smooth arcs lies in some $\varepsilon(Y_j)$, we can find a finite number of piecewise analytic paths $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_m$ in X_0 and a function $\beta : \{1, \ldots, m\} \to \{1, \ldots, k\}$ satisfying

(i) $\tilde{\gamma}_j \subseteq Y_{\beta(j)} \qquad j = 1, \ldots, m,$

(ii) $\tilde{\gamma}_1(0) = *'$,

(iii) $\gamma = (\varepsilon\circ\tilde{\gamma}_1)\cdot(\varepsilon\circ\tilde{\gamma}_2)\cdot\ldots\cdot(\varepsilon\circ\tilde{\gamma}_m)$.

Because $(\mathrm{cosk}_0\, X_0)_1 = X_0 \times_V X_0$ and since $X_1 \to X_0 \times_V X_0$ is surjective, we can find points $p_1, \ldots, p_m \in X_1$ such that

(i) $d_0(p_j) = \tilde{\gamma}_j(1),\ d_1(p_j) = \tilde{\gamma}_{j+1}(0),\ 1 \leq j < m,$

(ii) $d_0(p_m) = \tilde{\gamma}_m(1),\ d_1(p_m) = *'$.

If we let μ_j be the path in $|X_\bullet|$ which is the image of $p_j \times [0,1]$ $(\subseteq X_1 \times \Delta^1)$, then $\tilde{\gamma}_1 \cdot \mu_1 \cdot \tilde{\gamma}_2 \cdot \mu_2 \cdot \ldots \tilde{\gamma}_m \cdot \mu_m$ is a loop in $|X_\bullet|$, based at $*'$, which maps onto γ. This shows that $\pi_1(|X_\bullet|) \to \pi_1(V)$ is surjective.

To prove that $|X_\bullet|$ is connected, observe that $|X_\bullet|$ is obtained by attaching spaces to X_0. So it suffices to prove that each pair of points in X_0 is joined by a path in $|X_\bullet|$. This can be proved easily by an argument similar to the above.

The proof that $\pi_1(|X_\bullet|) \to \pi_1(V)$ is injective is long. The first step is to show that every element of $\pi_1(|X_\bullet|)$ can be represented by a path of a special form.

Lemma 3.2. Each element of $\pi_1(|X_\bullet|, *')$ can be represented by a path of the form $\gamma_1 \cdot \mu_1 \cdot \gamma_2 \cdot \ldots \cdot \gamma_m \cdot \mu_m$, where each γ_j is a piecewise real analytic path in X_0 and where μ_j is a path in $|X_\bullet|$ of the form $p_j \times [0,1]$ for some $p_j \in X_1$.

Proof. Since, for all spaces Y and $n \geq 2$, $\pi_1(Y\times\Delta^n, Y\times\partial\Delta^n) = 0$, every continuous path $\gamma : ([0,1], \{0,1\}) \to (|X_\bullet|,*')$ may be deformed so that it lands in the 1-skeleton, $X_0 \cup X_1 \times [0,1]$, of $|X_\bullet|$. Note that $\gamma^{-1}(X_1\times]1/3,2/3[)$ is a disjoint union of a countable number $\{I_n : n \geq 1\}$ of open subintervals of $]0,1[$. Of these, only a finite number $I_1, \ldots, I_k$ (say) have images that intersect $X_1\times\{1/2\}$. Since $X_1\times]0,1[$ is a manifold, the restriction of γ to each $\bar{I}_j$ $(1 \leq j \leq k)$ can be deformed (rel $\partial\bar{I}_j$) to a smooth path whose intersection with $X_1\times]1/3,2/3[$ is of the form $\{p_j\}\times]1/3,2/3[$. Since $X_1\times(]0,1/3]\cup[2/3,1[)$ is a deformation retraction of $X_1\times(]0,1[-\{1/2\})$, the restriction of γ to each $\bar{I}_j$ $(j > k)$ can be deformed to a path in $X_1\times\{1/3,2/3\}$. Glueing these homotopies together we see that γ can be deformed to a path whose intersection with $X_1\times]1/3,2/3[$ is of the form $\{p_1, \ldots, p_m\}\times]1/3,2/3[$. Such a path can further be deformed in $X_0\cup X_1\times[0,1]$ to a path whose image in $X_1\times]0,1[$ is $\{p_1, \ldots, p_m\}\times]0,1[$.

Thus every element of $\pi_1(|X_\bullet|,*')$ can be represented by a path of the form, $\gamma_1 \cdot \mu_1 \cdot \gamma_2 \cdot \ldots \cdot \gamma_m \cdot \mu_m$, where each γ_j is a path in X_0 and μ_j is a path in $|X_\bullet|$ of the form $p_j \times [0,1]$, where $p_j \in X_1$. Since there is a semi-algebraic triangulation of X_0 such that $d_0(p_j)$, $d_1(p_j)$, $1 \leq j \leq m$, are vertices, we can assume that each γ_j is a piecewise algebraic path in X_0. $\square$

The next step is to reduce to the case where $\dim Y_j = \dim \varepsilon(Y_j)$. It is convenient to introduce the following notation: $V_j = \varepsilon(Y_j)$, $n_j = \dim V_j$.

<u>Proposition</u> 3.3. For each j, there is an irreducible subvariety Z_j of Y_j of dimension n_j such that $\pi_j : Z_j \to V_j$ is surjective, where π_j is the restriction of ε_j to Z_j.

<u>Proof</u>. Since Y_j is smooth and $\varepsilon : Y_j \to V_j$ is surjective, we can find a point p in Y_j such that $\varepsilon(p)$ is a smooth point of V_j and ε has maximal rank at p. We can find an n_j dimensional plane $\prod$ in the tangent space of Y_j at p such that the restriction of ε_* to $\prod$ is injective. Let Z_j be a subvariety of Y_j of dimension n_j that is tangent to $\prod$ at p. Clearly $\varepsilon(Z_j)$ is a subvariety of V_j of dimension n_j. Since V_j is irreducible $\varepsilon(Z_j) = V_j$. That is, π_j is surjective. $\square$

Now suppose that $a \in \pi_1(|X_\bullet|,*')$ and that $\varepsilon_*(a) = 1$ in $\pi_1(V,*)$. Choose a representative γ of a in the form

$$\gamma = \gamma_1 \cdot \mu_1 \cdot \gamma_2 \cdot \mu_2 \cdot \ldots \cdot \gamma_m \cdot \mu_m$$

given by lemma 3.2. Each path γ_i can be written in the form

$$\gamma_i = \gamma_{i1} \cdot \gamma_{i2} \cdot \ldots \cdot \gamma_{in(i)},$$

where either $\varepsilon\circ\gamma_{ij} : [0,1] \to V$ is non singular on $]0,1[$, or else the image of $\varepsilon\circ\gamma_{ij}$ is a point. There is a function $\alpha : \{1, \ldots, m\} \to \{1, \ldots, k\}$ such that γ_{ij} is a path in $Y_{\alpha(i)}$.

According to Hironaka [29], there is a semi-algebraic triangulation of V such that

(a) each V_j is a subcomplex,

(b) each γ_{ij} is a subcomplex,

(c) the set of critical values of each of the maps $\pi_j : Z_j \to V_j$ is a subcomplex.

Since $\varepsilon \circ \gamma$ is nullhomotopic, there is a triangulation B of the disk and a simplicial map $\varphi : B \to V$ with $\partial\varphi = \varepsilon \circ \gamma$. Since the smooth points V_j^* of V_j are dense in V_j, each 2-simplex in the critical set of π_j is the face of a 3-simplex whose interior is contained in V_j^*. By passing to a barycentric subdivision of V if necessary, we can thus assume that for each 2-simplex σ of B on which φ is non degenerate, there is an integer $j(\sigma)$ such that $\varphi(\sigma) \subseteq V_{j(\sigma)}$ and that int $\varphi(\sigma)$ contains no critical values of $\pi_{j(\sigma)}$.

We will prove that γ is nullhomotopic by induction on the number of 2-simplices of B on which φ is non degenerate. When φ is degenerate on all 2-simplices of B, $\varepsilon \circ \gamma$ can be contracted to * within the 1-skeleton of V. That γ is nullhomotopic in this case is a direct consequence of the following three lemmas. First some notation: for $p \in X_1$, we shall denote the path $p \times [0,1]$ in $|X_\bullet|$ by μ_p.

<u>Lemma</u> 3.4. Suppose that $\gamma : [0,1] \to |X_\bullet|$ is a continuous path of the form $\gamma = \gamma_1 \cdot \mu_p \cdot \gamma_2$, where $p \in X_1$ and γ_1, γ_2 are piecewise algebraic paths in X_0. If $\varepsilon \circ \gamma_1 = (\varepsilon \circ \gamma_2)^{-1}$, then there exists $q \in X_1$ such that γ is homotopic to μ_q rel $\{0,1\}$.

<u>Proof</u>. Because $\varepsilon \circ \gamma_1 = (\varepsilon \circ \gamma_2)^{-1}$, $\varepsilon \circ \gamma_1(0) = \varepsilon \circ \gamma_2(1)$. Since $\operatorname{cosk}_0 X_0 = X_0 \times_V X_0$ and since $X_1 \to \operatorname{cosk}_0 X_0$ is onto, there exists $q \in X_1$ such that $d_0(q) = \gamma_1(0)$ and $d_1(q) = \gamma_2(1)$.

Form the split 2-truncated simplicial space $B'_\bullet$ whose non degenerate simplices are

$$B'_0 = \{\Delta_0, \Delta_1, \Delta_2\},\ N(B'_1) = \{I_{01}, I_{02}, I_{12}\},\ N(B'_2) = \{\sigma\},$$

where each Δ_j is a standard 2-simplex, each I_{ij} is a standard 1-simplex and where σ is a point. Define the face maps as in Figure 3.1.

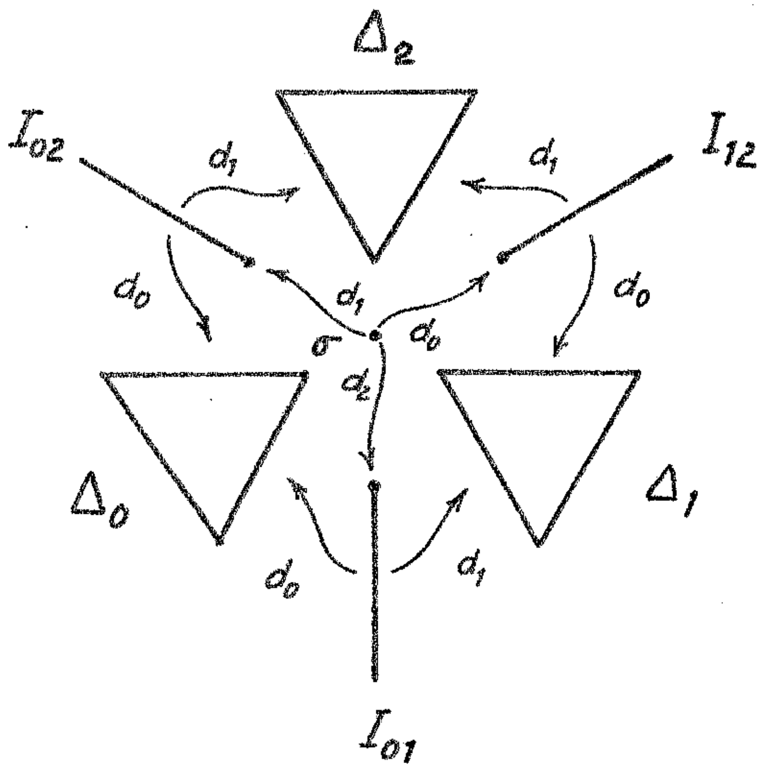

Figure 3.1.

Denote the common value of $\varepsilon\circ\gamma_1(0)$ and $\varepsilon\circ\gamma_2(1)$ by Q. We can define a map ε_B: $B'_\bullet \to V$ as in Figure 3.2. (There, the letter Q stands for the constant map whose value is Q. The image of the map $\Delta_j \to V$ $(j = 1,2)$ factors

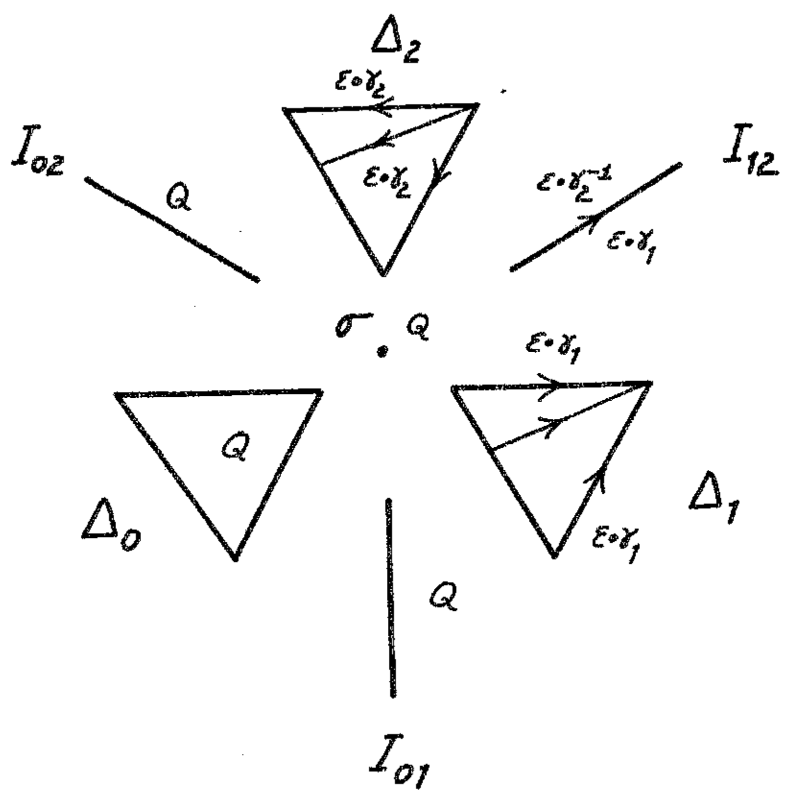

Figure 3.2.

through the map $\varepsilon\circ\gamma_j : [0,1] \to V$.) Thus $B'_\bullet$ can be viewed as a 2-truncated simplicial object in $\underline{\underline{\mathrm{Top}}}_V$. Let $B_\bullet = \mathrm{sk}_2 B'_\bullet$, which is a simplicial object in $\underline{\underline{\mathrm{Top}}}_V$. The geometric realization of $B_\bullet$ is a disk, so to complete the proof of the proposition, we have to construct a map $B_\bullet \to X_\bullet$ over V whose boundary is $\mu_q^{-1} \cdot \gamma_1 \cdot \mu_p \cdot \gamma_2$.

Since $\varepsilon \circ \gamma_1 = \varepsilon \circ \gamma_2^{-1}$, there is a piecewise algebraic path $\tau : [0,1] \to X_0 \times_V X_0$ such that the diagram

$$\begin{tikzcd}
{[0,1]} \arrow[dr, "\tau"] \arrow[drr, bend left, "\gamma_1"] \arrow[ddr, bend right, "\gamma_2^{-1}"'] & & \\
& X_0 \times_V X_0 \arrow[r, "d_0"] \arrow[d, "d_1"] & X_0 \arrow[d] \\
& X_0 \arrow[r, "\varepsilon"'] & V
\end{tikzcd}$$

commutes. Since $X_1 \to X_0 \times_V X_0$ is proper and surjective, and since τ is piecewise real algebraic, there exists a path $\tilde{\tau} : [0,1] \to X_1$ such that $d_0 \circ \tilde{\tau} = \gamma_1$ and $d_1 \circ \tilde{\tau} = \gamma_2^{-1}$. Define a map $f : r_1 B_\bullet \to r_1 X_\bullet$ as in Figure 3.3.

This map extends to a map $\hat{f} : B_\bullet \to \mathrm{cosk}_1 r_1 X_\bullet$. Since the map $r : X_2 \to (\mathrm{cosk}_1 r_1 X_\bullet)_2$ is surjective, there exists a point Σ in X_2 such that $r(\Sigma) = \hat{f}(\sigma)$. We can now extend f to a map $\bar{f} : r_2 B_\bullet \to r_2 X_\bullet$ by defining $\bar{f}(\sigma) = \Sigma$. But $r_2 B_\bullet = B_\bullet'$ and so $\bar{f}$ extends to a map $B_\bullet \to X_\bullet$ in $(\underline{\underline{\mathrm{Top}}}_V)^{\Delta}$. It is easy to check that $|B_\bullet|$ is a 2 disk and that $\partial\bar{f}$ is the path $\mu_q^{-1} \cdot \gamma$. □

A similar, but simpler, argument can be used to prove the following two lemmas.

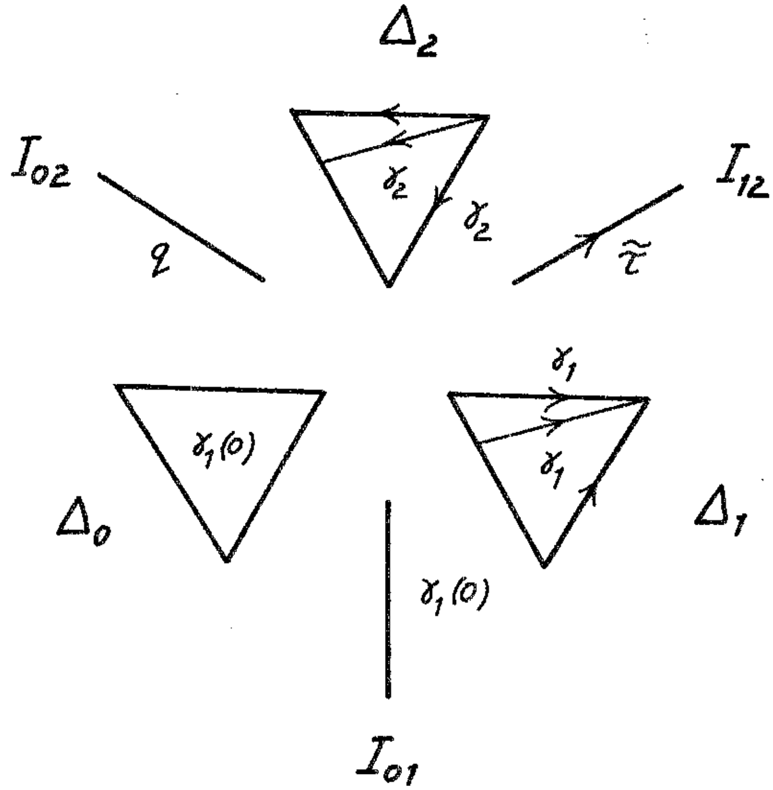

Figure 3.3

__Lemma__ 3.5. If γ is a path in X_0 such that $\varepsilon\circ\gamma$ is a constant path, then there exists $p \in X_1$ such that γ is homotopic to μ_p in $|X_\bullet|$ rel$\{0,1\}$. $\square$

__Lemma__ 3.5. If p, q, r are points in X_1 with $d_0(p) = d_0(r)$, $d_1(p) = d_0(q)$ and $d_1(q) = d_1(r)$, then μ_r is homotopic to $\mu_p \cdot \mu_q$ rel $\{0,1\}$ in $|X_\bullet|$. $\square$

To complete the proof of the theorem, we need to show that if $\varepsilon\circ\gamma$ is nullhomotopic, via a homotopy $\varphi : B \to V$ which is non degenerate on n 2-simplices of B, then γ is homotopic to a loop γ' in $|X_\bullet|$ such that $\varepsilon\circ\gamma'$ is nullhomotopic via a homotopy $\varphi' : B' \to V$ which is non degenerate on at most n-1 2-simplices.

To do this, first note that if $n \geq 1$, then we may assume that there is a 2-simplex σ of B on which φ is non degenerate and such that $\partial\sigma \cap \partial B \neq \emptyset$. For if not, we may deform γ using lemmas 3.4, 3.5, 3.6 until this condition is satisfied. Now suppose that σ is a 2-simplex of B on which φ is non degenerate and such that $\partial\sigma$ and ∂B have an edge e in common. There is an arc γ_{ij} of γ such that $\varepsilon\circ\gamma_{ij} = \varphi(e)$. By changing base points, we can assume that $\gamma_{ij} = \gamma_{11}$. Recall that there is an ℓ such that $\varphi(\sigma) \subseteq V_\ell$ and such that int $\varphi(\sigma)$ contains no critical values of $\pi_\ell : Z_\ell \to V_\ell$. Thus we can lift φ : int $\sigma \to V$ to a real algebraic map $\tilde{\varphi}$: int $\sigma \to Z_\ell$. The closure of im $\tilde{\varphi}$ is a semi algebraic polygon P in Z_ℓ. There is an edge $\tilde{\gamma}_{11}$ of P with $\varepsilon\circ\tilde{\gamma}_{11} = \varepsilon\circ\gamma_{11}$. Let γ'_{11} be the other half of ∂P. Note that P defines a homotopy from $\tilde{\gamma}_{11}$ to γ'_{11} rel $\{0,1\}$. It follows that the path

$$\gamma' = \gamma'_{11} \cdot \gamma_{12} \cdot \cdots \cdot \gamma_{m,n(m)} \cdot \mu_m$$

is a loop in $|X_\bullet|$ that is homotopic to γ. It is easy to check, after applying Lemma 3.4 to cancel the pair of edges lying above $\varphi(e)$, that if $B' = B - \sigma$ and $\varphi' = \varphi|B'$, then $\partial\varphi' = \varepsilon\circ\gamma'$. $\square$

4. The de Rham Theorem for Simplicial Manifolds

In this and subsequent sections, we assume that the reader is familiar with K.-T. Chen's notion of a differentiable space. A convenient reference is [24], pp. 22 - 26.

A topological differentiable space is a set together with a topology and a differentiable space structure. These structures are compatible in the sense that every plot $\alpha : U \to X$ is continuous. A map between two topological differentiable spaces is a function on the underlying sets that is continuous with respect to the topologies and differentiable with respect to the differentiable structures. Smooth manifolds are examples of topological differentiable spaces. We will see shortly that simplicial manifolds are also examples.

Denote the category of topological differentiable spaces and their maps by TDiff and the category of differentiable spaces and differentiable maps by Diff. There are obvious forgetful functors

$$u : \text{TDiff} \to \text{Diff}, \quad \tau : \text{TDiff} \to \text{Top}.$$

Every topological space can be regarded as a differentiable space ([24], p. 26); thus Top can be regarded as a subcategory of Diff. That is, we can view τ as a functor TDiff $\to$ Diff. For each topological differentiable space X, there is a natural map $uX \to \tau X$ that is induced by the identity.

To each differentiable space X, we can associate the chain complex $S_*(X)$, which is the free abelian group generated by the smooth maps $\sigma : \Delta^n \to X$. The boundary map $\partial : S_n(X) \to S_{n-1}(X)$ is defined in the usual way. Define the (singular) homology $H_*(X;R)$ of X with coefficients in the ring (or abelian group) R to be the homology of the complex $S_\bullet(X)\otimes_{\mathbb{Z}}R$ and the singular cohomology of X with coefficients in R to be the homology of the cochain complex

$$S^\bullet(X;R) = \mathrm{Hom}_{\mathbb{Z}}(S_\bullet(X),R).$$

Thus, to every topological differentiable space X we can associate the smooth singular homology $H_*(uX;R)$ of X and the ordinary singular homology $H_*(\tau X;R)$ of X. The natural map $uX \to \tau X$ induces a natural map

$$H_*(uX;R) \to H_*(\tau X;R).$$

Similarly, there is a natural map

$$H^*(\tau X;R) \to H^*(uX;R).$$

When X is a smooth manifold, these maps are isomorphisms [15]. A consequence of this and the following lemma is that the above maps are isomorphisms for spaces built up from manifolds in a nice way.

Lemma 4.1. The functors

$$X \rightsquigarrow H_*(uX;R),\ X \rightsquigarrow H_*(\tau X;R)$$

from TDiff into graded R-modules both satisfy the Eilenberg-Steenrod axioms for a homology theory in TDiff.[1]

Proof. One just mimics the usual proof for Top. None of the constructions used in the proof lead outside TDiff. □

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[1] There are numerous important consequences of 4.1 such as the universal coefficient theorems for $H^*(uX;R)$. A particularly important consequence is that $S_\bullet(X)$ a free and acyclic functor with models so that the Eilenberg Zilber theorem is true for $H_*(uX;R)$ (cf. [50] p. 165, p. 233)

The cohomology version of Lemma 4.1 follows from the universal coefficient theorem and the homology version of the lemma.

We will abbreviate $H_*(X;\mathbb{Z})$ and $H^*(X;\mathbb{Z})$ to $H_*(X)$ and $H^*(X)$, respectively.

The geometric realization of a simplicial object $X_\bullet$ in Diff has a natural differentiable space structure that we shall denote by re $X_\bullet$. It is the unique differentiable structure on $|X_\bullet|$ such that the natural map

$$\coprod_{n\geq 0} X_n \times \Delta^n \to \text{re } X_\bullet$$

is a quotient map in Diff. Similarly, the geometric realization re $X_\bullet$ of a simplicial object $X_\bullet$ in TDiff has the structure of a topological differentiable space: the underlying topological space is $|\tau X_\bullet|$. That is, $\tau(\text{re } X_\bullet) = |\tau X_\bullet|$.

The functor u : TDiff $\to$ Diff is an inclusion functor. To avoid cumbersome notation, we will regard TDiff as a subcategory of Diff and neglect to write the functor u.

Lemma 4.2. Suppose that $X_\bullet$ is a split simplicial object in TDiff. If, for each n, the map

$$H_*(X_n) \to H_*(\tau X_n)$$

is an isomorphism, then the natural maps

$$H_*(\text{re } X_\bullet) \to H_*(|\tau X_\bullet|)$$
$$H^*(|\tau X_\bullet|) \to H^*(\text{re } X_\bullet)$$

are isomorphisms.

Proof. It suffices to prove that the first map is an isomorphism, for it then follows from the universal coefficient theorem that the second map is an isomorphism.

It is not hard to check that

$$\mathrm{re}\, X_\bullet = \varinjlim \mathrm{re}\, \mathrm{sk}_n\, X_\bullet$$

in Diff, while it is well known that

$$|\tau X_\bullet| = \varinjlim |\mathrm{sk}_n\, \tau X_\bullet|$$

in Top. Since

$$S_\bullet(\varinjlim \mathrm{re}\, \mathrm{sk}_n\, X_\bullet) = \varinjlim S_\bullet(\mathrm{re}\, \mathrm{sk}_n\, X_\bullet)$$
$$S_\bullet(\varinjlim |\mathrm{sk}_n\, \tau X_\bullet|) = \varinjlim S_\bullet(|\mathrm{sk}_n\, \tau X_\bullet|),$$

and since homology commutes with direct limits, we need only show that the map

$$(*) \qquad H_*(\mathrm{re}\, \mathrm{sk}_n\, X_\bullet) \to H_*(|\mathrm{sk}_n\, \tau X_\bullet|)$$

is an isomorphism for all n. We prove this by induction. When n = 1, this is true because

$$H_*(X_1) \to H_*(\tau X_1)$$

is an isomorphism. Suppose that $n > 1$ and that the map (*) is an isomorphism for $n - 1$. Because $X_\bullet$ is split, $\mathrm{re}\, \mathrm{sk}_{n-1}\, X_\bullet$ has an open collar in $\mathrm{re}\, \mathrm{sk}_n X_\bullet$. Consequently,

$$H_*(\mathrm{re\,sk}_n\, X_\bullet,\ \mathrm{re\,sk}_{n-1}\, X_\bullet) \approx H_*(N(X_n) \times \Delta^n,\ X_n \times \partial\Delta^n)$$
$$\approx H_*(N(X_n)) \otimes H_*(\Delta^n, \partial\Delta^n).$$

Similarly,

$$H_*(|\mathrm{sk}_n\ \tau\, X_\bullet|,\ |\mathrm{sk}_{n-1}\ \tau\, X_\bullet|) \approx H_*(\tau N(X_n)) \otimes H_*(\Delta^n, \partial\Delta^n).$$

Since $H_*(X_m) \to H_*(\tau\, X_m)$ is an isomorphism for $m \leq n$ and since $X_\bullet$ is split in TDiff, it follows that $H_*(N(X_n)) \to H_*(\tau N(X_n))$ is an isomorphism. From a standard argument that uses the long exact sequence of the pair $(\mathrm{re\,sk}_n\ X_\bullet,\ \mathrm{re\,sk}_{n-1}\ X_\bullet)$ and the 5-lemma, it follows that the map (*) is an isomorphism for n. □

We shall denote the de Rham complex (of real valued forms) of the differentiable space X by $E^\bullet(X)$. For a topological differentiable space X, $E^\bullet(X)$ will denote the de Rham complex of the underlying differentiable space. The homology of the complex $E^\bullet(X)$ will be called the de Rham cohomology of X and will be denoted by $H^*_{dR}(X)$.

Suppose that X is a topological differentiable space. The integration map

$$\int : E^\bullet(X) \to \mathrm{Hom}(S_\bullet(X), \mathbb{R})$$
$$w \mapsto \{\sigma \mapsto \int_\sigma w\}$$

is a cochain map and consequently induces a map

$$H^*_{dR}(X) \to H^*(X; \mathbb{R}).$$

We will say that the de Rham theorem is true for X if the maps

$$H^*_{dR}(X) \to H^*(X;\mathbb{R}) \text{ and } H^*(\tau X) \to H^*(X),$$

are both isomorphisms.

Our next task is to understand the de Rham complex of the realization of a simplicial object in <u>Diff</u>.

Suppose that M,N are differentiable spaces and that $i : W \to N$ is a subspace of N. (That is, $W \subseteq N$ and W has the induced differentiable structure c.f. [24], p. 25.) Suppose that $f : W \to M$ is a differentiable map. Define $M \cup_f N$ to be the unique differentiable space such that

$$\begin{array}{ccc} W & \xrightarrow{i} & N \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ M & \xrightarrow[j]{} & M\cup_f N \end{array}$$

is a pushout in <u>Diff</u>. The natural maps $M \to M\cup_f N$ and $N \to M\cup_f N$ induce a differentiable map $M \coprod N \to M\cup_f N$ which is easily seen to be a quotient map in <u>Diff</u>. It induces a map

$$E^\bullet(M\cup_f N) \to \{(w_M,w_N) \in E^\bullet(M)\oplus E^\bullet(N) : f^*w_M = i^*w_N\}.$$

<u>Lemma</u> 4.3. With notation as above, the natural map

$$E^\bullet(M\cup_f N) \to \{(w_M,w_N) \in E^\bullet(M)\oplus E^\bullet(N) : f^*w_M = i^*w_N\}$$

is an isomorphism.

Proof. Denote the differentiable structures (i.e., families of plots) of M, N, W by $\mathcal{U}_M$, $\mathcal{U}_N$, $\mathcal{U}_W$, respectively. The differentiable structure on MU_fN is generated by the family $\mathcal{U}$ of plots on MU_fN, where

$$\begin{array}{ccc} \mathcal{U}_W & \xrightarrow{i_*} & \mathcal{U}_N \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ \mathcal{U}_M & \xrightarrow[j_*]{} & \mathcal{U} \end{array}$$

is a pushout square in Set. (This assertion can be proved by verifying that the set MU_fN together with the differentiable structure generated by $\mathcal{U}$ has the required universal mapping property.) Observe that i_* and j_* are injective.

A differential form w on MU_fN is determined by the family of forms $(w_\alpha)_{\alpha\in\mathcal{U}}$ associated to w. It follows that the natural map

$$E^\bullet(MU_fN) \to E^\bullet(M)\oplus E^\bullet(N)$$

is injective.

On the other hand, a family of forms $(w_\alpha)_{\alpha\in\mathcal{U}}$, where $w_\alpha \in E^*(U)$ and $\alpha : U \to MU_fN$, defines a form on MU_fN provided that $\theta^* w_\alpha = w_\beta$ whenever $\theta : V \to U$ is a smooth map for which the diagram

$$\begin{array}{ccc} U & \xrightarrow{\alpha} & MU_fN \\ {\scriptstyle \theta}\downarrow & \nearrow{\scriptstyle \beta} & \\ V & & \end{array}$$

commutes.

Suppose that $w_M \in E^{\bullet}(M)$, $w_N \in E^{\bullet}(N)$ and that $f^*w_M = i^*w_N$. We will define a form $w = (w_\alpha)_{\alpha\in\mathcal{U}}$ on MU_fN with $j^*w = w_M$ and $g^*w = w_N$. If $\alpha \in \mathcal{U}$, then either:

(a) there exists $\alpha' \in \mathcal{U}_M$ such that $j\circ\alpha' = \alpha$. In this case define $w_\alpha = (w_M)_{\alpha'}$.

(b) there exists $\alpha'' \in \mathcal{U}_N$ such that $g\circ\alpha'' = \alpha$. In this case define $w_\alpha = (w_N)_{\alpha''}$.

Observe that if both liftings α', α'' of α exist, then there exists $\beta \in \mathcal{U}_W$ such that $\alpha' = f\circ\beta$ and $\alpha'' = i\circ\beta$. Hence

$$(w_M)_{\alpha'} = (f^*w_M)_\beta = (i^*w_N)_\beta = (w_N)_{\alpha''}.$$

Consequently, the family $(w_\alpha)_{\alpha\in\mathcal{U}}$ is well defined.

Finally, suppose that $\theta : V \to U$ is a smooth map, that $\alpha, \beta \in \mathcal{U}$ and that the diagram

$$\begin{tikzcd} U \arrow[r, "\alpha"] & MU_fN \\ V \arrow[u, "\theta"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

commutes. To complete the proof, we must show that $\theta^*w_\alpha = w_\beta$. There are four possibilities:

(i) There exist $\alpha', \beta' \in \mathcal{U}_M$ with $\alpha = j\circ\alpha'$ and $\beta = j\circ\beta'$. Because $j_* : \mathcal{U}_M \to \mathcal{U}$ is injective, $\alpha'\circ\theta = \beta'$ and

$$\theta^*w_\alpha = \theta^*(w_M)_{\alpha'} = (w_M)_{\beta'} = w_\beta.$$

(ii) There exists α'', $\beta'' \in \mathcal{U}_N$ with $\alpha = g\circ\alpha''$, $\beta = g\circ\beta''$. Either $\alpha''\circ\theta = \beta''$, in which case an argument similar to that in (i) shows that $\theta^* w_\alpha = w_\beta$, or α'', $\beta'' \in i_*\mathcal{U}_W$. That is, there exist $\tilde{\alpha}$, $\tilde{\beta} \in \mathcal{U}_W$ such that $\beta'' = i\circ\tilde{\beta}$, $\alpha'' = i\circ\tilde{\alpha}$. Since $g_*(\alpha''\circ\theta) = g_*(\beta'')$ and $j_*f_* = g_*i_*$, it follows that $j_*f_*(\tilde{\alpha}\circ\theta) = j_*f_*(\tilde{\beta})$. Set $\alpha' = f\circ\tilde{\alpha}$, $\beta' = f\circ\tilde{\beta}$. Now $\alpha = j\circ\alpha'$, $\beta = j\circ\beta'$ and $\alpha'\circ\theta = \beta'$. That $\theta^* w_\alpha = w_\beta$ now follows from (i).

(iii) There exist $\alpha' \in \mathcal{U}_M$ and $\beta'' \in \mathcal{U}_N$ such that $\alpha = j\circ\alpha'$ and $\beta = g\circ\beta''$. Since the diagram

$$\begin{tikzcd}
 & V \arrow[dl, "\theta"'] \arrow[ddr, "\beta''"] & \\
U \arrow[ddr, bend right, "\alpha'"'] & & \\
 & W \arrow[r, "i"] \arrow[d, "f"'] & N \arrow[d, "g"] \\
 & M \arrow[r, "j"'] & M\cup_f N
\end{tikzcd}$$

commutes, β'' lands in W. Because W is a differentiable subspace of N, there is a plot $\gamma : V \to W$, $\gamma \in \mathcal{U}_W$, such that the diagram

$$\begin{tikzcd}
 & V \arrow[dl, "\theta"'] \arrow[dd, "\gamma"'] \arrow[ddr, "\beta''"] & \\
U \arrow[ddr, bend right, "\alpha'"'] & & \\
 & W \arrow[r, "i"] \arrow[d, "f"'] & N \arrow[d, "g"] \\
 & M \arrow[r, "j"'] & M \; {}_f N
\end{tikzcd}$$

commutes. Now

$$\theta^* w_\alpha = \theta^*(w_M)_{\alpha'} = (f^* w_M)_\gamma = (i^* w_N)_{\beta''} = w_\beta$$

as required.

(iv) There exist $\beta' \in \mathcal{U}_M$ and $\alpha'' \in \mathcal{U}_N$ such that $\beta = j\circ\beta'$ and $\alpha = g\circ\alpha''$. An argument similar to the one in (iii) can be used to show that $\theta^* w_\alpha = w_\beta$. $\square$

Suppose that $X_\bullet$ is a simplicial object in $\underline{\underline{\mathrm{Diff}}}$. The natural quotient map

$$\coprod_{n\geq 0} X_n \times \Delta^n \to \mathrm{re}\, X_\bullet$$

induces an injective map

$$\begin{aligned} \mathcal{E}^\bullet(\mathrm{re}\, X_\bullet) &\to \bigoplus_{n\geq 0} \mathcal{E}^\bullet(X_n \times \Delta^n) \\ w &\mapsto (w_n)_{n\geq 0}. \end{aligned}$$

If $f : [n] \to [m]$ is a map in Δ, then the diagram

$$\begin{tikzcd} & X_m \times \Delta^m \arrow[dr] & \\ X_m \times \Delta^n \arrow[ur, "\mathrm{id} \times |f|"] \arrow[dr, "\hat{f} \times \mathrm{id}"'] & & \mathrm{re}\, X_\bullet \\ & X_n \times \Delta^n \arrow[ur] & \end{tikzcd}$$

commutes in $\underline{\underline{\mathrm{Diff}}}$, where $|f| : \Delta^n \to \Delta^m$ denotes the affine map induced by f and $\hat{f} : X_m \to X_n$ denotes the map induced by f. It follows that if $(w_n)_{n\geq 0}$ is the pullback to $\bigoplus_{n\geq 0} \mathcal{E}^\bullet(X_n \times \Delta^n)$ of a form w on $\mathrm{re}\, X_\bullet$, then

$$(\mathrm{id}\times|f|)^* w_m = (\hat{f}\times\mathrm{id})^* w_n$$

in $\mathcal{E}^\bullet(X_m \times \Delta^n)$ for all maps f in Δ. The following lemma is a direct consequence of Lemma 4.3.

Lemma 4.4. If $X_\bullet$ is a split simplicial object in Diff, then the natural injection

$$E^\bullet(\text{re } X_\bullet) \to \{(w_n)_{n\geq 0} \in \bigoplus_{n\geq 0} E^\bullet(X_n \times \Delta^n) : (\text{id}\times|f|)^* w_m = (\hat{f}\times\text{id})^* w_n\}$$

is an isomorphism. □

Suppose that $X_\bullet$ is a simplicial object in Diff. It is convenient not to work with the full de Rham complex of re $X_\bullet$, but with a subcomplex of $E^\bullet(\text{re } X_\bullet)$ that we shall now describe. Denote by $E^\bullet(X_\bullet)$ the subcomplex of $E^\bullet(\text{re } X_\bullet)$ that consists of those forms on re $X_\bullet$ whose pullback $(w_n)_{n\geq 0}$ to $\coprod_{n\geq 0} X_n \times \Delta^n$ satisfies

$$w_n \in E^\bullet(X_n) \otimes E^\bullet(\Delta^n)$$

for all n. Denote the homology of the complex $E^*(X_\bullet)$ by $H^*_{dR}(X_\bullet)$.

Theorem 4.5. Suppose that $X_\bullet$ is a split simplicial object in TDiff. If the de Rham theorem is true for each X_n, then the integration map induces an isomorphism

$$H^*_{dR}(X_\bullet) \to H^*(\text{re } X_\bullet; \mathbb{R}).$$

Corollary 4.6. If V is an algebraic variety and $\varepsilon : X_\bullet \to V$ is a split hypercovering of V in Alg, then there is a natural isomorphism

$$H^*(V;\mathbb{R}) \approx H^*_{dR}(X_\bullet). \quad \square$$

The one remaining ingredient required in the proof of Theorem 4.5 is the following extension lemma.

Lemma 4.7. If $X_\bullet$ is a simplicial object in Diff, then the restriction map

$$E^\bullet(\mathrm{sk}_n\, X_\bullet) \to E^\bullet(\mathrm{sk}_{n-1}\, X_\bullet)$$

is surjective for all $n \geq 1$.

Proof. Two remarks are in order. First, let $Y_\bullet$ be a simplicial object in Diff. To determine whether an element $(w_m)_{m\geq 0}$ of $\bigoplus_{m\geq 0} E^\bullet(Y_m \times \Delta^m)$ is in $E^\bullet(\mathrm{re}\, Y_\bullet)$ using Lemma 4.4, it is sufficient to check the compatibility conditions only for face and degeneracy maps $f : [m] \to [m\pm 1]$. Second, if $Y_\bullet$ is also split, then a form on re $Y_\bullet$ is determined by its pullback to $\bigoplus_{m\geq 0} E^\bullet(N(Y_m) \times \Delta^m)$: an element $(\zeta_m)_{m\geq 0}$ of $\bigoplus_{m\geq 0} E^\bullet(N(Y_m) \times \Delta^m)$ corresponds to a form on re $Y_\bullet$ if and only if

$$(d_j \times id)^*\zeta_{m-1} = (id \times |d_j|)^*\zeta_m$$

in $E^\bullet(N(Y_m) \times \Delta^{m-1})$, for all m and j with $0 \leq j \leq m$. Finally, note that

$$N((\mathrm{sk}_\ell\, Y_\bullet)_m) = \begin{cases} N(Y_m) & m \leq \ell \\ \emptyset & m > \ell \end{cases}$$

Suppose that $w_m \in E^\bullet(N(X_m))\otimes E^\bullet(\Delta^m)$, $0 \leq m < n$, and that $(w_m)_{0\leq m<n}$ determines a form w on $\mathrm{sk}_{n-1}\, X_\bullet$. To extend w to a form on $\mathrm{sk}_n\, X_\bullet$, we need to find a form w_n in $E^\bullet(N(X_n))\otimes E^\bullet(\Delta^n)$ satisfying

$$(d_j^*\otimes id)\, w_{n-1} = (id\otimes|d_j|^*)w_n$$

in $E^\bullet(N(X_n)) \otimes E^\bullet(\Delta^{n-1})$, where $0 \leq j \leq n$.

Observe that

$$\{(d_j^* \otimes \mathrm{id})\, w_{n-1}\}_{0\leq j\leq n}$$

is an $E^\bullet(N(X_n))$ valued form η on the simplicial set $\partial\Delta^n$. That is, $\eta \in E^\bullet(N(X_n)) \otimes E^\bullet(\partial\Delta^n)$. Since the restriction map $E^\bullet(\Delta^n) \to E^\bullet(\partial\Delta^n)$ is surjective ([51], p. 297 and [25], Proposition 13.8), it follows that there exists $w_n \in E^\bullet(N(X_n)) \otimes E^\bullet(\Delta^n)$ whose restriction to $\partial\Delta^n$ is η. That is,

$$(\mathrm{id}\otimes|d_j|^*)\, w_n = (d_j^* \otimes \mathrm{id})\, w_{n-1}$$

as required. $\square$

For a differentiable space M with subspace N, denote by $E^\bullet(M,N)$ the kernel of the restriction map $E^\bullet(M) \to E^\bullet(N)$.

<u>Proof of Theorem</u> 4.5. Filter $E^\bullet(X_\bullet)$ and $S^\bullet(\mathrm{re}\, X_\bullet;\mathbb{R})$ by the skeleton filtrations:

$$G^n E^\bullet(X_\bullet) = E^\bullet(X_\bullet,\ \mathrm{sk}_{n-1} X_\bullet),$$
$$G^n S^\bullet(\mathrm{re}\, X_\bullet,\mathbb{R}) = S^\bullet(\mathrm{re}\, X_\bullet,\ \mathrm{re}\,\mathrm{sk}_{n-1}\ X_\bullet;\mathbb{R}).$$

The integration map $E^\bullet(X_\bullet) \to S^\bullet(X_\bullet;\mathbb{R})$ is filtration preserving. From the extension Lemma 4.7, it follows that the spectral sequence associated to $E^\bullet(X_\bullet)$ satisfies

$$\begin{aligned} E_0^n &= E^\bullet(\mathrm{sk}_n X_\bullet,\ \mathrm{sk}_{n-1} X_\bullet) \\ &\approx E^\bullet(N(X_n)) \otimes E^\bullet(\Delta^n,\partial\Delta^n), \\ E_1^{n,m} &= H^m_{dR}(N(X_n)). \end{aligned}$$

The spectral sequence associated to $S^\bullet(X_\bullet;\mathbb{R})$ satisfies

$$\begin{aligned} E_0^n &= S^\bullet(\operatorname{re}\operatorname{sk}_n X_\bullet,\ \operatorname{re}\operatorname{sk}_{n-1} X_\bullet;\mathbb{R}), \\ E_1^{n,m} &= H^{n+m}(\operatorname{re}\operatorname{sk}_n X_\bullet,\ \operatorname{re}\operatorname{sk}_n X_\bullet;\mathbb{R}) \\ &\approx H^{n+m}(N(X_n) \times (\Delta^n,\partial\Delta^n);\mathbb{R}) \\ &\approx H^m(N(X_n);\mathbb{R}). \end{aligned}$$

These isomorphisms are compatible with the integration map. Since $X_\bullet$ is split and since the de Rham theorem is true for each X_n, it is true for each $N(X_n)$. It follows that the map

$$E^\bullet(X_\bullet) \to S^\bullet(\operatorname{re} X_\bullet;\mathbb{R})$$

induces an isomorphism on homology. $\square$

5. Mixed Hodge Complexes

In order to put a mixed Hodge structure (M.H.S.) on the homotopy groups of an algebraic variety V, it is necessary to prove that the de Rham complex of a hypercovering of V is a mixed Hodge complex. In this section we prove this by modifying Deligne's construction [12]. We begin by recalling Deligne's definition of a mixed Hodge complex (8.1.5, [12]).

First recall that a map $K^\bullet \to L^\bullet$ between two cochain complexes is a quasi-isomorphism if it induces an isomorphism on cohomology. Two cochain complexes $K^\bullet$, $L^\bullet$ are said to be quasi-isomorphic if there is a sequence

$$K^\bullet \to K_1^\bullet \leftarrow K_2^\bullet \to \dots \to K_n^\bullet \leftarrow L^\bullet$$

of quasi-isomorphisms joining them.

A filtered complex $(K^\bullet, W_\bullet)$ consists of a cochain complex $K^\bullet$ and an increasing filtration $W_\bullet$:

$$\dots \subseteq W_\ell \subseteq W_{\ell+1} \subseteq \dots \quad .$$

The corresponding decreasing filtration, denoted by $W^\bullet$, is defined by $W^\ell = W_{-\ell}$. A map $(K^\bullet, W_\bullet) \to (L^\bullet, W_\bullet)$ between two filtered complexes is a quasi-isomorphism if it induces an isomorphism on the E_1 term of the spectral sequences associated to the decreasing filtrations $W^\bullet$. Two filtered complexes, $(K^\bullet, W_\bullet)$ and $(L^\bullet, W_\bullet)$, are said to be quasi-isomorphic if there is a sequence

$$(K^\bullet, W_\bullet) \to (K_1^\bullet, W_\bullet) \leftarrow (K_2^\bullet, W_\bullet) \to \dots \leftarrow (L^\bullet, W_\bullet)$$

of quasi-isomorphisms.

Let k be a field such that $\mathbb{Q} \subseteq k \subseteq \mathbb{R}$. A k-mixed Hodge complex (k-M.H.C.) is a triple $(K_{\mathbb{Z}}^\bullet, (K_k^\bullet, W_\bullet), (K_{\mathbb{C}}^\bullet, W_\bullet, F^\bullet))$ that satisfies

(i) $K^\bullet_{\mathbb{Z}}$ is a cochain complex and for each m, $H^m(K^\bullet_{\mathbb{Z}})$ is a finitely generated abelian group,

(ii) $(K^\bullet_k, W_\bullet)$ is a filtered cochain complex of k-vector spaces and $K^\bullet_{\mathbb{Z}} \otimes k$ is quasi-isomorphic with $K^\bullet_k$,

(iii) $(K^\bullet_{\mathbb{C}}, W_\bullet, F^\bullet)$ is a bifiltered cochain complex of complex vector spaces and $(K^\bullet_k \otimes \mathbb{C}, W_\bullet)$ is quasi-isomorphic with $(K^\bullet_{\mathbb{C}}, W_\bullet)$,

(iv) the differential d_0 of the E_0 term of the spectral sequence associated to the decreasing <u>weight filtration</u> $W^\bullet$ of $K^\bullet_{\mathbb{C}}$ (the weight spectral sequence) is strictly compatible with the induced <u>Hodge filtration</u> $F^\bullet$,

(v) with the induced Hodge filtration, the $E_1^{s,t}$ term of the weight spectral sequence has a Hodge structure of weight t.

The importance of M.H.C.'s lies in the following theorem.

<u>Theorem</u> 5.1 (Deligne [12]). If $(K^\bullet_{\mathbb{Z}}, (K^\bullet_k, W_\bullet), (K^\bullet_{\mathbb{C}}, W_\bullet, F^\bullet))$ is a M.H.C., then $H^*(K_{\mathbb{Z}})$ has a M.H.S. with weight filtration defined by

$$W_{\ell+m} H^m(K^\bullet_k) = \mathrm{im}\{H^m(W_\ell K^\bullet_k) \to H^m(K^\bullet_k)\}$$

and Hodge filtration defined by

$$F^p H^m(K^\bullet_{\mathbb{C}}) = \mathrm{im}\{H^m(F^p K^\bullet_{\mathbb{C}}) \to H^m(K^\bullet_{\mathbb{C}})\}. \qquad \square$$

A filtration $W_\bullet$ of a cochain complex $K^\bullet$ induces a filtration of $K^\bullet \otimes K^\bullet$:

$$W_\ell(K^\bullet \otimes K^\bullet) = \sum_{s+t=\ell} W_s K^\bullet \otimes W_t K^\bullet .$$

Suppose that $K^\bullet$ is a differential graded algebra (d.g.a.). A filtration $W_\bullet$ of $K^\bullet$ is said to be <u>multiplicative</u> if the product $K^\bullet \otimes K^\bullet \to K^\bullet$ is filtration preserving. A <u>filtered</u> <u>d.g.a</u>. $(K^\bullet, W_\bullet)$ consists of a d.g.a. $K^\bullet$ together with a multiplicative filtration $W_\bullet$. The notion of quasi-isomorphism can be extended to d.g.a.'s and (filtered) d.g.a.'s by requiring that all maps be d.g.a. homomorphisms.

In the sequel, $E^{\bullet}_{\mathbb{R}}(M)$ will denote the d.g.a. of real valued differential forms on the differentiable space M and $E^{\bullet}_{\mathbb{C}}(M)$ will denote the d.g.a. of complex valued forms. For the standard n-simplex we also have $E^{\bullet}_{\mathbb{Q}}(\Delta^n)$, the complex of rational polynomial forms on Δ^n.

Example 5.2. Suppose that X is a smooth quasi-projective variety. Appealing to Hironaka's resolution of singularities, we can assume that $X = Y - D$, where Y is a smooth projective variety and D is a divisor in Y with normal crossings.

Suppose that U is an open polydisk in Y with holomorphic coordinates $(z_1,\ldots,z_n)$. Suppose that $Y \cap D = \{z_1 = z_2 = \ldots = z_k = 0\}$.
Define

$$A^{\bullet}_{\mathbb{R}}(U) = E^{\bullet}_{\mathbb{R}}(U) \otimes \Lambda(\theta_1,\ldots,\theta_k)$$

$$A^{\bullet}_{\mathbb{C}}(U) = E^{\bullet}_{\mathbb{C}}(U) \otimes \Lambda(\frac{dz_1}{z_1}\ldots,\frac{dz_k}{z_k},\frac{d\bar{z}_1}{\bar{z}_1},\ldots,\frac{d\bar{z}_k}{\bar{z}_k}) \otimes \mathbb{C}[\log|z_1|,\ldots,\log|z_k|]$$

$$E^{\bullet}(U\log D) = E^{\bullet}_{\mathbb{C}}(U) \otimes \Lambda(\frac{dz_1}{z_1},\ldots,\frac{dz_k}{z_k})\ ,$$

where $\theta_j = \frac{1}{4\pi i}(dz_j/z_j - d\bar{z}_j/\bar{z}_j)$. Note that there are natural d.g.a. inclusions

$$A^{\bullet}_{\mathbb{R}}(U) \to E^{\bullet}_{\mathbb{R}}(U\backslash D)$$

$$A^{\bullet}_{\mathbb{C}}(U) \to E^{\bullet}_{\mathbb{C}}(U\backslash D)$$

$$E^{\bullet}(U\log D) \to E^{\bullet}_{\mathbb{C}}(U\backslash D),$$

that $A^{\bullet}_{\mathbb{R}}(U)$ is a real complex (as $\overline{\theta}_j = \theta_j$)

and that the diagram

$$\begin{array}{ccccc} A^\bullet_{\mathbb{R}}(U) & \longrightarrow & A^\bullet_{\mathbb{C}}(U) & \longleftarrow & E^\bullet(U\log D) \\ \downarrow & & \downarrow & & \downarrow \\ E^\bullet_{\mathbb{R}}(U\backslash D) & \longrightarrow & E^\bullet_{\mathbb{C}}(U\backslash D) & = & E^\bullet_{\mathbb{C}}(U\backslash D) \end{array}$$

commutes. Since

$$\Lambda(\frac{dz_1}{z_1},\ldots,\frac{dz_k}{z_k}) \otimes \Lambda(d\log|z_1|,\ldots,d\log|z_k|) \otimes \mathbb{C}[\log|z_1|,\ldots,\log|z_k|]$$
$$= \Lambda(\frac{dz_1}{z_1},\ldots,\frac{dz_k}{z_k}, \frac{d\bar{z}_1}{\bar{z}_1},\ldots, \frac{d\bar{z}_k}{\bar{z}_k}) \otimes \mathbb{C}[\log|z_1|,\ldots,\log|z_k|]$$
$$= \Lambda(\theta_1,\ldots,\theta_k) \otimes \Lambda(d\log|z_1|,\ldots,d\log|z_k|) \otimes \mathbb{C}[\log|z_1|,\ldots,\log|z_k|],$$

it follows from a standard spectral sequence argument that each of the vertical maps in the previous diagram is a d.g.a. quasi-isomorphism.[2]

Define a weight filtration $W_\bullet$ on

$$\Lambda = \Lambda(\frac{dz_1}{z_1},\ldots,\frac{dz_k}{z_k}, \frac{d\bar{z}_1}{\bar{z}_1},\ldots,\frac{d\bar{z}_k}{\bar{z}_k}) \otimes \mathbb{C}[\log|z_1|,\ldots,\log|z_k|]$$

by

$$W_\ell\Lambda = \{\text{elements of } \Lambda \text{ of polynomial degree} \leq \ell\}$$

i.e. each of dz_j/z_j, $d\bar{z}_j/\bar{z}_j$ and $\log|z_j|$ has weight 1. This filtration extends naturally to a weight filtration $W_\bullet$ on $A^\bullet_{\mathbb{C}}(U)$ where $E^\bullet_{\mathbb{C}}(U) \subseteq W_0$. Give $E^\bullet(U\log D)$ and $A^\bullet_{\mathbb{R}}(U)$ the induced filtrations. These filtrations are easily seen to be independent of the holomorphic coordinates $(z_1,\ldots,z_n)$ choosen for U. They are clearly multiplicative.

Define $A^\bullet_{\mathbb{R}}(Y,D)$, $A^\bullet_{\mathbb{C}}(Y,D)$ and $E^\bullet(Y\log D)$ to be the global sections of the sheaves $A^\bullet_{\mathbb{R}}(U)$, $A^\bullet_{\mathbb{C}}(U)$ and $E^\bullet(U\log D)$, respectively. The weight filtrations of these sheaves induce weight filtrations on each of these complexes. Moreover, these weight filtrations are multiplicative.

---

2 Filter each d.g.a. by the powers of its augmentation ideal. The resulting second quadrant spectral sequences have isomoprhic E_1 terms.

Standard sheaf theoretic arguments may be used to show that

$$(A^{\bullet}_{\mathbb{R}}(Y,D) \otimes \mathbb{C}, W_{\bullet}) \to (A^{\bullet}_{\mathbb{C}}(Y,D), W_{\bullet}) \leftarrow (E^{\bullet}(Y\log D), W_{\bullet})$$

is a quasi-isomorphism of the filtered d.g.a.'s $(A^{\bullet}_{\mathbb{R}}(Y,D) \otimes \mathbb{C}, W_{\bullet})$ and $(E^{\bullet}(Y\log D), W_{\bullet})$. Give $E^{\bullet}(Y\log D)$ it usual Hodge filtration $F^{\bullet}$ (see [21] or [38]). It now follows from standard facts about the log complex $E^{\bullet}(Y\log D)$ ([21],[38]) and the above discussion that

$$(S^{\bullet}(X), (A^{\bullet}_{\mathbb{R}}(Y,D), W_{\bullet}), (E^{\bullet}(Y\log D), W_{\bullet}, F^{\bullet}))$$

is an $\mathbb{R}$-M.H.C., with multiplicative filtrations, that is functorial in the pair (Y,D).

The general case 5.3. Suppose that $Y_{\bullet}$ is a simplicial variety and that $D_{\bullet}$ is a divisor in $Y_{\bullet}$ with normal crossings. Let $X_{\bullet} = Y_{\bullet} - D_{\bullet}$ and suppose that $X_{\bullet}$ is a split simplicial variety. Define

$$A^{\bullet}_{\mathbb{R}}(Y_{\bullet}, D_{\bullet}) = \{(w_n)_{n\geq 0} \in E^{\bullet}_{\mathbb{R}}(X_{\bullet}) : w_n \in A^{\bullet}_{\mathbb{R}}(Y_n, D_n) \otimes E^{\bullet}(\Delta^n)\},$$

where $E^{\bullet}_{\mathbb{R}}(X_{\bullet})$ is the subcomplex of $E^{\bullet}_{\mathbb{R}}(\mathrm{re}\, X_{\bullet})$ defined above theorem 4. . Note that $A^{\bullet}_{\mathbb{R}}(Y_{\bullet}, D_{\bullet})$ is a double complex:

$$A^{\bullet}_{\mathbb{R}}(Y_{\bullet}, D_{\bullet}) = \oplus_{s,t} A^{s,t}_{\mathbb{R}}(Y_{\bullet}, D_{\bullet}) ,$$

where

$$A^{s,t}_{\mathbb{R}}(Y_{\bullet}, D_{\bullet}) = \{(w_n)_{n\geq 0} : w_n \in A^{t}(Y_n, D_n) \otimes E^{s}(\Delta^n)\} .$$

Define a decreasing filtration $W^{\bullet}$ of each $A^{s,t}_{\mathbb{R}}(Y_{\bullet}, D_{\bullet})$ by

$$W^{\ell}E^{s,t}(X_{\bullet}) = \{(w_n)_{n\geq 0} : w_n \in W_{-\ell}A^{t}_{\mathbb{R}}(Y_n, D_n) \otimes E^{s}(\Delta^n)\}$$

and a decreasing weight filtration $W^\bullet$ of $A^\bullet_{\mathbb{R}}(Y_\bullet,D_\bullet)$ by

$$W^\ell A^\bullet_{\mathbb{R}}(Y_\bullet,D_\bullet) = \sum_{m+s\geq \ell} W^m A^{s,t}_{\mathbb{R}}(Y_\bullet,D_\bullet).$$

Define

$$\mathcal{E}^\bullet(Y_\bullet \log D_\bullet) = \{(w_n)_{n\geq 0} \in \mathcal{E}^\bullet_{\mathbb{C}}(X_\bullet) : w_n \in \mathcal{E}^\bullet(Y_n \log D_n) \otimes \mathcal{E}^\bullet(\Delta^n)\}$$

and $A^\bullet_{\mathbb{C}}(Y_\bullet,D_\bullet) = A^\bullet_{\mathbb{R}}(Y_\bullet,D_\bullet) \otimes \mathbb{C}$. As above we can define a weight filtration on $\mathcal{E}^\bullet(Y_\bullet \log D_\bullet)$. Define the Hodge filtration on it by

$$F^p\mathcal{E}^\bullet(Y_\bullet \log D_\bullet) = \{(w_n)_{n\geq 0} \in \mathcal{E}^\bullet_{\mathbb{C}}(X_\bullet) : w_n \in F^p\mathcal{E}^\bullet(Y_n \log D_n) \otimes \mathcal{E}^\bullet(\Delta^n)\}$$

It will follow from the corollary 5.6 that

$$(S^\bullet(\mathrm{re}\ X_\bullet),\ (A^\bullet_{\mathbb{R}}(Y_\bullet,D_\bullet),W_\bullet),\ (\mathcal{E}^\bullet(Y_\bullet \log D_\bullet),W_\bullet,F^\bullet))$$

is an $\mathbb{R}$-M.H.C.

It is expedient to first abstract the setting. Suppose that $X_\bullet$ is a simplicial object in TDiff and that, to each X_n, we can associate the k-M.H.C.

$$(S^\bullet(X_n),\ (A^\bullet_k(X_n),W_\bullet),\ (L_{\mathbb{C}}(X_n),W_\bullet,F^\bullet)),$$

where $A_k(X_n) \subseteq \mathcal{E}_{\mathbb{R}}(X_n)$, $L_{\mathbb{C}}(X_n) \subseteq \mathcal{E}_{\mathbb{C}}(X_n)$ and where $A_k(X_n) \otimes \mathbb{C} \subseteq \mathcal{E}_{\mathbb{C}}(X_n)$. Suppose further that if $f : [n] \to [m]$ is a map in Δ then the d.g.a. map

$$f^* : \mathcal{E}^\bullet(X_n) \to \mathcal{E}^\bullet(X_m)$$

induces maps

$$A^\bullet_k(X_n) \to A^\bullet_k(X_m) \quad \text{and} \quad L^\bullet_{\mathbb{C}}(X_n) \to L^\bullet_{\mathbb{C}}(X_m).$$

A <u>cosimplicial cochain complex</u> $C^\bullet[\ ]$ is a covariant functor from the simplicial category into the category of cochain complexes. Denote by $C^\bullet[n]$ the image of $[n]$. For example, if we define

$$A_k^\bullet[n] = A_k^\bullet(X_n) \ ,$$

then $A_k^\bullet[\ ]$ is a cosimplicial cochain complex. Similarly, we can define cosimplicial cochain complexes

$$W_\ell A_k^\bullet[\ ], \ W_\ell L_{\mathbb{C}}[\ ], \quad F^p L_{\mathbb{C}}[\ ], \text{ etc.}$$

The de Rham complex of a cosimplicial cochain complex $C^\bullet[\ ]$ is the double complex $\bigoplus_{s,t} E^s(C^t[\ ])$, where $E^s(C^t[\ ])$ consists of those $(c_n)_{n\geq 0} \in \bigoplus C^t[n] \otimes E^s(\Delta^n)$ that satisfy the compatibility condition

$$(f_*\otimes id)c_n = (id\otimes |f|^*)c_m \quad \text{in } C^t[m]\otimes E^s(\Delta^n)$$

for all maps $f : [n] \to [m]$ in Δ.

$$\begin{tikzcd}
C^t[n] \otimes E^s(\Delta^n) \arrow[dr, "f_* \otimes id"] & \\
& C^t[m] \otimes E^s(\Delta^n) \\
C^t[m] \otimes E^s(\Delta^m) \arrow[ur, "id \otimes |f|^*"'] &
\end{tikzcd}$$

Define cochain complexes $A_k^\bullet(X_\bullet)$ and $L_{\mathbb{C}}^\bullet(X_\bullet)$ by

$$A_k^\bullet(X_\bullet) = E_{\mathbb{Q}}^\bullet(A_k^\bullet[\ ]), \quad L_{\mathbb{C}}^\bullet(X_\bullet) = E^\bullet(L_{\mathbb{C}}^\bullet[\ ]).$$

Note that both are sub d.g.a.'s of $E_{\mathbb{C}}^\bullet(X_\bullet)$. Define descending weight filtrations on them by

$$W^{\ell}A_{k}^{\bullet}(X_{\bullet}) = \bigoplus_{s\geq 0} E_{\mathbb{Q}}^{s}(W^{\ell-s}A_{k}^{\bullet}[\ \ ]) ,$$

$$W^{\ell}L_{\mathbb{C}}^{\bullet}(X_{\bullet}) = \bigoplus_{s\geq 0} E^{s}(W^{\ell-s}L_{\mathbb{C}}^{\bullet}[\ \ ]) \ .$$

and a Hodge filtration on $L_{\mathbb{C}}^{\bullet}(X_{\bullet})$ by

$$F^{p}L_{\mathbb{C}}^{\bullet}(X_{\bullet}) = E^{\bullet}(F^{p}L_{\mathbb{C}}^{\bullet}[\ \ ]) \quad .$$

For subsequent use we record the following easily proved result.

Proposition 5.4. If the Hodge and weight filtrations of each $A_{k}^{\bullet}(X_{n})$ and $L_{\mathbb{C}}^{\bullet}(X_{n})$ are multiplicative, then the Hodge and weight filtrations of $A_{k}^{\bullet}(X_{\bullet})$ and $L_{\mathbb{C}}^{\bullet}(X_{\bullet})$ are multiplicative. □

An important technical ingredient needed to prove that the triple constructed in 5.3 is a M.H.C. is the following lemma. Its proof is a slight abstraction of the proof of lemma 4.7 and is omitted.

Suppose that $C^{\bullet} : \underline{\underline{\mathrm{Diff}}} \to \underline{\underline{\mathrm{Ab}}}^{\bullet}$ is a contravariant functor. Suppose that $X_{\bullet}$ is a simplicial object in $\underline{\underline{\mathrm{Diff}}}$. Let $C^{\bullet}[\ \ ]$ be the associated cosimplicial cochain complex and let $sk_{n}C^{\bullet}[\ \ ]$ be the cosimplicial cochain complex associated to $sk_{n}X_{\bullet}$. The natural map $sk_{n}X_{\bullet} \to X_{\bullet}$ induces a map

$$E^{\bullet}(C^{\bullet}[\ \ ]) \to E^{\bullet}(sk_{n}C^{\bullet}[\ \ ])$$

Lemma 5.5. If $X_{\bullet}$ is a split simplicial object in $\underline{\underline{\mathrm{Diff}}}$, then the restriction mapping

$$E^{\bullet}(C^{\bullet}[\ \ ]) \to E^{\bullet}(sk_{n}C^{\bullet}[\ \ ])$$

is surjective. □

Note that in the above discussion, that $C^{\bullet}$ only has to be defined on some subcategory of Diff, such as Alg.

Our basic result is the following lemma.

Lemma 5.6. Suppose that $X_{\bullet}$ is a split simplicial object in TDiff such that the de Rham theorem is true for each X_n. If $(S^{\bullet}(X_n), (A^{\bullet}_{k}(X_n),W_{\bullet}), (L^{\bullet}_{\mathbb{C}}(X_n),W_{\bullet},F^{\bullet}))$ is as in the preceding discussion 5.3, then

$$(S^{\bullet}(\mathrm{re}X_{\bullet}), (A^{\bullet}_{k}(X_{\bullet}),W_{\bullet}), (L^{\bullet}_{\mathbb{C}}(X_{\bullet}),W_{\bullet},F^{\bullet}))$$

is a k-M.H.C..

Proof. The E_r term of the spectral sequence associated to the weight filtration of $L^{\bullet}_{\mathbb{C}}(X_{\bullet})$ will be denoted by $E_r^{\ell,m}(X_{\bullet})$. The E_r term of the spectral sequence associated to $L^{\bullet}_{\mathbb{C}}(X_n)$ will be denoted by $E_r^{\ell,m}[n]$.

Our first task is to show that the natural map

$$(*) \qquad E_0^{\ell,m}(X_{\bullet}) \to \bigoplus_{s\geq 0} E^s(E_0^{\ell-s,m}[\ \ ])$$

is an isomorphism. This follows from the next proposition by taking

$$F[n] = W^{\ell-s}L_{\mathbb{C}}^{\ell+m-s}(X_n) \text{ and } G[n] = W^{\ell-s+1}L_{\mathbb{C}}^{\ell+m-s+1}(X_n)\ .$$

Proposition 5.7. Suppose that $F[\ \ ]$ and $G[\ \ ]$ are cosimplicial abelian groups associated to a split simplicial object in Diff (or a convenient subcategory) via a contravariant functors $\underline{\mathrm{Diff}} \to \underline{\mathrm{Ab}}$. If $G[n] \subseteq F[n]$ for all n, then the natural map

$$E^s(F[\ \ ])\big/E^s(G[\ \ ]) \longrightarrow E^s(F/G[\ \ ])$$

is an isomorphism.

Proof. It is easily seen to be injective. To prove that it is surjective, it suffices to prove that

$$\phi : \mathcal{E}^s(\mathcal{F}[\ \ ]) \to \mathcal{E}^s(\mathcal{F}/\mathcal{G}[\ \ ])$$

is surjective.

Suppose that $(e_n)_{n\geq 0} \in \mathcal{E}^s(\mathcal{F}/\mathcal{G}[\ \ ])$. Note that

$$\mathcal{E}^s(\mathcal{F}[0]) \to \mathcal{E}^s(\mathcal{F}/\mathcal{G}[0])$$

is surjective. Suppose that $m > 0$ and that $(f_n)_{n<m} \in \mathcal{E}^s(sk_{m-1}\mathcal{F}[\ \ ])$ with $\phi_n f_n = e_n$ for $n < m$. By lemma 5.5,

$$\mathcal{E}^s(sk_m\mathcal{F}[\ \ ]) \to \mathcal{E}^s(sk_{m-1}\mathcal{F}[\ \ ])$$

is surjective. Let $(f_n)_{n\leq m}$ be an extension of $(f_n)_{n<m}$ to the m-skeleton. Now $(e_n - \phi_n f_n)_{n\leq m}$ lies in the kernel of

$$\mathcal{E}^s(sk_m\mathcal{F}/\mathcal{G}[\ \ ]) \to \mathcal{E}^s(sk_{m-1}\mathcal{F}/\mathcal{G}[\ \ ])$$

which is

$$\mathcal{E}^s(\Delta^m, \partial\Delta^m) \otimes \mathcal{F}/\mathcal{G}(N(X_m)) .$$

Since $\mathcal{F}(N(X_M)) \to \mathcal{F}/\mathcal{G}(N(X_m))$ is surjective, we can find $(\tilde{f}_n)_{n\leq m}$ in $\mathcal{E}^s(sk_m\mathcal{F}[\ \ ])$ such that

$$\phi(\tilde{f}_n) = (e_n - \phi_n f_n) \qquad (n\leq m)$$

in $\mathcal{E}^s(sk_m\mathcal{F}/\mathcal{G}[\ \ ])$

Let $f_{\hat{n}} = \tilde{f}_n + f_n$ $(n\leq m)$. Then $\phi(f_{\hat{n}})_{n\leq m} = (e_n)_{n\leq m}$ in $\mathcal{E}^s(sk_m\mathcal{F}/\mathcal{G}[\ \ ])$. □

Similarly, one can prove the following proposition.

Proposition 5.8. Suppose that $F^\bullet[\ ]$ is a cosimplicial cochain complex associated to a split simplicial object in Diff (or a subcategory) via a functor Diff $\to$ $\mathrm{Ab}^\bullet$. Then the natural map

$$\delta E^s(F^\bullet[\ ]) \to E^s(\delta F^\bullet[\ ])$$

is an isomorphism, where δ denotes the internal differential of $F^\bullet[\ ]$. Consequently,

$$H^*(E^\bullet(F^\bullet[\ ])) = E^\bullet(H^*(F^\bullet[\ ])). \quad \square$$

Our next job is to show that d_0 is strictly compatible with the Hodge filtration on $E_0(X_\bullet)$. First observe that, with respect to the isomorphism (*), $d_0^{\ell,m} = \bigoplus_{s\geq 0} d_0^{\ell-s,m}$. So, to prove that d_0 is strictly compatible with $F^\bullet$, we need only show that

$$\frac{d_0 E^s(E_0^{\ell-s,m}[\ ]) \cap E^s(F^p E_0^{\ell-s,m}[\ ])}{d_0 E^s(F^p E_0^{\ell-s,m}[\ ])} = 0$$

for all $s \geq 0$. By proposition 5.8, the left hand side of this expression is

$$\frac{E^s(d_0 E_0^{\ell-s,m} \cap F^p E_0^{\ell-s,m}[\ ])}{E^s(d_0 F^p E_0^{\ell-s,m}[\ ])} \quad .$$

But, by proposition 5.7, this last expression is isomorphic to

$$E^s\left(\frac{d_0 E_0^{\ell-s,m} \cap F^p E_0^{\ell-s,m}}{d_0 F^p E_0^{\ell-s,m}}[\ ]\right) \quad .$$

However, since each $(L^\bullet_{\mathbb{E}}(X_n), W_\bullet, F^\bullet)$ is part of a M.H.C., this quotient is zero, as required.

The next step is to show that

$$E_1^{\ell,m} = \bigoplus_{s\geq 0} \mathcal{E}^s(E_1^{\ell-s,m}[\;]) \ .$$

The proof of this uses 5.7 and 5.8 and is omitted. It follows that $E_1^{\ell,m}$ has a Hodge structure of weight m.

Similarly, one can show that the E_1 term of the weight spectral sequence associated to $(A_k^\bullet(X_\bullet)\otimes\mathbb{C},W_\bullet)$ is isomorphic to the E_1 term of the weight spectral sequence associated to $(L_{\mathbb{C}}^\bullet,W_\bullet)$ calculated above. That is, $(A_k^\bullet(X_\bullet)\otimes\mathbb{C},W_\bullet)$ and $(L_{\mathbb{C}}^\bullet(X_\bullet),W_\bullet)$ are quasi-isomorphic.

Finally, since the de Rham theorem is true for each X_n, it is true for re $X_\bullet$ by theorem 4.5. A routine spectral sequence argument using the skeleton filtration can be used to show that the inclusion

$$A_k^\bullet(X\;) \to S^\bullet(X_\bullet,k)$$

induces an isomorphism on cohomology. This completes the proof of 5.6. □

<u>Corollary</u> 5.9. If $X_\bullet$ is a split simplicial object in <u>Alg</u> and $Y_\bullet$ is a completion of $X_\bullet$ such that $Y_\bullet - X_\bullet$ is a divisor $D_\bullet$ in $Y_\bullet$ with normal crossings, then the triple

$$(S^\bullet(X_\bullet),\ (A_{\mathbb{R}}^\bullet(Y_\bullet,D_\bullet),W_\bullet),\ (\mathcal{E}^\bullet(Y_\bullet\log D_\bullet),W_\bullet,F^\bullet)) \ ,$$

constructed in 5.3, is an $\mathbb{R}$-M.H.C.. Moreover, all the filtrations are multiplicative. □

<u>Remarks</u> 5.10

1) At this point of the paper, we could appeal to Morgan's results [38] to deduce that the homotopy groups of a quasi-projective variety have a natural $\mathbb{R}$-M.H.S.

2) It should be possible to define the weight filtration, of the complexes constructed in 5.3, over $\mathbb{Q}$. Morgan [38] has shown that this is the case when V is smooth. If V is complete and there is a hypercovering $X_\bullet \to V$ (or indeed, any nice resolution of V with the property that $|X_\bullet| \to V$ is a homotopy equivalence) where $reX_\bullet$ is smoothly triangulable, then the weight filtration on homotopy is defined over $\mathbb{Q}$. To see this replace $E^\bullet_{\mathbb{R}}(X_\bullet)$ by the rational polynomial forms on the triangulation of $reX_\bullet$. For example, using Carlson's resolutions [58], one can see that for complete surfaces, and divisors with normal crossings, the weight filtration on rational homotopy is defined over $\mathbb{Q}$.

There are two difficulties that one encounters when trying to define the weight filtration over $\mathbb{Q}$. First, not all algebraic maps are triangulable. This makes it hard to triangulate hypercoverings. The second is to find a $\mathbb{Q}$-M.H.C. of forms for a smooth variety that is functorial with respect to algebraic maps, Unfortunately, Morgan's complex is not quite functorial.

3) The M.H.S. that this complex defines on the cohomology of a quasi projective variety V agrees with that of Deligne [12]. To see this, choose a split hypercovering $(Y_\bullet - D_\bullet) \to V_\bullet$ Denote $E^\bullet(Y_n \log D_n)$ by $L^\bullet(X_n)$.

As remarked earlier, $L^\bullet(X_\bullet)$ is a double complex $\oplus L^{s,t}$, where $L^{s,t} = E^s(L^t[\ ])$. Deligne's complex is the double complex $A^{s,t}$ associated with the cosimplical cochain complexes $L^t[\ ]$. Specifically, $A^{s,t} = L^t(X_s)$ with the combinatorial differential $\delta: A^{s,t} \to A^{s+1,t}$ and the internal differential $d: A^{s,t} \to A^{s,t+1}$ (see section 2). The Hodge and weight filtrations on $L^{s,t}$ and $A^{s,t}$ are defined by

$$F^pL^{s,t} = \mathcal{E}^s(F^pL^\bullet[\quad]),\ W^\ell L^{s,t} = \mathcal{E}^s(W^{\ell-s}L^\bullet[\quad]),$$

$$F^pA^{s,t} = F^pL^t(X_s),\ W^\ell A^{s,t} = W^{\ell-s}L^t(X_s).$$

Define a linear map $\Phi: L^{s,t} \to A^{s,t}$ as follows: A typical element of $L^{s,t}$ is of the form $\omega = \sum_j \eta_j \otimes \omega_j$, where, for each m, the restriction of ω to Δ^m is $\sum_j \eta_j^m \otimes \omega_j^m$, with $\eta_j^m \in \mathcal{E}^s(\Delta^m)$ and $\omega_j^m \in L^t(X_m)$. Define Φ by

$$\Phi(\omega) = \sum\left(\int_{\Delta^s}\eta_j\right)\omega_j = \sum\left(\int_{\Delta^s}\eta_j^s\right)\omega_j^s.$$

Using Stokes' theorem and the fact that ω satisfies the compatibility conditions, it is straightforward to check that Φ commutes with the differential of bidegree $(1,0)$. It is easy to see that Φ commutes with the other differential and that Φ preserves both the Hodge and the weight filtrations. Consequently, Φ induces a morphism of M.H.S's

$$\Phi_*: H^*(L^\bullet(X_\bullet)) \to H^*(A^{\bullet\bullet}),$$

which is necessarily an isomoprhism, as both complexes have cohomology isomoprhic to $H^*(V;\mathbb{C})$.

6. Review of Homotopy Theory

Let X be a topological space having base point $*$. The loop space of $(X,*)$ is the set

$$\Omega_* X = \{\gamma : [0,1] \to X : \gamma \text{ is continuous and } \gamma(0) = \gamma(1) = *\}$$

endowed with the compact-open topology. It has a natural base point, namely the constant loop η_* at $*$ which is defined by $\eta_*(t) = *$ for all t. A function $u : U \to \Omega_* X$ is continuous if and only if its suspension

$$\begin{aligned} \hat{u} : ([0,1]\times U, \{0,1\}\times U) &\to (X,*) \\ (t,\xi) &\longmapsto u(\xi)(t) \end{aligned}$$

is continuous. In particular, taking U to be the p-sphere S^p, we obtain a one to one correspondence between continuous maps $f : (S^p,*) \to (\Omega_* X,\eta_*)$ and maps $\hat{f} : ([0,1]\ S^p, \{0,1\}\ S^p) \to (X,*)$. This inturn induces, for all $p \geq 0$, an isomorphism

$$\hat{(\ )} : \pi_p(\Omega_* X,\eta_*) \to \pi_{p+1}(X,*)$$

which we shall call the suspension isomorphism. The base points $*$ and η_* will usually be omitted.

There is a natural $\mathbb{Z}$-bilinear pairing

$$[\ ,\ ] : \pi_{p+1}(X) \otimes \pi_{q+1}(X) \to \pi_{p+q+1}(X)$$

called the Whitehead product (see [52].) Identifying $\pi_{p+1}(X)$ with $\pi_p(\Omega X)$ via the suspension isomorphism, we obtain a pairing

$$[\ ,\ ] : \pi_p(\Omega X) \otimes \pi_q(\Omega X) \to \pi_{p+q}(\Omega X).$$

This pairing gives $\pi_*(\Omega X)$ the structure of a graded Lie ring over $\mathbb{Z}$. That is, if $\alpha \in \pi_p(\Omega X)$, $\beta \in \pi_q(\Omega X)$, $\gamma \in \pi_r(\Omega X)$, then

$$[\alpha,\beta] + (-1)^{pq}[\beta,\alpha] = 0$$

$$[\alpha,[\beta,\gamma]] = [[\alpha,\beta],\gamma] + (-1)^{pq}[\beta,[\alpha,\gamma]].$$

(For a proof of these identities, see [52].)

The singular homology of ΩX has an associative (but generally noncommutative) product

$$H_p(\Omega X)\otimes H_q(\Omega X) \to H_{p+q}(\Omega X)$$

induced by the map $\Omega X \times \Omega X \to \Omega X$ that takes a pair of loops (γ,μ) to their product $\gamma\cdot\mu$. It is called the Pontrjagin product. A basic fact, see [52], is that if $\alpha \in \pi_p(\Omega X)$ and $\beta \in \pi_q(\Omega X)$, then

$$h([\alpha,\beta]) = h(\alpha)\,h(\beta) - (-1)^{pq}h(\beta)\,h(\alpha)\ ,$$

where $h : \pi_*(\Omega X) \to H_*(\Omega X)$ denotes the Hurewicz homomorphism. Thus $H_*(\Omega X)$ looks like the universal enveloping algebra of $\pi_*(\Omega X)$. This is the case when we tensor everything in sight by $\mathbb{Q}$.

To make the last remark precise, recall that $H_*(\Omega X,\mathbb{Q})$ is a graded Hopf algebra. That is, in addition to its algebra structure, $H_*(\Omega X;\mathbb{Q})$ has a coproduct, or diagonal

$$\Delta : H_r(\Omega X;\mathbb{Q}) \to \bigoplus_{p+q=r} H_p(\Omega X;\mathbb{Q})\otimes H_q(\Omega X;\mathbb{Q})$$

which is induced by the diagonal map $\Omega X \to \Omega X \times \Omega X$ and is dual to the cup product on cohomology. The product and coproduct of a Hopf algebra are required to satisfy some natural compatibility conditions. For a precise definition of a graded Hopf algebra, see [37].

Another example of a graded Hopf algebra is the universal enveloping algebra $U\mathfrak{g}$ of a graded Lie algebra $\mathfrak{g}$ (over $\mathbb{Q}$). The coproduct $\Delta : U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$ is induced by the diagonal map $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$: the enveloping algebra of $\mathfrak{g} \oplus \mathfrak{g}$ is $U\mathfrak{g} \otimes U\mathfrak{g}$ and the natural inclusion takes (x,y) to $x \otimes 1 + 1 \otimes y$. Thus, if $x \in \mathfrak{g}$, then $\Delta x = x \otimes 1 + 1 \otimes x$.

<u>Theorem 6.1</u> (Borel-Serre [37]). If X is a simply connected space having the homototopy type of a CW-complex, then the rational Hurewicz homomorphism

$$h : \pi_*(\Omega X) \otimes \mathbb{Q} \to H_*(\Omega X; \mathbb{Q})$$

induces a natural Hopf algebra isomorphism

$$U\pi_*(\Omega X) \otimes \mathbb{Q} \to H_*(\Omega X; \mathbb{Q})$$

In particular, the rational Hurewicz homomorphism is injective. □

In fact the Hopf algebra structure of $H_*(\Omega X;\mathbb{Q})$ determines the Lie algebra structure of $\pi_*(\Omega X) \otimes \mathbb{Q}$. A graded Hopf algebra A is <u>cocommutative</u> if the diagram

$$\begin{tikzcd} & A \otimes A \arrow[d, "\tau"] \\ A \arrow[ur, "\Delta"] \arrow[r, "\Delta"'] & A \otimes A \end{tikzcd}$$

commutes, where τ is the interchange map $a \otimes b \longmapsto (-1)^{\deg a . \deg b}\, b \otimes a$. Examples of cocommutative Hopf algebras include $H_*(\Omega X;\mathbb{Q})$ and enveloping algebras of graded Lie algebras. For a cocommutative Hopf algebra A, the set of its <u>primitive elements</u>

$$PA = \{x \in A : \Delta x = x \otimes 1 + 1 \otimes x\}$$

is a graded Lie algebra with bracket $[x,y] = xy - (-1)^{\deg x \cdot \deg y} yx$.

Theorem 6.2 (Milnor-Moore [37]).

(a) If A is a cocommutative Hopf algebra over a field of characteristic zero, then there is a natural isomorphism

$$UPA \longrightarrow A.$$

(b) If $\mathfrak{g}$ is a graded Lie algebra over a field of characteristic zero, then there is a natural isomorphism

$$\mathfrak{g} \longrightarrow PU\mathfrak{g} . \qquad \Box$$

Corollary 6.3. If k is a field of characteristic zero and if X is a simply connected space with the homotopy type of a CW-complex, then the Hurewicz homomorphism induces a natural Lie algebra isomorphism

$$\pi_*(\Omega X) \otimes k \to PH_*(\Omega X; k). \qquad \Box$$

When X is simply connected and has finite Betti numbers, ΩX also has finite Betti numbers (see [50]). The rational cohomology $H^*(\Omega X;\mathbb{Q})$ is then the dual of $H_*(\Omega X;\mathbb{Q})$ and is also a Hopf algebra. The inclusion of the base point $\{\eta_*\} \to \Omega X$ induces an augmentation $H^*(\Omega X;\mathbb{Q}) \to \mathbb{Q}$. Since X is simply connected, its kernel is

$$\bigoplus_{p>0} H^p(\Omega X;\mathbb{Q}) ,$$

which we denote by $IH^*(\Omega X;\mathbb{Q})$. Dual to the primitive elements of $H_*(\Omega X;\mathbb{Q})$ is

$$IH^*(\Omega X;\mathbb{Q}) \big/ IH^*(\Omega X;\mathbb{Q}) \wedge IH^*(\Omega X;\mathbb{Q}) ,$$

the set of <u>indecomposables</u> of $H^*(\Omega X;\mathbb{Q})$, which we denote by $QH^*(\Omega X;\mathbb{Q})$. The coproduct of $H^*(\Omega X;\mathbb{Q})$ induces a Lie coproduct $\Delta : Q \to Q \otimes Q$ (see [24], p. 65).

<u>Corollary</u> 6.4. If k is a field of characteristic zero and if X is a simply connected space with finite Betti numbers and the homotopy type of a CW-complex, then there is a natural isomorphism

$$QH^*(\Omega X;k) \to \mathrm{Hom}(\pi_{p+1}(X),k).$$

Moreover, the Lie coproduct of $QH^*(\Omega X;k)$ is dual to the Whitehead product. □

For spaces that are not simply connected, there is an analogue of the preceding discussion. In it $H_*(\Omega X;\mathbb{Q})$ is replaced by the graded ring $\mathrm{gr}\,\mathbb{Q}\,\pi_1(X)$ associated to the filtration

$$\mathbb{Q}\pi_1(X) \supseteq J \supseteq J^2 \supseteq \dots$$

of the rational group ring of $\pi_1(X)$ by the powers of its augmentation ideal J. The analogue of $\pi_*(\Omega X)\otimes\mathbb{Q}$ is the graded Lie algebra $(\mathrm{gr}\,\pi_1(X))\otimes\mathbb{Q}$ associated to the lower central series filtration of $\pi_1(X)$. This is explained below in detail.

It is easy to see that, for any ring R, there is a natural isomorphism

$$H_0(\Omega_* X;R) \simeq R\pi_1(X,*)$$

of R-algebras that commute with the usual augmentations

$$\varepsilon : H_0(\Omega_* X;R) \to R \quad\text{and}\quad \varepsilon : R\pi_1(X,*) \to R.$$

Under this isomorphism the coproduct

$$\Delta : H_0(\Omega_* X;R) \to H_0(\Omega_* X;R) \otimes H_0(\Omega_* X;R)$$

induced by the diagonal $\Omega_* X \to \Omega_* X \times \Omega_* X$ corresponds to the standard coproduct

$$\Delta : R\pi_1(X,*) \to R\pi_1(X,*) \otimes R\pi_1(X,*)$$

defined by $g \longmapsto g \otimes g$ for all $g \in \pi_1(X,*)$.

If π is any (discrete) group, then the powers J^s $(s \geq 1)$ of the augmentation ideal J $(= \ker \varepsilon)$ of the rational group ring $\mathbb{Q}\pi$ of G define a topology on $\mathbb{Q}\pi$, called the J-adic topology. Denote the J-adic completion of $\mathbb{Q}\pi$ by $\mathbb{Q}\pi\hat{}$ and the completion of J by $\hat{J}$. The reason for completing $\mathbb{Q}\pi$ is that

$$\log : 1 + \hat{J} \to \hat{J} \quad \text{and} \quad \exp : \hat{J} \to 1 + \hat{J}$$

are now defined. The standard diagonal $\Delta : \mathbb{Q}\pi \to \mathbb{Q}\pi \otimes \mathbb{Q}\pi$ extends to a diagonal

$$\hat{\Delta} : \mathbb{Q}\pi\hat{} \to \mathbb{Q}\pi\hat{} \,\hat{\otimes}\, \mathbb{Q}\pi\hat{} \ ,$$

where $\hat{\otimes}$ denotes completed tensor product (see appendix A of [45].) The set $\mathcal{G}$ of <u>group like elements</u> is defined by

$$\mathcal{G} = \{x \in 1 + \hat{J} \ : \ \hat{\Delta}x = x \hat{\otimes} x\}$$

It forms a group under multiplication. The set $\mathcal{P}$ of <u>primitive elements</u> is defined by

$$\mathcal{P} = \{x \in \mathbb{Q}\pi\hat{} \ : \ \hat{\Delta}x = x \hat{\otimes} 1 + 1 \hat{\otimes} x\} \ .$$

It is a Lie algebra with the commutator as bracket. The functions exp and log define bijections

$$\log : G \to P \quad \text{and} \quad \exp : P \to G$$

(see appendix A of [45].) Both G and P have natural filtrations

$$G = G^1 \geq G^2 \geq \dots$$
$$P = P^1 \geq P^2 \geq \dots$$

defined by $G^s = G \cap (1 + \hat{J}^{s+1})$ and $P^s = P \cap \hat{J}^s$. Denote the lower central series filtration of π by

$$\pi = \pi^1 \geq \pi^2 \geq \dots \quad .$$

The natural inclusion of π into $\mathbb{Q}\pi$ induces group homomorphisms $\pi^s \to G^s$.

The following theorems are essentially due to Malcev [36]. Proofs of them, as stated, may be found in [10], [44] and appendix A of [45].

<u>Theorem</u> 6.5. The functions log and exp induce a Lie algebra isomorphism

$$\mathrm{gr}\, G \approx \mathrm{gr}\, P \ .$$

If π is finitely presented, then there is a natural isomorphism

$$(\mathrm{gr}\,\pi) \otimes \mathbb{Q} \approx \mathrm{gr}\, G \qquad \square$$

<u>Theorem</u> 6.6. If π is finitely presented, then G/G^{s+1} is a nilpotent Lie group over $\mathbb{Q}$ with Lie algebra P/P^{s+1}. If G_s denotes the real form of

G/G^{s+1}, then G_s is a simply connected, nilpotent, real Lie group and the quotient space $\pi\backslash G_s$ is a compact nilmanifold with fundamental group π/D^{s+1}, where

$$D^s = \{g \in \pi \;\; g-1 \in J^s\} \qquad \square$$

The group π/D^s is a torsion free nilpotent group and G_s is its Malcev completion. The group G is called the Malcev completion of π.

Corollary 6.7. If X is a space having the homotopy type of a CW-complex with finite Betti numbers, then the Hopf algebra structure of $H_0(\Omega_* X;\mathbf{Z})$ determines $\mathbf{Z}\pi_1(X,*)$ and the Malcev completion of $\pi_1(X,*)$. $\square$

7. Homotopy Versus Smooth Homotopy

Let X be a topological differentiable space. In this section we give conditions under which the natural map

$$\pi_k(X) \to \pi_k(\tau X)$$

is an isomorphism. These and other results in this section allow us to state cleaner versions ((8.2) (a), (8.2) (b)) of Chen's loop space de Rham theorems (2.3.1, [11] and 5.3, [9]). The philosophy behind the proofs in this section is the same as that behind 4.1: many constructions used to prove theorems about homotopy and singular homology in Top are valid in Diff and can be used to prove analogous results there.

First we clarify a minor technicality that arises in the proofs of several results in this section. We have been viewing $\partial\Delta^n$ as a differentiable subspace of $\Delta^n (\subseteq \mathbb{R}^{n+1})$. However, using this point of view makes it hard to determine when a function $f : \partial\Delta^n \to M$ is smooth. If the natural map

$$\coprod_{j=0}^{n} \Delta^{n-1} \to \partial\Delta^n$$

were a quotient map, then $f : \partial\Delta^n \to M$ would be smooth if and only if the restriction of f to each face Δ^{n-1} of Δ^n were smooth. This is in fact the case.

Proposition 7.1. If $\Delta(n)$ denotes the simplicial set that corresponds to the standard n-simplex (i.e., $\Delta(n)_m = \{f : [m] \to [n]\}$), then the natural map

$$\Delta^n \to \mathrm{re}\Delta(n)$$

is a diffeomorphism.

Proof. Let $\xi_n : [n] \to [n]$ be the identity. The composite

$$\Delta^n \to \{\xi_n\} \times \Delta^n \to \coprod_{m\geq 0} \Delta(n)_m \times \Delta^m \to \mathrm{re}\Delta(n)$$

is a smooth bijection. On the other hand each element $\sigma : [m] \to [n]$ of $\Delta(n)_m$ defines a smooth map $|\sigma| : \Delta^m \to \Delta^n$. These induce a smooth map

$$\varphi : \coprod_{m\geq 0} \Delta(n)_m \times \Delta^m \to \Delta^n$$

which inturn induces a smooth map $\mathrm{re}\Delta(n) \to \Delta^n$. $\square$

Corollary 7.2. The natural map

$$\mathrm{re\,sk}_{n-1}\,\Delta(n) \to \partial\Delta^n$$

is a diffeomorphism. Consequently, the natural map

$$\coprod_{j=0}^{n} \Delta^{n-1} \to \partial\Delta^n$$

is a quotient map in Diff.

Proof. To prove the first assertion we need only check that $\mathrm{re\,sk}_{n-1}\Delta(n)$ has the subspace differentiable structure induced by the inclusion $\mathrm{re\,sk}_n\Delta(n) \to \mathrm{re}\Delta(n)$. This follows immediately from the fact that

$$\mathrm{re}\Delta(n) = (\mathrm{re\,sk}_{n-1}\Delta(n)) \cup_\partial \Delta^n.$$

The second assertion follows from 7.1 and the fact that

$$\coprod_{j=0}^{n} \Delta(n-1) \to sk_{n-1}\Delta(n)$$

is a quotient map in $(\underline{\underline{\text{Set}}})^{\Delta}$. □

Let X be a differentiable space and * a point in X. Denote by Ω_*X (or, sometimes ΩX) the space of piecewise smooth loops $\gamma : ([0,1],\{0,1\}) \to (X,*)$ on X endowed with its usual differentiable space structure (see [11], p. 838 or [24], p. 40).

A <u>pair of differentiable spaces</u> (X,A) consists of a differentiable space X and a differentiable subspace A of X. A <u>pointed pair</u> (X,A,*) of differentiable spaces is a pair of differentiable spaces (X,A) and a point * in A.

If (X,A,*) is a pointed pair of differentiable spaces, then the natural map $\Omega_*A \to \Omega_*X$ is the inclusion of a subspace, so that $(\Omega_*X,\Omega_*A,\eta_*)$ is also a pointed pair. Here η_* denotes the constant loop at *. Define $\pi_k(X,A,*)$ to be the set of smooth homotopy classes of smooth maps

$$(I^{k-1},\partial I^{k-1},I^{k-2}\times\{0\} \cup \partial I^{k-2}\times I) \to (\Omega_*X,\Omega_*A,\eta_*),$$

where I denotes the unit interval. The H-space structure of Ω_*X gives $\pi_k(X,A,*)$ the structure of a group when $k \geq 2$. When $k > 2$, this group is abelian. We shall denote $\pi_k(X,*,*)$ by $\pi_k(X,*)$. As usual, $\pi_1(X,*)$ is a group. For each pointed pair (X,A,*) of differentiable spaces, there is a long exact sequence

$$\ldots \to \pi_{k+1}(X,A,*) \to \pi_k(A,*) \to \pi_k(X,*) \to \pi_k(X,A,*) \to \ldots$$

which is functorial in the pair (X,A,*).

The pointed pair $(X,A,*)$ of differentiable spaces is said to be n-connected ($n \geq 1$) if X and A are both path connected, as differentiable spaces, and if $\pi_k(X,A,*) = 0$ whenever $k \leq n$.

If $(X,A,*)$ is a pair of topological differentiable spaces (i.e., $(X,A,*)$ is a pointed pair, both in Top and Diff), then there is a natural map

$$\pi_*(X,A,*) \to \pi_*(\tau X,\tau A,*)$$

Suppose that $(X,A,*)$ is a pair of pointed differentiable spaces. For each $k \geq 0$, define chain complexes $S^{(k)}_\bullet(X,A)$ by defining $S^{(k)}_p(X,A)$ to be the free abelian group generated by the smooth singular simplices $\sigma : \Delta^p \to X$ which map the vertices of Δ^p to $*$ and the k-skeleton of Δ^p into A.

Lemma 7.3. If $(X,A,*)$ is an n-connected pair of pointed differentiable spaces and if $0 \leq k \leq n$, then the inclusion

$$S^{(k)}_\bullet(X,A) \to S_\bullet(X)$$

induces an isomorphism on homology.

Proof. The proof follows directly from (lemma 7, [50], p. 392), the proof of (theorem 8, [50], p. 392) and the following.

Using the techniques in the proof of (proposition 8.4, [24]) and 7.2, one can show that the pairs

$$(\Delta^q\times\{0\} \cup \partial\Delta^q\times I, \partial\Delta^q\times\{1\}) \text{ and } (E^q, S^{q-1})$$

are smoothly homotopy equivalent, so that the natural map

$$\pi_*(E^q, S^{q-1}) \to \pi_*(\tau E^q, \tau S^{q-1})$$

is an isomorphism. Consequently

$$\pi_k(\Delta^q \times \{0\} \cup \partial\Delta^q \times I, \partial\Delta^q \times \{1\}) = 0$$

whenever $0 \leq k < q$. □

Corollary 7.4. Suppose that X is a topological differentiable space with base point * and suppose that τX is connected. If the natural map $H_*(X) \to H_*(\tau X)$ is an isomorphism, then the inclusion

$$S_\bullet^{(0)}(X) \to S_\bullet(X)$$

induces an isomorphism on homology.

Proof. Since τX is connected and $H_0(X) = H_0(\tau X)$, it follows that X is 0-connected (i.e., every pair of points of X can be joined by a smooth path). The result now follows from 7.3. □

By "smoothing" the arguments on pp. 394 - 397 of [50], one can prove the following smooth version of the Hurewicz isomorphism theorem.

Theorem 7.5. Suppose that $(X,A,*)$ is a pair of pointed differentiable spaces. If $\pi_1(A,*)$ acts trivially on $\pi_*(X,A,*)$ and if $(X,A,*)$ is n-connected with $n \geq 2$, then $H_q(X,A) = 0$ when $q < n$ and $\pi_n(X,A,*) \approx H_n(X,A)$.

Conversely, if X and A are simply connected and $H_q(X,A) = 0$ when $q < n$, then $\pi_q(X,A,*) = 0$ for $q < n$ and $\pi_n(X,A,*) \approx H_n(X,A)$. □

For us a useful result is the following.

Corollary 7.6. Suppose that X is a connected topological differentiable space with base point *. Suppose that $\pi_1(X,*) = \pi_1(\tau X,*) = 0$. If the natural map $H_*(X) \to H_*(\tau X)$ is an isomorphism, then the natural map

$$\pi_*(X) \to \pi_*(\tau X)$$

is an isomorphism.

Proof. Denote the natural map $X \to \tau X$ by f. The mapping cylinder C(f) of f is the unique topological differentialbe space such that

$$\begin{array}{ccc} X \times \{0\} \cup \{*\} \times [0,1] & \hookrightarrow & X \times [0,1] \\ {\scriptstyle f}\downarrow & & \downarrow \\ \tau X & \longrightarrow & C(f) \end{array}$$

is a pushout in TDiff. As expected, τX and C(f) are smoothly homotopy equivalent and X is a subspace of C(f). Since $H_*(X) \to H_*(\tau X)$ is an isomorphism, $H_*(C(f),X) = 0$. Since X and C(f) are simply connected, it follows from 7.5 that $\pi_*(C(f),X) = 0$. From the homotopy long exact sequence, it follows that

$$\pi_*(X) \approx \pi_*(C(f)) \approx \pi_*(\tau X). \quad \square$$

As one would expect, for each differentiable space X, there is a natural isomorphism

$$\pi_{k+1}(X,*) \approx \pi_k(\Omega X,\eta_*).$$

Corollary 7.7. Suppose that X is a connected topological differentiable space with base point $*$. Suppose that $\pi_1(X,*) = \pi_1(\tau X,*) = 0$. If the natural map $H_*(X) \to H_*(\tau X)$ is an isomorphism, then the natural map

$$H_*(\Omega_* X) \to H_*(\Omega_* \tau X)$$

is an isomorphism.

Proof. According to 7.6, the natural map $\pi_*(X,*) \to \pi_*(\tau X,*)$ is an isomorphism. Consequently, the natural map

$$(\dagger) \qquad \pi_*(\Omega_* X,\eta_*) \to \pi_*(\Omega_* \tau X,\eta_*)$$

is also an isomorphism. Let C denote the mapping cylinder of the map

$$\Omega_* X \to \Omega_* \tau X.$$

As in 7.6, $\Omega_* \tau X$ is a smooth deformation retraction of C and $\Omega_* X$ is a differentiable subspace of C. It follows from (†) that $\pi_k(C,\Omega_* X,\eta_*) = 0$ for all k and 7.5 (a) now implies that

$$H_*(C,\Omega_* X) = 0.$$

The result follows from the fact that $H_*(C)$ is isomorphic with $H_*(\Omega_* \tau X)$. □

It should now be apparent that we need to understand when the natural map

$$\pi_1(X) \to \pi_1(\tau X)$$

is an isomorphism, both in the case when $\pi_1(\tau X) = 0$ (c.f. 7.6 and 7.7) and when τX is not simply connected.

Theorem 7.8. If $X_\bullet \to V$ is a hypercovering in Alg, then the natural map

$$\pi_1(reX_\bullet) \to \pi_1(|\tau X_\bullet|)$$

is an isomorphism.

Proof. According to 3.1, the augmentation induces an isomorphism $\pi_1(|\tau X_\bullet|) \to \pi_1(V)$. So it suffices to prove that the natural map $\pi_1(reX_\bullet) \to \pi_1(V)$ is also an isomorphism. In fact, to prove this, we only need to prove the smooth version of lemma 3.2: each element of $\pi_1(reX_\bullet,*')$ can be represented by a path of the form $\gamma_1 \cdot \mu_1 \cdot \gamma_2 \cdot \ldots \cdot \mu_m$, where each γ_j is a piecewise real analytic path in X_0 and where μ_j is a path in $reX_\bullet$ of the form $p_j \times [0,1]$ for some $p_j \in X_1$. We will now prove this.

As in the proof of 3.2, every smooth path $([0,1],\{0,1\}) \to (reX_\bullet,*')$ can be deformed, via a smooth homotopy, to a piecewise smooth path in $X_0 \cup X_1 \times [0,1]$ and that $\gamma^{-1}(X_1 \times]1/3,2/3[)$ is a disjoint union of a countable number of open intervals $\{I_n : n \geq 1\}$ of which only a finite number $I_1, \ldots, I_k$ (say) have image that intersects $X_1 \times \{1/2\}$. Since $X_1 \times]0,1[$ is a smooth manifold and the restriction of γ to each $\bar{I}_j$ $(1 \leq j \leq k)$ is a smooth path, $\gamma|\bar{I}_j$ can be smoothly homotoped to a smooth path whose intersection with $X_1 \times]1/2-\delta_j, 1/2+\delta_j[$ is of the form $p_j \times]1/2-\delta_j, 1/2+\delta_j[$, for some $\delta_j > 0$.

Pick a smooth function $\zeta : [0,1] - \{1/2\} \to [0,1/3] \cup [2/3,1]$ such that $\zeta(0) = 0$, $\zeta(1) = 1$, $\zeta^{-1}(1/3) = [1/3, 1/2[$, $\zeta^{-1}(2/3) =]1/2,2/3]$. Define a smooth map

$$g : X_0 \cup X_1 \times ([0,1]-\{1/2\}) \to X_0 \cup X_1 \times ([0,1/3] \cup [2/3,1])$$

by defining it to be the identity on X_0 and $\mathrm{id} \times \zeta$ on $X_1 \times ([0,1]-\{1/2\})$. One can easily check that g is smoothly homotopic to the identity map of $X_1 \times ([0,1]-\{1/2\})$. Thus, the restriction of γ to $[0,1] - \bigcup_{j=1}^{k} I_j$ can be smoothly homotoped to the path $g\circ(\gamma|[0,1] - \bigcup_{j=1}^{k} I_j)$.

It follows from the previous two paragraphs that γ can be smoothly homotoped to a piecewise smooth path $\tilde{\gamma}$ whose image intersects $X_1 \times]1/2-\delta,1/2+\delta[$ in the set $\{p_1, \ldots, p_k\} \times]1/2-\delta,1/2+\delta[$, where $0 < \delta \leq 1/6$. Pick a smooth function $\varphi : [0,1] \to [0,1]$ such that $\varphi^{-1}(0) \supseteq [0,1/2-\delta]$ and $\varphi^{-1}(1) \supseteq [1/2+\delta,1]$. The function φ is smoothly homotopic to the identity via the maps $\varphi_s : t \mapsto st + (1-s)\varphi(t)$ $(0 \leq s \leq 1)$. The map φ extends naturally to a smooth map

$$\Phi : X_0 \cup X_1 \times [0,1] \to X_0 \cup X_1 \times [0,1]$$

which is smoothly homotopic to the identity. The path $\Phi\circ\tilde{\gamma}$ is smoothly homotopic to γ and intersects $X_1 \times]0,1[$ in the set $\{p_1, \ldots, p_k\} \times]0,1[$. Finally, it follows from the fact that X_0 has a semi analytic triangulation [29] and the simplicial approximation theorem that $\Phi\circ\tilde{\gamma}$ can be smoothly homotoped to a path of the required form. $\square$

8. Review of Iterated Integrals

This is a terse account of iterated integrals and Chen's de Rham theorems. More detailed references include [11] and [24].

Suppose that $A^\bullet$ is a (not necessarily commutative) differential graded algebra (d.g.a.) with augmentation ideal $IA^\bullet$. It is convenient to define

$$Ja = (-1)^{\deg a}\, a$$

for all $a \in A^\bullet$. The <u>bar construction</u> on $A^\bullet$ is the double complex $\oplus B^{s,t}$ where,

$$B^{-s,t} = (\otimes^s IA^\bullet)^t.$$

A typical element of $B^{-s,t}$ will be denoted $[a_1|a_2|\ldots|a_s]$, where $a_j \in IA^\bullet$. The external differential $d_E : B^{-s,t} \to B^{-s,t+1}$ is defined by

$$d_E : [a_1|\ldots|a_s] \mapsto \sum_{i=1}^{s} (-1)^i [Ja_1|\ldots|Ja_{i-1}|da_i|\ldots|a_s].$$

The internal differential $d_I : B^{-s,t} \to B^{-s+1,t}$ is defined by

$$d_I : [a_1|\ldots|a_s] \mapsto \sum_{i=1}^{s-1} (-1)^{i+1} [Ja_1|\ldots|Ja_{i-1}|Ja_i \wedge a_{i+1}|\ldots|a_s].$$

One can check that $d_Id_E + d_Ed_I = 0$. When we refer to the bar construction, we will mean the associated total complex $\mathcal{B}(A^\bullet) = (\oplus B^{s,t},\ d_I + d_E)$. With the diagonal

$$\Delta : \mathcal{B}(A^\bullet) \to \mathcal{B}(A^\bullet)\otimes\mathcal{B}(A^\bullet)$$

$$[a_1|\ldots|a_s] \mapsto \sum_{i=0}^{s} [a_1|\ldots|a_i]\otimes[a_{i+1}|\ldots|a_s],$$

$\mathcal{B}(A^\bullet)$ becomes a d.g. coalgebra. If $A^\bullet$ is also commutative[3], then equipped with the shuffle product

$$\wedge : \mathcal{B}(A^\bullet)\otimes\mathcal{B}(A^\bullet) \to \mathcal{B}(A^\bullet)$$

$$[a_1|\ldots|a_r]\otimes[a_{r+1}|\ldots|a_{r+s}] \mapsto \sum_\sigma \varepsilon(\sigma)[a_{\sigma(1)}|\ldots|a_{\sigma(r+s)}],$$

where σ ranges over all shuffles of type (r,s) and where $\varepsilon(\sigma)$ is ± 1, $\mathcal{B}(A^\bullet)$ becomes a d.g. Hopf algebra.

The second quadrant spectral sequence obtained from the filtration

$$\mathcal{B}^{-s}(A^\bullet) = \bigoplus_{u\leq s} B^{-u,v}$$

of $\mathcal{B}(A^\bullet)$ is called the <u>Eilenberg-Moore</u> <u>spectral</u> <u>sequence</u> and satisfies

$$E_1 = \mathcal{B}(H^*(A^\bullet)).$$

The loop space Ω_*X of the differentiable space X with base point $*$ is defined by

$$\Omega_*X = \{\gamma : [0,1] \to X : \gamma \text{ is piecewise smooth, } \gamma(0) = \gamma(1) = *\} .$$

The differentiable structure of Ω_*X is described in [24], p.40. As in section 6, we will often suppress the basepoint.

Since $E^\bullet(*) = \mathbb{R}$, restricting forms on X to the basepoint defines an augmentation $E^\bullet(X) \to \mathbb{R}$.

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3 Here we mean commutative in the graded sense: $a \wedge b = (-1)^{st}\, b \wedge a$, where $a \in A^s$, $b \in A^t$.

Iterated integrals on X are special forms on ΩX constructed from forms on X: if $w_1, \ldots, w_s$ are forms on X of degree ≥ 1, then $\int w_1 \ldots w_s$ is a form on ΩX of degree $-s + \sum \deg w_j$. (For details, see [11] or [24].) If $A^\bullet$ is a sub d.g.a. of $E^\bullet(X)$, then the iterated integrals $\int w_1 \ldots w_r$, where $w_j \in A^\bullet$, form a sub d.g.a. of $E^\bullet(\Omega X)$ that we shall denote by $\int A^\bullet$. In fact, the surjective map

$$\Phi : B(A^\bullet) \to \int A^\bullet$$

that takes $[w_1|\ldots|w_s]$ to $\int w_1 \ldots w_s$ is a d.g.a. map whose kernel is a Hopf ideal of $B(A^\bullet)$. It follows that $\int A^\bullet$ is a d.g. Hopf algebra. Give $\int A^\bullet$ the natural filtration:

$$N^{-s} \int A^\bullet = \Phi\left(B^{-s}(A^\bullet)\right).$$

For later use we record the following technical facts.

<u>Lemma</u> 8.1 ([11]). If $A^\bullet$ is a sub d.g.a. of $E^\bullet(X)$ such that $H^0(IA^\bullet) = \tilde{H}_{dR}(X) = 0$ and $H^1(A^\bullet) \to H^1_{dR}(X)$ is injective, then the map induced by Φ from the Eilenberg-Moore spectral sequence of $B(A^\bullet)$ to the E_1 term of the spectral sequence of $(\int A^\bullet, N^\bullet)$ is an isomorphism. In particular, Φ induces an isomorphism

$$H^*(B^{-s}(A^\bullet)) \to H^*(N^{-s} \int A^\bullet)$$

for all s, and if $H^1(A^\bullet) = 0$, then Φ induces a Hopf algebra isomorphism

$$H^*(B(A^\bullet)) \to H^*(\int A^\bullet). \qquad \square$$

The following statement of Chen's loop space de Rham theorems combines Chen's statements (2.3.1, [11] and §3, [9]) with the results of section 7.

Theorem 8.2 (Chen). Suppose that $(X,*)$ is a connected, pointed topological differentiable space with finite Betti numbers for which the de Rham theorem is true. Suppose that $A^\bullet$ is a sub d.g.a. of $E^\bullet(X)$ for which the inclusion induces an isomorphism on cohomology.

(a) If X is simply connected[4] then the integration map induces a natural Hopf algebra isomorphism

$$H^*(\int A^\bullet) \to H^*(\Omega_* X; \mathbb{R}).$$

(b) If $\pi_1(X,*) \to \pi_1(\tau X,*)$ is an isomorphism, then the integration map induces a coalgebra isomorphism

$$H^0(N^{-s} \int A) \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}\pi_1(X)/J^{s+1}, \mathbb{R}),$$

where J denotes the augmentation ideal of $\mathbb{Z}\pi_1(X,*)$. Moreover, the natural map

$$H^0(\int A^\bullet) \otimes \mathbb{R}\pi_1(X,*) \to \mathbb{R}$$

defined by integration is a pairing of Hopf algebras. □

Restriction of the iterated integrals on X to the constant loop η_* defines an augmentation $\epsilon : \int A^\bullet \to \mathbb{R}$ which in turn induces an augmentation $H^*(\int A^\bullet) \to \mathbb{R}$, the kernel of which will be denoted by $IH^*(\int A^\bullet)$. The set of indecomposable elements of $H^*(\int A^\bullet)$ is defined by

$$QH^*(\int A^\bullet) = IH^*(\int A^\bullet)/_{I^2H^*(\int A^\bullet)}$$

and has the structure of a Lie coalgebra ([24], p.65). The filtration $N^\bullet$ of $\int A^\bullet$ induces a filtration of $IH^*(\int A^\bullet)$ which induces the filtration

$$0 \subseteq Q^{-1}H^*(\int A^\bullet) \subseteq Q^{-2}H^*(\int A^\bullet) \subseteq \dots$$

4 That is, $\pi_1(X) = \pi_1(\tau X) = 0$.

of $QH^*(\int A^\cdot)$ by sub Lie coalgebras. We will usually lower indices and denote $Q^{-s}H^*(\int A^\cdot)$ by $Q_sH^*(\int A\cdot)$.

Corollary 8.3. With the assumptions of 8.2:

(a) If X is simply connected, then integration induces a natural Lie coalgebra isomorphism

$$QH^*(\int A^\cdot) \to \mathrm{Hom}(\pi_*(\Omega_*X), \mathbb{R}).$$

(b) If the map $\pi_1(X,*) \to \pi_1(\tau X,*)$ is an isomorphism, then the integration map induces a Lie coalgebra isomorphism

$$Q_sH^0(\int A^\cdot) \to \mathrm{Hom}(\mathfrak{g}_s, \mathbb{R}),$$

where $\mathfrak{g}_s$ denotes the Lie algebra of the Malcev completion of $\pi_1(X,*)/\Gamma^{s+1}$. (Here, Γ^{s+1} denotes the (s+1)-st term of the lower central series of $\pi_1(X,*)$.)

Proof. Assertion (a) is an immediate consequence of 6.4 and 8.2(a), while (b) follows directly from 8.2(b) and the following lemma with $H = H^0(\int A^\cdot)$ and $A = \mathbb{R}\pi_1(X,*)$. $\square$

Let k be a field of characteristic zero. Suppose that H is a commutative Hopf algebra over k with augmentation ideal I and that H has a bimultiplicative filtration

$$0 \subseteq N_1 \subseteq N_2 \subseteq \cdots$$

by finite dimensional subspaces. Denote the filtration induced on the set of indecomposables $Q = I/I^2$ of H by $Q_1 \subseteq Q_2 \subseteq \cdots$. Suppose that A is a cocommutative Hopf algebra over k with augmentation ideal J. Denote by $gr_J A$ the graded Hopf algebra associated to the filtration of A by the powers of J. Let

$$\langle , \rangle : H \otimes A \to k$$

be a pairing of Hopf algebras. That is,

$$\langle \Delta h, a \otimes b \rangle = \langle h, a \cdot b \rangle , \quad \langle h \otimes k, \Delta a \rangle = \langle h \cdot k, a \rangle$$

where $h, k \in H$ and $a, b \in A$.

Lemma 8.4. If for all $s \geq 0$ the pairing $\langle \; , \; \rangle$ satisfies $\langle N^s, J^{s+1} \rangle = 0$ and the induced pairing

$$N^s \otimes A/J^{s+1} \to k$$

is non-singular, and if the map $gr_J PA \to Pgr_J A$ is an isomorphism, then the induced pairing

$$Q_s \otimes PA/(PA \cap J^{s+1}) \to k$$

is non-singular.

Proof. First observe that the map

$$Q_s \to (PA/PA \cap J^{s+1})^*$$
$$\bar{h} \mapsto \langle h, - \rangle$$

induced by the pairing is surjective. (Here, $(-)^*$ denotes $\mathrm{Hom}(-, k)$.) For if $x \in PA$ and $\langle \bar{y}, x \rangle = 0$ for all $\bar{y} \in Q_s$, then $\langle y, x \rangle = 0$ for all $y \in I \cap N_s$. Since the pairing $I \cap N_s \otimes J/J^{s+1} \to k$ is non singular, this implies that $x \in PA \cap J^{s+1}$.

Next, consider the graded Hopf algebras $gr^N_\bullet H$ and $gr^\bullet_J A$. Denote the graded dual of $gr^\bullet_J A$ by $(gr^\bullet_J A)^*$. The pairing induces a Hopf algebra homomorphism

$$\varphi : gr^N_\bullet H \to (gr^\bullet_J A)^* .$$

Since the pairing

$$N_s \otimes A/J^{s+1} \to k$$

is non singular for all $s \geq 1$, φ is an isomorphism.

Since the map $I \to Q$ is strictly compatible with the filtration $N_\bullet$, the induced map $gr^N_\bullet I \to gr_\bullet Q$ is surjective. Since $gr^N_\bullet I = Igr^N_\bullet H$, the map $Qgr^N_\bullet H \to gr_\bullet Q$ is surjective. On the other hand, since $gr^\bullet_J PA = Pgr^\bullet_J A$, it follows that $Q(gr^\bullet_J A)^* = (gr^\bullet_J PA)^*$. Because the map $Q_s \to (PA/PA \cap J^{s+1})^*$ is surjective for all s, it follows that the composite

$$Qgr^N_\bullet H \to gr_\bullet Q \to (gr^\bullet_J PA)^* \to Q(gr^\bullet_J A)^*$$

is surjective. However, φ is an isomorphism, so it follows that the above composite is an isomorphism and that the map $gr_\bullet Q \to (gr^\bullet_J PA)^*$ is an isomorphism. Consequently,

$$Q_s \to (PA/PA \cap J^{s+1})^*$$

is an isomorphism for all s. $\square$

9. Mixed Hodge Structures on Homotopy Groups

In this section we show that the homotopy groups of a simply connected quasi-projective variety have a natural mixed Hodge structure (M.H.S.) and that the tower of nilpotent Lie algebras associated to the fundamental group of an arbitrary quasi-projective variety has a system of compatible M.H.S.'s. Granted 3.1 and 5.9, these results follow directly from Morgan's results (section 8, [38]). However, in certain geometric situations, it is useful to put a M.H.S. on the iterated integrals on a variety. In section 11 we will show that Morgan's M.H.S. on homotopy agrees with ours.

Let X be a topological differentiable space. We define a <u>de Rham M.H.C.</u> on X to be an $\mathbb{R}$-M.H.C.

$$(S^\bullet(X),\ (A^\bullet,\ W_\bullet),\ (L^\bullet,\ W_\bullet,\ F^\bullet)),$$

where

(a) $S^\bullet(X)$ is the singular cochain complex of X,

(b) $A^\bullet$ and $L^\bullet$ are augmented d.g.a.'s,

(c) $A^\bullet$ is quasi-isomorphic with $E^\bullet_{\mathbb{R}}(X)$ as an augmented d.g.a.

(d) all filtrations are multiplicative, and all quasi-isomorphisms preserve the augmentation.

Suppose that $*$ is a base point of X. Let $E^\bullet(X) \to \mathbb{R}$ $(= E^\bullet(\{*\}))$ be corresponding augmentation. For such a M.H.C. on $(X,*)$, we can form the triple

$$(B(S^\bullet(X)),\ B(A^\bullet),\ B(L^\bullet)),$$

where $B(\ )$ denotes the bar construction described in section 8. The descending weight filtration $W^\bullet$ on $A^\bullet$ extends to a weight filtration on $\otimes^s IA^\bullet$. Define a weight filtration on $B(A^\bullet)$ by [1]

$$W_\ell B(A^\bullet) = \bigoplus_{s\geq 0} W_{\ell-s} \otimes^s IA^\bullet .$$

[1] That is, we have taken the convolution of the tensor product weight filtration with the bar filtration (c.f. [59]).

In a similar way, we can define a weight filtration on $\mathcal{B}(L^\bullet)$. The Hodge filtration on $L^\bullet$ extends to a Hodge filtration on $\otimes^s IL^\bullet$ by

$$F^p\mathcal{B}(L^\bullet) = \bigoplus_{s,t} F^pB^{-s,t} .$$

These filtrations induce weight filtrations on $\mathcal{B}^{-s}(A^\bullet)$, $\mathcal{B}^{-s}(L^\bullet)$ and a Hodge filtration on $\mathcal{B}^{-s}(L^\bullet)$.

Lemma 9.1. Suppose that $(X,*)$ is a path connected, pointed topological differentiable space for which $S_\bullet(X) \to S_\bullet(\tau X)$ is a quasi-isomorphism. If $(S_\bullet(X), (A^\bullet,W_\bullet), (L^\bullet,W_\bullet,F^\bullet))$ is a de Rham M.H.C. on X, then, for all $s \geq 0$

$$(\mathcal{B}^{-s}(S^\bullet(X)),\ (\mathcal{B}^{-s}(A^\bullet),W_\bullet),\ (\mathcal{B}^{-s}(L^\bullet),W_\bullet,F^\bullet))$$

is an $\mathbb{R}$-M.H.C.

Proof. The proof consists of two parts. The first is to establish properties (iii), (iv), (v) of the definition of an $\mathbb{R}$-M.H.C. (see section 5). The second is to construct a chain map $B^{-s}(\mathcal{E}^\bullet(X)) \to \mathcal{B}^{-s}(S^\bullet(X)) \otimes \mathbb{R}$ – this is not so easy, as the integration map $\mathcal{E}^\bullet(X) \to S^\bullet(X) \otimes \mathbb{R}$ is not an algebra map – and verify conditions (i) and (ii) in the definition of an $\mathbb{R}$-M.H.C.

The weight spectral sequence associated to $(\mathcal{B}^{-s}(A^\bullet),W_\bullet)$ will be denoted by $E_r^{\bullet\bullet}(\mathcal{B}^{-s}(A^\bullet))$, the weight spectral sequence associated to $\otimes^s IA^\bullet$ by $E_r^{\bullet\bullet}(\otimes^s IA^\bullet)$, and so on.

A straightforward calculation shows that the E_0 term of the weight spectral sequence associated to $\mathcal{B}^{-s}(L^\bullet)$ is

$$(*) \qquad E_0^{\ell,m}(\mathcal{B}^{-s}(L^\bullet)) = \bigoplus_{t=0}^{s} E_0^{\ell+t,m}(\otimes^t IL^\bullet)$$

and the differential d_0 is just the direct sums of the d_0's of the summands.

Since the tensor product of M.H.C.'s is a M.H.C. (8.1.24, [12]), the differential d_0 of $E_0^{\ell+t}(\otimes^t IL^\bullet)$ is strictly compatible with the Hodge filtration. Thus the differential d_0 of $E_0^{\ell}(B^{-s}(L^\bullet))$ is also strictly compatible with the Hodge filtration.

It also follows from the isomorphism (*) that

$$E_1^{\ell,m}(B^{-s}(L^\bullet)) = \bigoplus_{t=0}^{s} E_1^{\ell+t,m}(\otimes^t IL^\bullet).$$

Since $\otimes^t IL^\bullet$ is part of a M.H.C., the Hodge filtration induces a Hodge structure of weight m on $E_1^{\ell+t,m}(\otimes^t IL^\bullet)$. Consequently, the Hodge filtration induces a Hodge structure of weight m on $E_1^{\ell,m}(B^{-s}(L^\bullet))$.

Similarly, one can show that

$$E_1^{\ell,m}(B^{-s}(A^\bullet)) = \bigoplus_{t=0}^{s} E_1^{\ell+t,m}(\otimes^t IA^\bullet)$$

from which it follows that

$$(B^{-s}(A^\bullet)\otimes\mathbb{C},W_\bullet) \quad \text{and} \quad (B^{-s}(L^\bullet),W_\bullet)$$

are quasi-isomorphic. This completes the first part of the proof.

It follows from our hypotheses and 7.3 that the inclusion

$$S_\bullet^{(0)}(X) \to S_\bullet(X)$$

induces an isomorphism on homology. For a ring R, let

$$S_{(0)}^\bullet(X;R) = \mathrm{Hom}(S_\bullet^{(0)}(X),R)$$

The restriction map

$$S^\bullet(X;R) \to S_{(0)}^\bullet(X;R)$$

is a d.g.a. quasi-isomorphism. It now follows from a standard spectral sequence argument that the induced map

$$\mathcal{B}^{-s}(S^{\bullet}(X;R)) \to \mathcal{B}^{-s}(S^{\bullet}_{(0)}(X;R))$$

is a quasi-isomorphism.

Denote the cobar construction on the coalgebra $S^{(0)}_{\bullet}(X)$ by $\mathcal{F}(S^{(0)}_{\bullet}(X))$ (see [11], pp. 869-876) and its augmentation ideal by I. Note that there is a natural quasi-isomorphism

$$\mathcal{B}^{-s}(S^{\bullet}_{(0)}(X,R)) \to \mathrm{Hom}(\mathcal{F}(S^{(0)}_{\bullet}(X))/I^{s+1},R).$$

The integration map

$$\int : \mathcal{B}^{-s}(E^{\bullet}) \to \mathrm{Hom}(\mathcal{F}(S^{(0)}_{\bullet}(X))/I^{s+1}, \mathbb{R})$$

$$[w_1|\ldots|w_r] \mapsto \{c \mapsto \left\langle \int w_1 \ldots w_r,\ c \right\rangle\}$$

is a chain map - in fact, a d.g. coalgebra map. Filtering each by the bar filtration, we obtain spectral sequences whose respective E_1 terms are

$$E_1 = \mathcal{B}^{-s}(H^*(E^{\bullet})) \quad \text{and} \quad E_1 = \mathcal{B}^{-s}(H^*(S^{\bullet}_{(0)}(X;\mathbb{R}))).$$

The map induced between them is induced by the integration map

$$E^{\bullet} \to S^{\bullet}_{(0)}(X;\mathbb{R})$$

which is a quasi-isomorphism. It follows that the two E_1 terms are isomorphic and that $\int$ is a quasi-isomorphism. One can easily check, using the Eilenberg-Moore spectral sequence, that $H^*(\mathcal{B}^{-s}(S^{\bullet}(X)))$ is a finitely generated abelian group. □

Suppose that k is a field and that $\mathbb{Q} \subseteq k \subseteq \mathbb{R}$. By a k-<u>mixed Hodge structure</u> (M.H.S.) on a finitely generated abelian group, we mean a M.H.S. on H where the weight filtration is defined over k.

If $(X,*)$ is a pointed topological space, then $H_0(\Omega_* X) = \mathbb{Z}\pi_1(X,*)$ and $H^0(\Omega_* X;\ R) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\pi_1(X,*),\ R)$. Denote the augmentation ideal of $\mathbb{Z}\pi_1(X,*)$ by J. The filtration of $\mathbb{Z}\pi_1(X,*)$ by the powers of J induces the filtration

$$\cdots \subseteq N_s H^0(\Omega_* X;R) \subseteq N_{s+1} H^0(\Omega_* X;R) \subseteq \cdots$$

on $H^0(\Omega_* X;\mathbb{R})$, where

$$N_s H^0(\Omega_*;R) = \mathrm{Hom}_{\mathbb{Z}}(\pi_1(X)/J^{s+1},R).$$

The suspension map

$$([0,1] \ltimes \Omega_* X,\ \{0,1\} \ltimes \Omega_* X) \to (X,*)$$

$$(t,\gamma) \mapsto \gamma(t)$$

induces a natural map

$$\Sigma : \tilde{H}^{k+1}(X) \to H^k(\Omega_* X) \otimes H^1([0,1],\{0,1\}) \approx H^k(\Omega_* X)$$

that we shall also call the <u>suspension map</u>. It is an easy exercise to check that the Hurewicz homomorphism

$$h : \pi_k(\Omega_* X) \to H_{k+1}(X)$$

is dual to the suspension map. That is,

$$\langle \Sigma(u), \phi \rangle = \langle u, h(\phi) \rangle , \quad \text{where}$$

$$u \in H^{k+1}(X) \quad \text{and} \quad \phi \in \pi_k(\Omega_* X).$$

Dual to the suspension map is

$$\begin{aligned} i: \mathcal{E}^\bullet(X) &\to \mathcal{B}(\mathcal{E}^\bullet(X)) \\ w &\mapsto [w] \end{aligned}$$

which has degree -1 (c.f. [24],(5.8)). Note that $di = -id$, so that i induces a map

$$H^{k+1}(\mathcal{E}^\bullet(X)) \to H^k(\mathcal{B}(\mathcal{E}^\bullet(X))).$$

Thoerem 9.2. Suppose that $(V,*)$ is a pointed topological differentiable space with finite Betti numbers for which the de Rham theorem is true. Suppose that $(S^\bullet(V), A^\bullet(V), L^\bullet(V))$ is a de Rham M.H.C.

(a) If V is simply connected, then $H^*(\Omega_* V)$ has an $\mathbb{R}$-M.H.S. for which the product

$$H^*(\Omega_* V;\mathbb{C}) \otimes H^*(\Omega_* V;\mathbb{C}) \to H^*(\Omega_* V;\mathbb{C}),$$

the coproduct

$$H^*(\Omega_* V;\mathbb{C}) \to H^*(\Omega_* V;\mathbb{C}) \otimes H^*(\Omega_* V;\mathbb{C})$$

and the suspension

$$H^{k+1}(V;\mathbb{C}) \to H^{k}(\Omega_{*}V;\mathbb{C})$$

are morphisms of M.H.S.'s.

(b) For all $s \geqq 0$, $N_sH^0(\Omega_*V)$ has an $\mathbb{R}$-M.H.S. for which the natural inclusions

$$N_sH^0(\Omega_*V) \to N_tH^0(\Omega_*V) \qquad s \leqq t,$$

the product

$$N_sH^0(\Omega_*V) \otimes N_tH^0(\Omega_*V) \to N_{s+t}H^0(\Omega_*V),$$

the coproduct

$$N_sH^0(\Omega_*V) \to N_sH^0(\Omega_*V) \otimes N_sH^0(\Omega_*V),$$

and the suspension

$$H^1(V) \to N_sH^0(\Omega_*V)$$

induce morphisms of M.H.S.'s.

<u>Corollary</u> 9.3. (a) If V is simply connected, then the homotopy groups $\pi_*(V)$ of V have an $\mathbb{R}$-M.H.S. and the Whitehead product

$$\pi_{p+1}(V) \otimes \pi_{q+1}(V) \to \pi_{p+q+1}(V)$$

and the Hurewicz homomorphism

$$\pi_k(V) \to H_k(V)$$

induce morphisms of M.H.S.'s.

(b) If $\pi_1(V,*) \to \pi_1(\tau V,*)$ is an isomorphism, then for each $s \geqq 0$, the Malcev completion $\mathfrak{g}_s$ of $\pi_1(V)/\Gamma^{s+1}$ has an $\mathbb{R}$-M.H.S. Moreover, the Lie bracket and the natural projections $\mathfrak{g}_s \to \mathfrak{g}_t$ $(t \leqq s)$ induce morphisms of M.H.S.'s. In particular, the "Hurewicz homomorphism" $\mathfrak{g}_s \to \mathfrak{g}_1 = H_1(V;\mathbb{Q})$ induces a morphism of M.H.S.'s.

Proof. To prove (a), note that the augmentation $H^*(\Omega_* V;\mathbb{C}) \to \mathbb{C}$ is a morphism of M.H.S.'s so that $IH^*(\Omega_* V;\mathbb{C})$ has an $\mathbb{R}$-M.H.S. Since the product $I \otimes I \to I$ is a morphism of M.H.S.'s, its cokernel $QH^*(\Omega_* V;\mathbb{C})$ has an $\mathbb{R}$-M.H.S. and, by 8.3(a), $\pi_*(V)$ has an $\mathbb{R}$-M.H.S. Because the coproduct of $H^*(\Omega_* V)$ is a morphism of M.H.S.'s, the induced Lie cobracket of $QH^*(\Omega_* V;\mathbb{C})$ is a morphism of M.H.S.'s. Since this cobracket is dual to the Whitehead product of $\pi_*(V)$, the Whitehead product is a morphism of M.H.S.'s. Finally since the composite

$$H^{k+1}(V;\mathbb{C}) \to H^k(\Omega_* V;\mathbb{C}) \to QH^k(\Omega_* V;\mathbb{C})$$

is a morphism of M.H.S.'s and is dual to the Hurewicz homomorphism

$$\pi_{k+1}(V) \otimes \mathbb{C} = PH_k(\Omega_* V;\mathbb{C}) \to H_{k+1}(V;\mathbb{C}),$$

it follows that the Hurewicz homomorphism is a morphism of M.H.S.'s.

The proof of (b) is similar. It uses 8.3(b) and 9.2(b). □

<u>Proof of</u> 9.2. Suppose that V is simply connected. Since $H^k(B^{-s}(S^\bullet(V))) = H^k(B(S^\bullet(V)))$ whenever $s > k$, it follows from 8.1, 9.2(a) and 9.1 that

$$(B(S^\bullet(V)),\ B(A^\bullet),\ B(L^\bullet))$$

is an $\mathbb{R}$-M.H.C. and that $H^*(B(L^\bullet))$ has an $\mathbb{R}$-M.H.S.

Recall that the tensor product of two M.H.C.'s is a M.H.C. ([12], (8.1.24)). It follows that the natural isomorphism

$$H^*(B(L^\bullet)) \otimes H^*(B(L^\bullet)) \approx H^*(B(L^\bullet) \otimes B(L^\bullet))$$

is a M.H.S. isomorphism. Since the shuffle product

$$B(L^\bullet) \otimes B(L^\bullet) \to B(L^\bullet)$$

and the coproduct

$$B(L^\bullet) \to B(L^\bullet) \otimes B(L^\bullet)$$

both preserve the Hodge and the weight filtrations, the product and coproduct of $H^*(B(L^\bullet))$ are morphisms of M.H.S.'s.

Because the map $L^\bullet \to B(L^\bullet): w \mapsto [w]$ preserves the Hodge filtration, is of degree -1 and shifts the weight filtration by 1, the suspension map

$$H^{k+1}(L^\bullet) \to H^k(B(L^\bullet))$$

preserves the Hodge and weight filtrations and is, therefore, a morphism of M.H.S.'s.

From 8.1 and 8.2(a), it follows that there is a natural Hopf algebra isomorphism

$$H^*(B(L^\bullet)) \approx H^*(\Omega_* V;\mathbb{C})$$

This completes the proof of (a). Similar arguments can be used to prove (b). □

Denote the category of pointed algebraic varieties and base point preserving regular maps by $\underline{\underline{\mathrm{Alg}}}_*$.

Theorem 9.4. Suppose the $(V,*)$ is a pointed quasi-projective algebraic variety.

(a) If V is simply connected, then the homotopy groups $\pi_*(V,*)$ of V have a natural $\mathbb{R}$-M.H.S. that are functorial with respect to maps in $\underline{\underline{\mathrm{Alg}}}_*$. The Whitehead product and Hurewicz homomorphism induce morphisms of M.H.S.'s. All the weights on $\pi_*(V)$ are non-positive.

(b) For each $s \geq 0$, the Malcev completion $\mathfrak{g}_s$ of $\pi_1(V,*)/\Gamma^{s+1}$ has an $\mathbb{R}$-M.H.S. with non-positive weights that is functorial with respect to maps in $\underline{\underline{\mathrm{Alg}}}_*$. The natural projections $\mathfrak{g}_s \to \mathfrak{g}_t$ $(t \leq s)$ induce morphisms of M.H.S.'s.

(c) For each $s \geq 0$, the quotient $\mathbb{Z}\pi_1(V,*)/J^{s+1}$ has an $\mathbb{R}$-M.H.S. that is functional with respect to algebraic maps.

Proof. Choose a split hypercovering $\varepsilon: X_\bullet \to V$ and a base point $x \in X_0$ with $\varepsilon(x) = *$. Let $(S^\bullet(X_\bullet), A^\bullet, L^\bullet)$ be the de Rham M.H.C. for $X_\bullet$ constructed in 5.3. The base point x defines augmentations $S^\bullet(X_\bullet) \to \mathbb{Z}$, $A^\bullet \to \mathbb{R}$, $L^\bullet \to \mathbb{C}$.

Suppose now that V is simply connected. From 3.1 it follows that $|X_\bullet|$ is simply connected and that ε induces a Hopf algebra isomorphism

$$H^*(\Omega_x|X_\bullet|) \overset{\approx}{\to} H^*(\Omega_* V).$$

According to 4.5, the natural map $H_*(\mathrm{re}\ X_\bullet) \to H_*(|X_\bullet|)$ is an isomorphism so that 7.7 and 7.8 together imply that the natural map

$$H^*(\Omega_x|X_\bullet|) \to H^*(\Omega_x \mathrm{re}\ X_\bullet)$$

is a Hopf algebra isomorphism. By 9.2, $H^*(\Omega_x\ \mathrm{re}\ X_\bullet)$ has a M.H.S. whose product, coproduct and suspension are morphisms of M.H.S. Thus, each choice of a pointed hypercovering $\varepsilon:(X_\bullet,x) \to (V,*)$ determines a M.H.S. on $H^*(\Omega_* V)$ such that the product, coproduct and suspension are morphisms of M.H.S.'s. We have to prove that this M.H.S. does not depend on the choice of base point x or the hypercovering $X_\bullet \to V$.

First the base point: If $y \in X_0$ and $\varepsilon(x) = \varepsilon(y) = *$, then, since $X_\bullet \to V$ is a hypercovering, there exists $q \in X_1$ such that $d_0 q = x, d_1 q = y$. The path $\alpha: t \mapsto (q,t) \in X_1 \times [0,1] \subseteq |X_\bullet|$ is smooth and joins x to y in $\mathrm{re}\ X_\bullet$. The base points x and y induce different augmentations on $L^\bullet$. Denote the associated augmented algebras by $L^\bullet_x$ and $L^\bullet_y$. Denote the complex of iterated integrals on $\Omega_x\ \mathrm{re}\ X_\bullet$ by $(\int L^\bullet)_x$ and on $\Omega_y\ \mathrm{re}\ X_\bullet$ by $(\int L^\bullet)_y$. Observe that the M.H.S. on $H^*(\Omega_x\ \mathrm{re}\ X_\bullet)$ can be defined via the Hodge and weight filtrations on $(\int L^\bullet)_x$ induced by the natural map $B(L^\bullet_x) \to (\int L^\bullet)_x$, and similarly for $H^*(\Omega_y\ \mathrm{re}\ X_\bullet)$.

The path α defines a smooth map

$$\begin{aligned} \Omega_y \text{ re } X_\bullet &\to \Omega_x \text{ re } X_\bullet \\ \gamma &\mapsto \alpha\gamma\alpha^{-1} \quad . \end{aligned}$$

This induces a d.g. algebra map

$$\Phi_\alpha : \left(\int L^\bullet\right)_x \to \left(\int L^\bullet\right)_y$$

$$\int w_1 \ldots w_r \mapsto \sum_{0 \le i < j \le r} \left(\int_\alpha w_1 \ldots w_i\right) \int w_{i+1} \ldots w_j \left(\int_{\alpha^{-1}} w_{j+1} \ldots w_r\right).$$

To prove that the M.H.S. on $H^*(\Omega_* V)$ is independent of the choice of base point of $X_\bullet$, it suffices to show that this map is filtration preserving.

Note that if $v_1, \ldots, v_k \in L^\bullet$, then $\int_\alpha v_1 \ldots v_k \neq 0$ implies that each $v_j \in L^1$. Since $W_{-2} L^1 = 0$, it follows that if $\int_\alpha v_1 \ldots v_k \neq 0$, then $\int_\alpha v_1 \ldots v_k \notin W_{-1} \int L^\bullet$ and it follows that Φ_α preserves the weight filtration,

Recall from the discussion preceding 5.4 that

$$L^1(X_\bullet) = E^1(L^0[\quad]) \oplus E^0(L^1[\quad]).$$

and note that if $v \in E^0(L^1[\quad])$, then $\alpha^* v = 0$. Thus, if $u_j \in E^1(L^0[\quad])$ and $v_j \in E^0(L^1[\quad])$, then

$$\int_\alpha (u_1 + v_1) \ldots (u_k + v_k) = \int_\alpha u_1 \ldots u_k.$$

Since $F^1 \cap E^1(L^0[\quad]) = 0$, it follows that Φ_α preserves the Hodge filtration.

Next we show that the M.H.S. on $H^*(\Omega_* V)$ does not depend on the choice of the hypercovering. Suppose that $\varepsilon_X : X_\bullet \to V$ and $\varepsilon_Y : Y_\bullet \to V$ are split hypercoverings of V. By 2.1(2), there is a split hypercovering $\varepsilon_Z : Z_\bullet \to V$ and maps $\eta_X : Z_\bullet \to X_\bullet$, $\eta_Y : Z_\bullet \to Y_\bullet$ such that the diagram

$$\begin{tikzcd}[column sep=small] & Z_\bullet \arrow[dl, "\eta_X"'] \arrow[d, "\varepsilon_Z"] \arrow[dr, "\eta_Y"] & \\ X_\bullet \arrow[dr, "\varepsilon_X"'] & V & Y_\bullet \arrow[dl, "\varepsilon_Y"] \\ & & \end{tikzcd}$$

commutes. We can thus assume that $Y_\bullet = Z_\bullet$. Choose $z \in Z_0$ such that $\varepsilon_Z(z) = *$. Set $x = \eta_X(z)$. The map η_X induces a filtration preserving d.g. Hopf algebra map

$$\mathcal{B}(L(X_\bullet)) \to \mathcal{B}(L(Z_\bullet))$$

and an isomorphism $H^*(\Omega_x \text{ re } X_\bullet) \to H^*(\Omega_z \text{ re } Z_\bullet)$ such that the diagram

$$\begin{tikzcd}[column sep=small] H^*(\Omega_x \text{ re } X_\bullet) \arrow[rr] & & H^*(\Omega_z \text{ re } Z_\bullet) \\ & H^*(\Omega_* V) \arrow[ul, "(\Omega\varepsilon_x)^*"] \arrow[ur, "(\Omega\varepsilon_z)^*"'] & \end{tikzcd}$$

commutes. Consequently, the M.H.S. on $H^*(\Omega_* V)$, and hence $\pi_*(V)$, is independent of the choice of pointed hypercovering.

The naturality of the M.H.S. is proved by a similar argument as is the corresponding result for $\pi_1(V)$ when V is not simply connected. □

Remark 9.4. In 11.8 we prove, when V is simply connected, that the M.H.S. on $\pi_*(V,*)$ is independent of the base point. When V is not simply connected, this is false. For example, let V be a smooth projective curve of genus $\geqslant 2$ and w_1, w_2 two linearly independent holomorphic 1-forms on V. Choose two distinct points p,q on V and a smooth path $\alpha\colon [0,1] \to V$

that joins p to q. We can assume that $\int_\alpha w_1$ and $\int_\alpha w_2$ are both non zero. Since $w_1 \wedge w_2 = 0$ and each w_j is closed, $\int w_1 w_2$ is a closed iterated integral. The map

$$\Psi : (\int E^{\bullet}(V))_p \to (\int E^{\bullet}(V))_q,$$

induced by the map $\Omega_q V \to \Omega_p V : \gamma \to \alpha\gamma\alpha^{-1}$, takes $\int w_1 w_2$ to

$$\int w_1 w_2 + (\int_\alpha w_1)\int w_2 - (\int_\alpha w_2)\int w_1.$$

Since $\int w_1 w_2 \in F^2$ and since w_1, w_2 are independent and $\int_\alpha w_1, \int_\alpha w_2$ are non zero, Ψ does not preserve the Hodge filtration.

10. Review of Chen's Formal Connections

Here we briefly review Chen's formal connections. More leisurely introductions to Chen's Lie algebra methods can be found in [11] and [24].

Throughout this section, k will denote a field of characteristic zero. Suppose that $A^\bullet$ is a commutative d.g. algebra with augmentation $\varepsilon: A^\bullet \to k$ and that each $H^m(A^\bullet)$ is finite dimensional over k. Denote the augmentation ideal of $H^*(A^\bullet)$ by $IH^*(A^\bullet)$ and set

$$\tilde{H}_*(A^\bullet) = \mathrm{Hom}(IH^*(A^\bullet), k).$$

For example, if $A^\bullet$ is the de Rham complex of a manifold M, then $\tilde{H}_*(A^\bullet)$ is the reduced real homology of M.

The free Lie algebra over k generated by the desuspension $s^{-1}\tilde{H}_*(A^\bullet)$ of $\tilde{H}_*(A^\bullet)$ will be denoted by $\mathbb{L}(s^{-1}\tilde{H}_*(A^\bullet))$. Since this expression is lengthy, we will often abbreviate it by $\mathbb{L}(A^\bullet)$, $\mathbb{L}_k$ or simply $\mathbb{L}$. To better understand $\mathbb{L}$, choose a graded basis $X_1, X_2, \ldots$ of $s^{-1}\tilde{H}_*(A^\bullet)$. Then $\mathbb{L}(A^\bullet)$ is the free graded Lie algebra $\mathbb{L}(X_1, X_2, \ldots)$ over k generated by $X_1, X_2, \ldots$. The basis $X_1, X_2, \ldots$ of $s^{-1}\tilde{H}_*(A^\bullet)$ can be extended by a graded basis (X_I) of $\mathbb{L}$, where each X_I is a homogeneous polynomial in the X_i's. Denote the polynomial degree of X_I by $|I|$. For example, if $X_I = [X_1,[X_2,X_3]]$, then $|I| = 3$, regardless of the homological degrees of the X_j.

We shall denote the lower central series of a Lie algebra $\mathfrak{g}$ by

$$\mathfrak{g} = I\mathfrak{g} \geqslant I^2\mathfrak{g} \geqslant I^3\mathfrak{g} \geqslant \ldots,$$

where $I^{k+1}\mathfrak{g} = [I^k\mathfrak{g},\mathfrak{g}]$. The quotient $\mathfrak{g}/I^{s+1}\mathfrak{g}$ is a nilpotent Lie algebra of class s that we shall denote by $\mathfrak{g}(s)$. Denote the I-adic completion of $\mathfrak{g}$ by $\mathfrak{g}^\wedge$.

Let s be a positive integer. A <u>formal connection on</u> $A^\bullet$ <u>of order</u> s consists of a differential δ on $\mathbb{L}(A^\bullet)(s)$ of degree -1 and an element

$$\omega = \sum_{|I|\leqslant s} w_I X_I$$

of $A^\bullet \otimes \mathbb{L}$ that satisfy

(i) if $X_I \in \mathbb{L}_n$, then $w_I \in A^{n+1}$,

(ii) if $\omega = \sum w_i X_i + \sum_{2\leqslant |I| \leqslant s} w_I X_I$,

then each w_i is closed and their cohomology classes $(\{w_i\})$ form a basis of $IH^*(A^\bullet)$ dual to the basis (sX_j) of $\tilde{H}_*(A^\bullet)$,

(iii) if $\bar{\omega}$ denotes the reduction of ω mod $A^\bullet \otimes I^{s+1}\mathbb{L}$, then $\delta\bar{\omega} + d\bar{\omega} - \frac{1}{2}[J\bar{\omega},\bar{\omega}] = 0$, where $\delta\bar{\omega} = \sum w_I(\delta X_I)$, $d\bar{\omega} = \sum (dw_I)X_I$ and $[J\bar{\omega},\bar{\omega}] = \sum (Jw_I)\wedge w_J[X_I,X_J]$.

A sequence (ω_s,δ_s) of formal connections on $A^\bullet$, where (ω_s,δ_s) is of order s, is <u>compatible</u> if $\omega_s \equiv \omega_t$ mod $A^\bullet \otimes I^{t+1}\mathbb{L}$ and $\delta_t X_j \equiv \delta_s X_j$ mod $I^{t+1}\mathbb{L}$ whenever $0 < t \leqslant s$. If we set $\delta = \lim \delta_s$ and $\omega = \lim \omega_s$, then $\omega \in A^\bullet \otimes \mathbb{L}^\wedge$, $\delta : \mathbb{L}^\wedge \to \mathbb{L}^\wedge$ is a differential that is continuous with respect to the I-adic topology and

$$\delta\omega + d\omega - \frac{1}{2}[J\omega,\omega] = 0$$

in $A^\bullet \otimes \mathbb{L}^\wedge$. Such a pair (ω,δ) is called a formal (power series) connection on $A^\bullet$.

Compatible sequences of formal connections (and hence connections) on $A^\bullet$ always exist (c.f. [10], theorem 1.3.1). A recipe for computing them is given in ([24], chapter 2). The connection form ω extracts the homotopical structure of $A^\bullet$ and records it in δ.

A formal connection on $A^\bullet$ defines a d.g. Lie algebra homomorphism from the dual of the indecomposables $Q\mathcal{B}(A^\bullet)$ of the bar construction on $A^\bullet$ into $\mathbb{L}(A^\bullet)^\wedge$. It is convenient to set

$$L(A^\bullet) = \mathrm{Hom}_k(Q\mathcal{B}(A^\bullet),k).$$

Since $Q\mathcal{B}(A^\bullet)$ is a d.g. Lie coalgebra, $L(A^\bullet)$ is a d.g. Lie algebra. When $A^\bullet$ is of finite type, $L(A^\bullet)$ is simply Quillen's L-construction ([45], apprendix B) on the commutative coalgebra $\mathrm{Hom}(A^\bullet,k)$.

Suppose that (ω,δ) is a formal connection on $A^\bullet$. The homomorphism $L(A^\bullet) \to \mathbb{L}(A^\bullet)^\wedge$ associated with ω is defined by the Lie transport T_L of ω, which we now define. Denote the universal enveloping algebra of $\mathbb{L}$ by $U\mathbb{L}$ and its completion with respect to the powers of its augmentation ideal by $U^\wedge\mathbb{L}$. There is a continuous, natural inclusion $\mathbb{L}^\wedge \to U^\wedge\mathbb{L}$. The transport of (ω,δ) is the element

$$T = 1 + [\omega] + [\omega|\omega] + [\omega|\omega|\omega] + \ldots$$

of $\mathcal{B}(A^\bullet) \otimes U^\wedge\mathbb{L}$. If $\omega = \sum w_i X_i + \sum w_{ij} X_i X_j + \ldots,$ then

$$T = 1 + \sum [w_i] X_i + \sum ([w_i|w_j] + [w_{ij}]) X_i X_j + \ldots$$

We can regard $\mathcal{B}(A^{\bullet}) \otimes U^{\wedge}\mathbb{L}$ as a complete $\mathcal{B}(A^{\bullet})$ algebra. The Lie transport T_L of ω is the element

$$\log T = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(T-1)^m}{m}$$

of $\mathcal{B}(A^{\bullet}) \otimes U\mathbb{L}^{\wedge}$. It follows from the proof of theorem A.1 in the appendix that

$$T_L \in Q\mathcal{B}(A^{\bullet}) \otimes \mathbb{L}^{\wedge} ,$$

where we are identifying $Q\mathcal{B}(A^{\bullet})$ with its image in $\mathcal{B}(A^{\bullet})$ via the map s defined in theorem A.1. (That this definition and the one given in [24], chapter 6 agree follows from the proof of theorem A.1)

Suppose that $T_L = \sum v_j V_j$, where $v_j \in Q\mathcal{B}(A^{\bullet})$ and $V_j \in \mathbb{L}$. Define

$$\Theta: L(A^{\bullet}) \to \mathbb{L}(A^{\bullet})^{\wedge}$$

by $\Theta(\phi) = \phi(v_j)V_j$. The basic facts about Θ are summarized by the following theorem.

<u>Theorem</u> 10.1. Suppose that $IH^k(A^{\bullet}) = 0$ whenever $k \leqslant 0$. If (ω,δ) is a formal connection on $A^{\bullet}$, then

(a) the map

$$\Theta: L(A^{\bullet}) \to \mathbb{L}(A^{\bullet})^{\wedge}$$

is a d.g. Lie algebra homomorphism,

(b) the induced map

$$H_0(L(A^\bullet)) \;/\; I^{s+1} \to H_0(\mathbb{L}(s), \delta_s)$$

is a Lie algebra isomorphism for all s,

(c) if, in addition, $H^1(A^\bullet) = 0$, then Θ lands in $\mathbb{L}(A^\bullet)$, $\delta : \mathbb{L} \to \mathbb{L}$ and the induced map

$$H_*(L(A^\bullet)) \to H_*(\mathbb{L})$$

is a Lie algebra isomorphism.

Proof. The proofs of (a) and (c) can be found in ([24], chapter 6). The proof of (b) is similar to the proof of (c) and is omitted.□

The interesting situation occurs when $A^\bullet$ is a sub d.g.a. of the de Rham complex of a differentiable space and $k = \mathbb{R}$ or $\mathbb{C}$. Suppose that $(M,*)$ is a pointed, connected, topological differentiable space with finite Betti numbers for which the de Rham theorem is true and for which $\pi_1(M,*) = \pi_1(\tau M,*)$. Suppose that $A^\bullet$ is a sub d.g.a. of $E^\bullet(M)$ and that the inclusion $A^\bullet \to E^\bullet(M)$ induces a cohomology isomorphism.

Part (a) of the following theorem is proved in [10] and part (b) is proved in ([24], chapter 6).

Theorem 10.2. (Chen, Hain) Suppose that $(M,*)$ is a pointed topological differentiable space with finite Betti numbers and that $A^{\bullet}$ is a sub d.g.a. of $E^{\bullet}(M)$ for which the inclusion is a quasi isomorphism. Suppose that (ω,δ) is a formal power series connection on $A^{\bullet}$.

(a) If M is simply connected, then there is a graded Lie algebra isomorphism

$$s^{-1}\pi_*(M,*) \otimes \mathbb{R} \to H_*(\mathbb{L}(A^{\bullet}),\delta).$$

(b) If the natural map $\pi_1(M,*) \to \pi_1(\tau M,*)$ is an isomorphism, then, for all $s \geq 1$, there is a Lie algebra isomorphism

$$\mathfrak{g}_s \to H_0(\mathbb{L}(A^{\bullet})(s),\delta),$$

where $\mathfrak{g}_s$ is the Lie algebra of the real Malcev completion of the s^{th} nilpotent quotient of $\pi_1(M,*)$. □

11. A M.H.S. on the Lie Algebra Model

In this section we show that the real (and sometimes rational) minimal Lie algebra model $(\mathbb{L}, \delta)$ of a quasi-projective variety V has a M.H.S. whose differential δ respects the Hodge and weight filtrations. Consequently, $H_*(\mathbb{L})$ has a M.H.S. According to 10.2, $H_*(\mathbb{L})$ is isomorphic to $\pi_*(V) \otimes \mathbb{C}$ when V is simply connected and, when V is not simply connected, $H_0(\mathbb{L})$ is the Malcev Lie algebra assoicated with $\pi_1(V)$. So, for example, we now have three ways to define a M.H.S. on $\pi_*(V)$ when V is simply connected: Morgan's method via Sullivan's minimal models [38], via Chen's iterated integrals (section 9), and via the Lie algebra model. We show that these three M.H.S.'s are the same and prove the analogous result for non simply connected varieties.

Since the integral part of a de Rham M.H.C. is not in general a commutative d.g.a., we will forget about it: $((K_k^\bullet, W_\bullet), (K_{\mathbb{C}}^\bullet, W_\bullet, F^\bullet))$ is a k-M.H.C. if it satisfies conditions (iii)-(v) of the definition of M.H.C. given in section 5 and if each $H^m(K_k^\bullet)$ is a finite dimensional k vector space. We will say that the k-M.H.C. $((K_k^\bullet, W_\bullet), (K_{\mathbb{C}}^\bullet, W_\bullet, F^\bullet))$ is a <u>multiplicative</u> M.H.C. if $K_k^\bullet$ and $K_{\mathbb{C}}^\bullet$ are augmented, commutative d.g. algebras with multiplicative Hodge and weight filtrations. The augmentations are required to be filtration preserving, where $W_{-1}k = 0$, $W_0 k = k$ and where $\mathbb{C}$ has the Hodge structure of type $(0,0)$.

A filtration $W_\bullet$ of a graded vector space $A^\bullet$ is <u>bounded below</u> if, for each n, there is an integer $\ell(n)$ such that $W_\ell A^n = 0$ whenever $\ell \leq \ell(n)$.

Lemma 11.1. Suppose that $((K^\bullet_{k},W_\bullet),(K^\bullet_{\mathbb{C}},W_\bullet,F^\bullet))$ is a k-M.H.C. whose weight filtrations are bounded below.

(a) If u is a closed element of $(F^p \cap W^\ell)K^\bullet_{\mathbb{C}}$ whose coholmology class $\{u\}$ in $H^*(K^\bullet_{\mathbb{C}})$ is trivial, then there exists $v \in (F^p \cap W_{\ell+1})K^\bullet_{\mathbb{C}}$ such that $dv = u$.

(b) If u is a closed element of $W_\ell K^\bullet_{k}$ whose cohomology class in $H^*(K^\bullet)$ is trivial, then there exists $v \in W_{\ell+1}K^\bullet$ such that $dv = u$.

Proof. Suppose that $u \in F^p \cap W^{-\ell} \cap K^n_{\mathbb{C}}$, $du = 0$ and that $\{u\} = 0$ in $H^*(K^\bullet_{\mathbb{C}})$. Since the weight filtration is bounded below, there is an integer $\ell(n)$ such that $W^{-\ell} = 0$ if $\ell \leqslant \ell(n)$. We prove the lemma by induction on $s = \ell-\ell(n)$.

If $s = 0$, then $W^{-\ell} = 0$ and so we may take $v = 0$. Suppose that $s > 0$ and that the result is true for all $t < s$. Denote the weight spectral sequence by $E^\bullet_r$ and the image of $x \in W^{-\ell}K^\bullet_{\mathbb{C}}$ in $E^{-\ell}_r$ by x_r, when it exists. Because $d_1 : E^{-\ell-1}_1 \to E^{-\ell}_1$ is a morphism of Hodge structures, because the weight spectral sequence degenerates at E_2 and because $\{u\} = 0$, there exists $v' \in F^p \cap W^{-\ell-1} \cap K^{n-1}_{\mathbb{C}}$ such that $d_1 v'_1 = u_1$ in $E^{-\ell}_1$. That is, $u_1 - d_1v'_1 = (u - dv')_1 = 0$ in $E^{-\ell}_1$. It follows that $(u - dv')_0$ is exact in $E^\bullet_0$. Because d_0 is strictly compatible with the Hodge filtration, we can find $v'' \in F^p \cap W^{-\ell} \cap K^{n-1}_{\mathbb{C}}$ such that $d_0 v''_0 = (u - dv')_0$. That is, $(u - d(v'+v''))_0 = 0$ and $u - d(v'+v'') \in W^{-\ell+1} \cap F^p \cap K_{\mathbb{C}}$. But $[u - d(v'+v'')] = \{u\} = 0$ in $H^n(K^\bullet_{\mathbb{C}})$ and by our induction hypothesis, we can find $v''' \in W^{-\ell} \cap F^p \cap K^{n-1}_{\mathbb{C}}$ such that $u - d(v'+v''+v''') = 0$ The result follows because $v' + v'' + v''' \in F^p \cap W^{-\ell-1} \cap K^{n-1}_{\mathbb{C}}$.

The proof of (b) is similar, but simpler. One just forgets the Hodge filtration.□

Another preliminary result that we need is the following proposition

<u>Proposition</u> 11.2. If $((A^\bullet_k, W_\bullet), (A^\bullet_{\mathbb{C}}, W_\bullet, F_\bullet))$ is a multiplicative k - M.H.C., then

$$((QB(A^\bullet_k), W_\bullet), (QB(A^\bullet_{\mathbb{C}}), W_\bullet, F^\bullet))$$

is a k-M.H.C. and the cobracket

$$QB \to QB \otimes QB$$

preserves the filtrations.

<u>Proof</u>. Let $\gamma\colon IB(A^\bullet_{\mathbb{C}}) \to IB(A^\bullet_{\mathbb{C}})$ be the canonical idempotent defined in theorem A.1 in the appendix. From the formula for γ, it is clear that γ is filtration preserving. Consequently, the filtration on $\operatorname{im}\gamma$ induced by the canonical inclusion $s\colon QB(A^\bullet_{\mathbb{C}}) \to IB(A^\bullet_{\mathbb{C}})$ and the canonical projection $\pi\colon IB(A^\bullet_{\mathbb{C}}) \to QB(A^\bullet_{\mathbb{C}})$ agree. Similarly, the Hodge and weight filtrations on $I^2B(A^\bullet_{\mathbb{C}})$ induced by the projection id-γ: $IB(A^\bullet_{\mathbb{C}}) \to I^2B(A^\bullet_{\mathbb{C}})$ and the inclusion $I^2B(A^\bullet_{\mathbb{C}}) \to IB(A^\bullet_{\mathbb{C}})$ agree. It follows that, as bifiltered complexes,

$$B(A^\bullet_{\mathbb{C}}) = \mathbb{C} \oplus QB(A^\bullet_{\mathbb{C}}) \oplus I^2B(A^\bullet_{\mathbb{C}}).$$

Similarly, as a filtered complex

$$B(A^\bullet_k) = k \oplus QB(A^\bullet_k) \oplus I^2B(A^\bullet_k).$$

If we denote the E_r term of the weight spectral sequence of $B(A^\bullet_{\mathbb{C}})$ by $E_r(B)$ etc., then

$$E_r(\mathcal{B}) = \mathbb{C} \oplus E_r(Q) \oplus E_r(I^2)$$

It follows immediately that

$$((Q\mathcal{B}(A^\bullet_k), W_\bullet), (Q\mathcal{B}(A^\bullet_{\mathbb{C}}), W_\bullet, F^\bullet))$$

is a k-M.H.C. Since the diagonal $\Delta:\mathcal{B}(A^\bullet_{\mathbb{C}}) \to \mathcal{B}(A^\bullet_{\mathbb{C}}) \otimes \mathcal{B}(A^\bullet_{\mathbb{C}})$ is filtration preserving, the cobracket $Q \to Q \otimes Q$ is also filtration preserving. □

As in section 10, we denote the vector space dual of $Q\mathcal{B}(A^\bullet_k)$ by $L(A^\bullet_k)$. The filtration of $L(A^\bullet_k)$ dual to the weight filtration of $Q\mathcal{B}(A^\bullet_k)$ is defined by

$$W_{-\ell}L(A^\bullet_k) = \mathrm{Hom}(Q\mathcal{B}(A^\bullet_k) \,/\, W_{\ell-1}, k)$$

Similarly, Hodge and weight filtrations can be defined on $L(A^\bullet_{\mathbb{C}})$.

<u>Corollary</u> 11.3. If $((A^\bullet_k, W_\bullet), (A^\bullet_{\mathbb{C}}, W_\bullet, F^\bullet))$ is a multiplicative k-M.H.C., then

$$((L(A^\bullet_k), W_\bullet), (L(A^\bullet_{\mathbb{C}}), W_\bullet, F^\bullet))$$

is a k-M.H.C. and the bracket

$$[\ ,\ ]: L \otimes L \to L$$

is filtration preserving.

<u>Proof</u>. The result follows directly from 11.2 together with the easily verified fact that the dual of a M.H.C. is a M.H.C. Note that because $H_n = H^{-n}$, the weight filtration on $H_*(L)$ is defined by

$$W_{\ell-n}H_n(L) = \mathrm{im}\{H_n(W_\ell L) \to H_n(L)\}. \ \square$$

Suppose that $((A_k^\bullet, W_\bullet), (A_{\mathbb{C}}^\bullet, W_\bullet, F^\bullet))$ is a multiplicative k-M.H.C. with connected homology. That is, $IH^m(A_k^\bullet) = 0$ when $m \leq 0$. The dual $\tilde{H}_*(A_k^\bullet)$ of $IH^*(A_k^\bullet)$ has a k-M.H.S. Consequently we have the canonical bigraded decomposition

$$\tilde{H}_*(A_{\mathbb{C}}^\bullet) = \oplus A^{p,q}$$

of $\tilde{H}_*(A_{\mathbb{C}}^\bullet)$, with the properties that

$$W_\ell \tilde{H}_*(A_{\mathbb{C}}^\bullet) = \bigoplus_{p+q \leq \ell} A^{p,q} \ , \ F^p \tilde{H}_*(A_{\mathbb{C}}^\bullet) = \bigoplus_{s \geq p} A^{s,\bullet}$$

(See [12], (1.2.8) or [21], (1.12).)

This bigrading extends naturally to a bigrading $\oplus B^{p,q}$ of the free Lie algebra $\mathbb{L}(A_{\mathbb{C}}^\bullet)$ in such a way that the bracket preserves the bigrading. Define the Hodge and weight filtrations on $\mathbb{L}(A_{\mathbb{C}}^\bullet)$ in the natural way. Namely,

$$W_\ell \mathbb{L} = \bigoplus_{p+q \leq \ell} B^{p,q} \ , \ F^p \mathbb{L} = \bigoplus_{s \geq p} B^{s,\bullet}$$

Note that we have not yet defined a k structure on $\mathbb{L}(A_{\mathbb{C}}^\bullet)$. We will do this later in the section. In general it will not be the one induced by the natural inclusion $\tilde{H}_*(A_k^\bullet) \to \tilde{H}_*(A_{\mathbb{C}}^\bullet)$.

The Lie algebra $\mathbb{L}(A_{\mathbb{C}}^\bullet)$ is quadragraded: Denote the elements of $\mathbb{L}$ of homological degree n and polynomial degree s by $\mathbb{L}_{n,s}$. Set $\mathbb{L}_{n,s}^{p,q} = \mathbb{L}_{n,s} \cap B^{p,q}$. Note that $\tilde{H}_*(A_{\mathbb{C}}^\bullet)$ is trigraded: it is $\oplus A_n^{p,q}$ where $A_n^{p,q} = \tilde{H}_n \cap A^{p,q}$. Choose a trigraded basis X_i of $s^{-1}\tilde{H}_*(A_{\mathbb{C}}^\bullet)$ and extend it to a qaudragraded basis $(X_I)_I$ of $\mathbb{L}(A_{\mathbb{C}}^\bullet)$. As in section

10, we shall denote the polynomial degree of X_I by $|I|$.

The main step in showing that $\mathbb{L}(A^\bullet_{\mathbb{C}})$ has a M.H.S. is the following lemma.

Lemma 11.4. If $((A^\bullet_k, W_\bullet), (A^\bullet_{\mathbb{C}}, W_\bullet, F^\bullet))$ is a multiplicative k-M.H.C. with connected cohomology and whose weight filtrations are bounded below, then there exists a formal power series connection (ω, δ) on $A^\bullet_{\mathbb{C}}$, where

$$\omega = \sum w_I X_I \ , \ \delta X_j = \sum a^I_j X_I, \ w_I \in A^\bullet_{\mathbb{C}} \text{ and } a^I_j \in \mathbb{C},$$

satisfying:

(i) if $X_I \in \mathbb{L}_n^{-p,-\ell+p}$, then $w_I \in F^p \cap W_{\ell-n-1} A^{n+1}_{\mathbb{C}}$,

(ii) δ is filtration preserving.

Proof. It suffices to construct a sequence (ω_s, δ_s) of compatible formal connections on $A^\bullet_{\mathbb{C}}$. The proof proceeds by induction on s. Suppose that $s = 1$. Choose a basis $u_1, \ldots, u_N$ of $IH^*(A^\bullet_{\mathbb{C}})$ dual to the basis $X_1, \ldots, X_N$ of $s^{-1}\tilde{H}_*(A^\bullet_{\mathbb{C}})$. If $X_j \in \mathbb{L}_n^{p,-\ell+p}$, then we can choose a representative w_j of u_j with $w_j \in F^p \cap W_{\ell-n-1} A^{n+1}_{\mathbb{C}}$. Set $\omega_1 = \sum w_j X_j$ and $\delta_1 = 0$. We have constructed a connection of order (i) satisfying (i) and (ii).

Suppose now that $s > 1$ and that $(\omega_t, \delta_t)_{t<s}$ that satisfy (i) and (ii) have been constructed. To construct ω_s, we have to find, for all I of length s, complex numbers a^j_I and elements w_I of $A^\bullet_{\mathbb{C}}$ that satisfy the conditions:

(a) if $X_I \in \mathbb{L}_{n+1}^{-p,-\ell+p}$, then $w_I \in F^p \cap W_{\ell-n-1} A^{n+1}_{\mathbb{C}}$

(b) if $X_j \in \mathbb{L}_{n+1}^{-p,-\ell+p}$ and $a_I^j \neq 0$ then $X_I \in F^{-p} \cap W_{-\ell}\mathbb{L}_n$,

(c) $\sum_{|I|=s} (dw_I + \sum_j a_I^j w_j) X_I \equiv -\partial_{s-1}\omega_{s-1} + \frac{1}{2}[J\omega_{s-1}, \omega_{s-1}]$

mod $A_{\mathbb{C}}^{\bullet} \otimes I^{s+1}\mathbb{L}$.

If such w_I and a_I^j exist and if we set

$$\omega_s = \omega_{s-1} + \sum_{|I|=s} w_I X_I \ , \ \delta_s X_j = \delta_{s-1} X_j + \sum_{|I|=s} a_I^j X_I,$$

then (ω_s, δ_s) is a formal connection satisfying conditions (i) and (ii).

For $\xi = \sum z_I X_I$ in $A_{\mathbb{C}}^{\bullet} \otimes \mathbb{L}$, denote by $\xi_{(s)}$ the part $\sum_{|I|=s} z_I X_I$ of ξ that is homogeneous of degree s.

Since $(\omega_{s-1}, \delta_{s-1})$ satisfies (i) and (ii), since the filtrations of $A_{\mathbb{C}}^{\bullet}$ are multiplicative, and since

$$\Omega = (\partial_{s-1}\omega_{s-1} + d\omega_{s-1} - \tfrac{1}{2}[J\omega_{s-1}, \omega_{s-1}])_{(s)}$$

$$= (\partial_{s-1}\omega_{s-1} - \tfrac{1}{2}[J\omega_{s-1}, \omega_{s-1}]_{(s)},$$

it follows that $\Omega = \sum_{|I|=s} v_I X_I$, where $X_I \in \mathbb{L}_n^{-p,-\ell+p}$ implies that $v_I \in F^p \cap W_{-\ell-n-2} A_{\mathbb{C}}^{n+2}$. The usual argument (c.f. the proof of 1.3.1 in [10]) implies that if $|I| = s$, then $dv_I = 0$. Thus there are complex numbers a_I^j such that

$$\{v_I\} = \sum a_I^j \{w_j\} \quad \text{in} \quad H^*(A_{\mathbb{C}})$$

Moreover, if $X_I \in \mathbb{L}_n^{-p,-\ell+p}$, then $a_I^j = 0$ whenever $X_j \in \mathbb{L}^{-s,-m+s}$ with either $s > p$ or $m < \ell$. This establishes (b). Having chosen a_I^j note that if $X_I \in \mathbb{L}_n^{-p,-\ell+p}$, then

$$v_I - \sum_j a_I^j w_j$$

is an exact element of $F^{-p} \in W_{\ell-n-2}A_{\mathbb{C}}^{n+2}$ whose cohomology class is trivial. Thus, by 10.1, we can find $w_I \in F^{-p} \cap W_{\ell-n-1}A_{\mathbb{C}}^{n+1}$ such that

$$dw_I + \sum_j a_I^j w_j = v_I.$$

This establishes (a) and (c), and completes the proof.□

Corollary 11.5. For all $s \geqslant 0$, the Lie algebra homomorphism

$$\Theta: L(A_{\mathbb{C}}^{\bullet}) \to \mathbb{L}(A_{\mathbb{C}}^{\bullet})(s),$$

induced by the connection ω constructed in 11.4, preserves the Hodge filtration and satisfies $\Theta(W_\ell L_n(A_{\mathbb{C}}^{\bullet})) \leqslant W_{\ell-n}\mathbb{L}(A_{\mathbb{C}}^{\bullet})(s)$. Consequently, the map induced by Θ on homology preserves both the Hodge and the weight filtrations.

Proof. Suppose that $\omega = \sum w_I X_I$ is the connection form given by 11.4 The Lie transport T_L of ω is given by the formula

$$T_L = \sum_{(I_1, \ldots, I_s)} \gamma[w_{I_1}|\ldots|w_{I_t}] \; X_{I_1} \ldots X_{I_s},$$

where γ is the idempotent defined in theorem A.1. Since $X_I \in \mathbb{L}_n^{-q,-m+q}$ implies that $w_I \in F^q \cap W_{m-n-1} A_{\mathbb{C}}^{n+1}$, it follows that if $X_{I_1} \ldots X_{I_s} \in \mathbb{L}_n^{-p,-\ell+p}$, then $[w_{I_1}|\ldots| w_{I_s}] \in F^p \cap W_{\ell-n}B(A_{\mathbb{C}}^{\bullet})^n$. Since γ preserves the filtrations, $\gamma[w_{I_1}|\ldots|w_{I_s}] \in F^p \cap W_{\ell-n}QB(A_{\mathbb{C}}^{\bullet})$.

If $\phi \in F^{-p} \cap W_{n-\ell}(A_{\mathbb{C}}^{\bullet})$, then ϕ vanishes on $F^{p+1} \cap W_{\ell-n-1}QB(A_{\mathbb{C}}^{\bullet})^n$. It follows that, for all s, $\Theta(\phi) \in F^p \cap W_\ell \mathbb{L}(A_{\mathbb{C}}^{\bullet})(s)$. Because

$$W_{-n}H_n(L) = \operatorname{im}\{H_n(W_\ell L) \to H_n(L)\}$$

and

$$F^p H_n(L) = \operatorname{im}\{H_n(F^p L) \to H_n(L)\} ,$$

it follows that $\Theta_*: H_*(L) \to H_*(\mathbb{L}\,(s))$ is filtration preserving. $\square$

To define a k-structure on $\mathbb{L}\,(A_{\mathbb{C}}^{\bullet})$ so that $(\mathbb{L}\,(A_{\mathbb{C}}^{\bullet}), W_{\bullet}, F^{\bullet})$ is a k-M.H.S., we need to assume that $A_{\mathbb{C}}^{\bullet} \subseteq A_k^{\bullet} \otimes_k \mathbb{C}$. Such an $\mathbb{R}$-M.H.C. for a quasi-projective variety U can be constructed as follows: First, it suffices to consider the case where $U = Y - D$, where Y is a smooth projective variety and D is a divisor in Y with normal crossings. Let $A_{\mathbb{C}}^{\bullet} = E^{\bullet}(Y \log D)$. Let $A_{\mathbb{R}}^{\bullet}$ be the complex of forms that are locally elements of

$$E^{\bullet}(U) \otimes \Lambda(\theta_1, \ldots, \theta_k, d\log r_1, \ldots, d\log r_k) \otimes \mathbb{R}\,[\log r_1, \ldots, \log r_k] ,$$

where U is a polydisk in Y with coordinates $(z_1, \ldots, z_n)$, $U \cap D\{z_1 = \ldots = z_k = 0\}$ and

$$\theta_j = \frac{1}{4\pi i}\left(\frac{dz_j}{z_j} - \frac{d\bar{z}_j}{\bar{z}_j}\right), \quad r_j = |z_j| .$$

Define a weight filtration on $A_{\mathbb{R}}^{\bullet}$ by defining $W_0 A_{\mathbb{R}}^{\bullet} = \operatorname{im}\{E^{\bullet}(Y) \to A_{\mathbb{R}}^{\bullet}\}$ and by defining the weight of each θ_j, $d\log r_j$, $\log r_j$ to be 1 and extending multiplicatively. With this weight filtration, $A_{\mathbb{R}}^{\bullet}$ is filtered quasi-isomorphic with the complex $A_{\mathbb{R}}^{\bullet}(Y,D)$ defined in 5.2. and $E^{\bullet}(Y \log D) \subseteq A_{\mathbb{R}}^{\bullet} \otimes \mathbb{C}$.

<u>Theorem 11.6</u>. If $((A_k^{\bullet}, W_{\bullet}), (A_{\mathbb{C}}^{\bullet}, W_{\bullet}, F^{\bullet}))$ is a multiplicative k-M.H.C. whose weight filtrations are bounded below and such that $A_{\mathbb{C}}^{\bullet} \subseteq A_k^{\bullet} \otimes_k \mathbb{C}$, then, for all $s \geq 0$, the Lie algebras $((\mathbb{L}\,(A_{\mathbb{C}}^{\bullet})(s), \delta)$ each have a k-M.H.S. for which the differential δ is a morphism of M.H.S.'s and for which the induced maps

$$\Theta_*: H_0(L(A_{\mathbb{C}}^{\bullet})(s) \to H_0(\mathbb{L}\,(A_{\mathbb{C}}^{\bullet})(s))$$

and, when $A^{\bullet}_{\mathbb{C}}$ has simply connected homology, the map

$$\theta_*: H_*(L(A^{\bullet}_{\mathbb{C}})) \to H_*(\mathbb{L}\,(A^{\bullet}_{\mathbb{C}}))$$

are isomorphisms of M.H.S.'s.

Corollary 11.7. Suppose that $(V,*)$ is a pointed quasi-projective variety.

(a) If V is simply connected, then the complex Lie algebra $(\mathbb{L}_{\mathbb{C}}(V),\delta)$ of V has an $\mathbb{R}$-M.H.S. for which the differential δ is a morphism of M.H.S.'s.

(b) If $(V,*)$ is not simply connected, then its completed complex Lie algebra model $(\mathbb{L}^{\wedge}_{\mathbb{C}}(V),\delta)$ has an $\mathbb{R}$-M.H.S. for which the differential δ is a morphism of M.H.S.'s. □

Proof of 11.6. Choose a formal power series connection (ω,δ) on $A^{\bullet}_{\mathbb{C}}$. Choose cycles $\phi_1, \phi_2, \ldots$ in $\mathrm{Hom}_k(A^{\bullet}_k,k)$ whose homology classes $\{\phi_1\}, \{\phi_2\}, \ldots$ form a graded basis of $\tilde{H}_*(A_{\mathbb{C}})$ with the property that their images $\overline{\{\phi_1\}}, \overline{\{\phi_2\}}, \ldots$ in $\mathrm{gr}^W_{\bullet}\tilde{H}_*(A^{\bullet}_{\mathbb{C}})$ (the graded object associated with the weight filtration) are linearly independent. By adjusting ϕ_j by a boundary if necessary, we can assume that if $\overline{\{\phi_j\}}$ has weight ℓ_j in $H_{n_j}(A^{\bullet}_k)$, then $\phi_j \in W_{\ell_j+n_j}\mathrm{Hom}_k(A^{\bullet}_k,k)$.

Composing the natural inclusion $L(A^{\bullet}_k) \to L(A^{\bullet}_k \otimes_k \mathbb{C})$ with the projection $L(A^{\bullet}_k \otimes \mathbb{C}) \to L(A^{\bullet}_{\mathbb{C}})$ induced by the inclusion, we obtain a d.g. Lie algebra homomorphism $\Psi: L(A^{\bullet}_k) \to L(A^{\bullet}_{\mathbb{C}})$ that is strictly compatible with the weight filtration. Since desuspension shifts the weight filtration by 1, $s^{-1}\phi_j \in W_{\ell_j+n_j-1}L(A^{\bullet}_k)$.

Set $U_j = \Theta \circ \Psi(s^{-1}\phi_j)$, where

$$\Theta : L(A^{\bullet}_{\mathbb{C}}) \to \mathbb{L}^{\wedge}(A^{\bullet}_{\mathbb{C}})$$

is the holonomy map associated with the connection. Denote the free k-Lie algebra $\mathbb{L}(U_1, U_2, \ldots)$ by $\mathbb{L}_k$. Define a weight filtration $W_\circ$ on $\mathbb{L}_k$ by defining the weight of U_j to be ℓ_j and extending the filtration to $\mathbb{L}_k$ multiplicatively. The natural inclusion $\mathbb{L}_k \subseteq \mathbb{L}(A^{\circ}_{\mathbb{C}})$ extends to a $\mathbb{C}$ linear isomorphism

$$\mathbb{L}_k \otimes_k \mathbb{C} \to \mathbb{L}(A^{\bullet}_{\mathbb{C}})$$

and the weight filtration on $\mathbb{L}_k$ defined above agrees with the weight filtration induced on $\mathbb{L}_k$ by the weight filtration on $\mathbb{L}(A^{\bullet}_{\mathbb{C}})$. That is, the weight filtration of $\mathbb{L}(A^{\bullet}_{\mathbb{C}})$ is defined over k.

Next, we have to show that for each $s \geqslant 0$, that $((\mathbb{L}_k(s), W_\bullet), (\mathbb{L}(A^{\bullet}_{\mathbb{C}})(s), W_\bullet, F^{\bullet}))$ is a k-M.H.S. We do this by induction on s. When $s = 1$ $(\mathbb{L}_k(1), W_\bullet) \approx (s^{-1}\tilde{H}_*(A^{\bullet}_k), W_\bullet)$ and $(\mathbb{L}(A^{\bullet}_{\mathbb{C}})(1), W_\bullet, F^{\circ}) \approx (s^{-1}\tilde{H}_*(A^{\bullet}_{\mathbb{C}}), W_\bullet, F^{\bullet})$. Since the inclusion $\mathbb{L}_k(1) \to \mathbb{L}(A^{\bullet}_{\mathbb{C}})(1)$ corresponds to the inclusion $s^{-1}\tilde{H}_*(A^{\bullet}_k) \to s^{-1}\tilde{H}_*(A^{\bullet}_{\mathbb{C}})$, the result is true when $s = 1$. When $s > 1$, there is a short exact sequence

$$0 \to I^s/I^{s+1} \to \mathbb{L}(A^{\bullet}_{\mathbb{C}})(s) \to \mathbb{L}(A^{\bullet}_{\mathbb{C}})(s-1) \to 0.$$

Using a standard basis of I^s/I^{s+1}, one can show that I^s/I^{s+1} has a k-M.H.S. Since $\mathbb{L}(A^{\bullet}_{\mathbb{C}})(s)$ is bifiltered and the maps in this short exact sequence are filtration preserving, it follows, by induction, that $\mathbb{L}(A^{\bullet}_{\mathbb{C}})(s)$ has a k-M.H.S. (c.f. [21],(1.16)).

Our last task is to show that the holonomy map of the connection induces a morphism of M.H.S.'s on homology. We will prove this in the simply connected case ($IH^m(A^\bullet_{\mathbb{C}}) = 0$ when $m \leqslant 1$). In the non-simply connected case, one can verify directly that

$$\Theta(H_0(L(A^\bullet_k))(s) \subseteq H_0(\mathbb{L}_k(s)) \quad \text{for all} \quad s \geqslant 0.$$

Since $\Theta \circ \Psi : L(A^\bullet_k \otimes_k \mathbb{C}) \to \mathbb{L}_{\mathbb{C}}$ is a quasi-isomorphism and since $H_*(L(A^\bullet_k)) \to H_*(L(A^\bullet_k \otimes_k \mathbb{C}))$ is injective, it follows that if u is a cycle in $\ker\Theta \cap L(A^\bullet_k)$, then there exists $v \in \ker\Theta \cap L(A^\bullet_k)$ such that $\delta v = u$. Using this one can show that there is a sub d.g. Lie algebra L of $L(A^\bullet_k)$ that contains each $s^{-1}\phi_j$ and such that $\Theta \circ \Psi(L) \subseteq \mathbb{L}_k$ and the restriction of $\Theta \circ \Psi$ to L is a quasi-isomorphism $L \to \mathbb{L}_k$. Note that the inclusion $L \subseteq L(A^\bullet_k)$ is also a quasi-isomorphism. Since the diagram

$$\begin{array}{ccc}
H_*(L) & \xrightarrow{(\Theta\circ\Psi)_*} & H_*(\mathbb{L}_k) \\
\| & & \\
H_*(L(A^\bullet_k)) & & \big\downarrow \\
\psi_* \big\downarrow & & \\
H_*(L(A^\bullet_{\mathbb{C}})) & \xrightarrow{\Theta_*} & H_*(\mathbb{L}_{\mathbb{C}})
\end{array}$$

commutes, it follows that the M.H.S.'s on $H_*(L(A^\bullet_{\mathbb{C}}))$ and $H_*(\mathbb{L}_{\mathbb{C}})$ are isomorphic. □

We can now prove that when V is simply connected, the M.H.S. on $\pi_*(V,*)$ is independent of the base point.

<u>Proposition</u> 11.8. Suppose that V is a simply connected quasi-projective variety. If $x,y \in V$ and $\alpha:[0,1] \to V$ is a path with $\alpha(0) = x$, $\alpha(1) = y$, then the natural isomorphisms

$$\phi_\alpha : \pi_*(V,y) \otimes_{\mathbb{Z}} \mathbb{C} \to \pi_*(V,x) \otimes_{\mathbb{Z}} \mathbb{C},$$

$$\phi_\alpha : H^*(\Omega_x V;\mathbb{C}) \to H^*(\Omega_y V;\mathbb{C})$$

are isomorphisms of M.H.S.'s.

<u>Proof</u>. We need only prove the assertion for $H^*(\Omega V)$. Denote the canonical bigraded decomposition of $H^*(\Omega_x V;\mathbb{C})$ by $\oplus A^{p,q}$. It suffices to show that

$$\phi_\alpha(A^{p,q}) \subseteq F^p \cap W_{p+q}\, H^*(\Omega_y V;\mathbb{C}).$$

If necessary, choose a hypercovering $\varepsilon: X_\bullet \to V$ and $x',y' \in X_0$ such that $\varepsilon(x') = x$, $\varepsilon(y') = y$. The path α can be lifted to path $\alpha':[0,1] \to |X_\bullet|$ with $\alpha'(0) = x'$ and $\alpha'(1) = y'$. Since ϕ_α depends only on the homotopy class of α, we can, by 7.8, assume that α' is smooth. The map ϕ_α is then induced by the map

$$(\int E(\mathrm{re}X_\bullet))_{x'} \to (\int E(\mathrm{re}X_\bullet))_{y'}$$

$$\int w_1 \ldots w_r \to \sum_{0\leqslant i<j\leqslant r} \int_\alpha' w_1 \ldots w_i \int w_{i+1} \ldots w_j \int_{\alpha^{-1}} w_{j+1} \ldots w_r.$$

This map preserves the weight filtration but does not preserve the Hodge filtration, as "dz's can be integrated out."

To complete the proof, it suffices to show that if $u \in A^{p,q}$, then there exists a closed iterated integral $\sum I_j \in F^p \int L_{\mathbb{C}}(X\ )$ that represents u, where $I_j = \int w_1^j \ldots w_{r_j}^j$ and each w_i^j has degree $\geqslant 2$.

To see that this can be done, choose a formal connection (ω,δ) on $L^\bullet_{\mathbb{C}}(X_\bullet)$ (see 5.3) satisfying the conditions of 11.4. There is a bigraded decomposition

$$U\mathbb{L}(L^\bullet_{\mathbb{C}}) = H \oplus M \oplus \delta M,$$

where $\delta|H \equiv 0$, the map $H \to H_*(U\mathbb{L})$ is an isomorphism, and consequently $\delta|M: M \to \delta M$ is an isomorphism. Such a bigraded decomposition gives a decomposition

$$T = 1 + T_H + T_M + T_{\delta M}$$

of the transport of ω. (c.f. [24], (6.22).)

Choose a bigraded basis $U_1, U_2, \ldots$ of $UH_*(\mathbb{L})$ such that u is an element of the corresponding dual basis of $H^*(\Omega_x V;\mathbb{C})$. Let $Z_1, Z_2, \ldots$ be the corresponding basis of H. We can write

$$T_H = \Sigma z_j Z_j ,$$

where $z_j \in (\int L^{\bullet}_{\mathbb{C}}(X.))_x$. As in [24], (7.1), one can show that each z_j is closed and their cohomology classes $\{z_j\}$ form the basis of $H^*(\Omega_x V;\mathbb{C})$ dual to the basis $U_1, U_2, \ldots$ of $H_*(\Omega_x V;\mathbb{C})$. Since

$$T\colon \operatorname{Hom}(\mathcal{B}(L^{\bullet}_{\mathbb{C}}(X.)),\mathbb{C}) \to \mathbb{L}(L^{\bullet}_{\mathbb{C}})$$

preserves the Hodge filtration (c.f. (11.5)), it follows that if $Z_j \in (U\mathbb{L})^{-p,-\ell+p}$, then $z_j \in F^p(\int L^{\bullet}_{\mathbb{C}}(X.))_x$. Since ω has no coeficients that are 1-forms, z_j has no 1-forms and so $\phi_\alpha(z_j) = z_j$. This completes the proof.□

Our final task in this section is to show that our M.H.S. on homotopy agrees with Morgan's.

Proposition 11.9. If $((A^{\bullet}_{\mathbb{R}}, W_{\bullet}), (A^{\bullet}_{\mathbb{C}}, W_{\bullet}, F^{\bullet}))$ is a multiplicative $\mathbb{R}$-M.H.C. with 1-connected homology that satisfies $A^{\bullet}_{\mathbb{C}} \subseteq A^{\bullet}_{\mathbb{R}} \otimes \mathbb{C}$, then the M.H.S.'s on the real homotopy groups of $A^{\bullet}_{\mathbb{C}}$ obtained by Morgan agrees with the M.H.S. on the homotopy groups that we constructed in section 9. In particular, if V is a simply connected smooth projective variety, then the $\mathbb{R}$-M.H.S. on the homotopy groups of V constructed by Morgan [38] and by us, in section 9, agree.

Proof. Choose minimal models $\rho_1\colon (M^{\bullet}_{\mathbb{R}}, W_{\bullet}) \to (A^{\bullet}_{\mathbb{R}}, W_{\bullet})$, $\rho_2\colon N^{\bullet\bullet}_{\mathbb{C}} \to A^{\bullet}_{\mathbb{C}}$ such that

$$\rho_1(W_\ell M^{\bullet}_{\mathbb{R}}) \subseteq W_\ell A^{\bullet}_{\mathbb{R}}$$

and

$$\rho_2(N_{\mathbb{C}}^{p,\ell-p})^n \subseteq F^p \cap W_{\ell-n}A_{\mathbb{C}}^n .$$

Note that the weight filtration on $M_{\mathbb{R}}^{\bullet}$ used by Morgan is $\operatorname{Dec} W_\ell$. In this case

$$(\operatorname{Dec} W_\ell)M_{\mathbb{R}}^n = W_{\ell-n}M_{\mathbb{R}}^n .$$

Define a weight filtration on $N_{\mathbb{C}}^{\bullet\bullet}$ by

$$W_\ell(N_{\mathbb{C}}^{\bullet\bullet})^n = \bigoplus_p (N_{\mathbb{C}}^{p,\ell+n-p})^n$$

so that

$$\rho_2(W_\ell N_{\mathbb{C}}^{\bullet\bullet}) \subseteq W_\ell A_{\mathbb{C}}^{\bullet} .$$

Set $M_{\mathbb{C}}^{\bullet} = M_{\mathbb{R}}^{\bullet} \otimes \mathbb{C}$. According to Morgan, the filtered vectorspaces $(QM_{\mathbb{C}}^{\bullet}, W_\bullet)$ and $(QN_{\mathbb{C}}^{\bullet\bullet}, W_\bullet)$ are isomorphic. Choose $W_\bullet$ preserving splittings

$$QM_{\mathbb{C}}^{\bullet} \to M_{\mathbb{C}}^{\bullet}, \quad QN_{\mathbb{C}}^{\bullet\bullet} \to N_{\mathbb{C}}^{\bullet\bullet}$$

of the natural maps $M_{\mathbb{C}}^{\bullet} \to QM_{\mathbb{C}}^{\bullet}$ and $N_{\mathbb{C}}^{\bullet\bullet} \to QN_{\mathbb{C}}^{\bullet\bullet}$. We identify $(M_{\mathbb{C}}^{\bullet}, W_\bullet)$ and $(N_{\mathbb{C}}^{\bullet\bullet}, W_\bullet)$ with $(\Lambda(QM_{\mathbb{C}}^{\bullet}), W_\bullet)$.

Set

$$F^{\bullet} = M_{\mathbb{C}}^{\bullet} \otimes N_{\mathbb{C}}^{\bullet\bullet} \otimes \Lambda(s^{-1}QM_{\mathbb{C}}^{\bullet}) ,$$

where $s^{-1}Q$ denotes the desuspension of Q . Define a differential on $F^{\bullet}$ by

$$d(x\otimes 1\otimes 1) = dx\otimes 1\otimes 1,\ d(1\otimes y\otimes 1) = 1\otimes dy\otimes 1,\ d(1\otimes 1\otimes s^{-1}v) = (v\otimes 1\otimes 1 - 1\otimes v\otimes 1) .$$

Define a weight filtration on $F^{\bullet}$ by extending the weight filtration on the generators multiplicatively to $F^{\bullet}$. One can show that the natural inclusions

$$(M^{\bullet}_{\mathbb{C}}, W_{\bullet}) \to (F^{\bullet}, W_{\bullet}), \qquad (N^{\bullet\bullet}_{\mathbb{C}}, W_{\bullet}) \to (F^{\bullet}, W_{\bullet})$$

are quasi-isomorphisms of filtered d.g.a.'s.

The d.g.a. homomorphisms

$$\rho_1 : M^{\bullet}_{\mathbb{C}} \to A^{\bullet}_{\mathbb{R}} \otimes \mathbb{C}, \ \rho_2 : N^{\bullet\bullet}_{\mathbb{C}} \to A^{\bullet}_{\mathbb{C}} \subseteq A^{\bullet}_{\mathbb{R}} \otimes \mathbb{C}$$

induce a d.g.a. homomorphism

$$\rho_1 \otimes \rho_2 : M^{\bullet}_{\mathbb{C}} \otimes N^{\bullet\bullet}_{\mathbb{C}} \to A^{\bullet}_{\mathbb{R}} \otimes \mathbb{C} \quad .$$

Using the fact that

$$d(W_{\ell} M^{\bullet}) \subseteq W_{\ell-1} M^{\bullet}, \quad d(W_{\ell} N^{\bullet\bullet}) \subseteq W_{\ell-1} N^{\bullet\bullet}$$

and 11.1, one can show that $\rho_1 \otimes \rho_2$ extends to a filtered d.g.a. homomorphism

$$\rho : (F^{\bullet}, W_{\bullet}) \to (A^{\bullet}_{\mathbb{R}} \otimes \mathbb{C}, W_{\bullet})$$

which is necessarily a filtered quasi-isomorphism.

Moreover the diagram

$$\begin{array}{ccccc} M^{\bullet}_{\mathbb{R}} & \longrightarrow & F^{\bullet} & \longleftarrow & N^{\bullet\bullet}_{\mathbb{C}} \\ \rho_1 \downarrow & & \downarrow \rho & & \downarrow \rho_2 \\ A^{\bullet}_{\mathbb{R}} & \longrightarrow & A^{\bullet}_{\mathbb{R}} \otimes \mathbb{C} & \longleftarrow & A^{\bullet}_{\mathbb{C}} \end{array}$$

commutes. Applying the functor $QH^*(\mathcal{B}\,\_\,)$, one obtains a commutative diagram of filtered coalgebras

$$\begin{array}{ccccc} QH^*(\mathcal{B}(M^{\bullet}_{\mathbb{R}})) & \hookrightarrow & QH^*(\mathcal{B}(F^{\bullet})) & \overset{\approx}{\leftarrow} & QH^*(\mathcal{B}(N^{\bullet\bullet}_{\mathbb{C}})) \\ \downarrow & & \downarrow & & \downarrow \\ QH^*(\mathcal{B}(A^{\bullet}_{\mathbb{R}})) & \hookrightarrow & QH^*(\mathcal{B}(A^{\bullet}_{\mathbb{R}})) \otimes \mathbb{C} & \overset{\approx}{\leftarrow} & QH^*(\mathcal{B}(A^{\bullet}_{\mathbb{C}})) \end{array}$$

To complete the proof, note that the natural isomorphism

$$s^{-1}QN^{\bullet\bullet} \xrightarrow{\approx} QH^{*}(B(N_{\mathbb{C}}^{\bullet\bullet}))$$

of coalgebras preserves the bigrading. □

12. Applications

1. Generalized Riemann-Hilbert problem

Our first application is to the following generalization of the Riemann-Hilbert (i.e. Riemann's 21st.) problem: Suppose that U is a smooth quasi projective variety. By Hironaka's resolution of singularities, we can assume that $U = X - D$, where X is a smooth projective variety and D is a divisor in X with normal crossings. The problem is to characterize those linear representations $\rho\colon \pi_1(U) \to GL(m,\mathbb{C})$ that occur as the monodromy representation of a flat algebraic connection, with regular singularities along D, on the trivial vector bundle $\mathbb{C}^m \times U \to U$. We by no means solve this problem, but we do characterize the unipotent representations of $\pi_1(U)$ that arise as the monodromy representation of a nilpotent, regular, flat, holomorphic connection on a trivial bundle. The characterication obtained suggests that the Hodge filtration on the Malcev completion of $\pi_1(U)$ plays a significant role.

Denote the complex form of the Malcev completion of $\pi_1(U)$ by $\mathcal{G}$ and its lower central series by

$$\mathcal{G} = \mathcal{G}^1 \geq \mathcal{G}^2 \geq \cdots \quad .$$

Denote its Lie algebra by $\mathfrak{g}$ and its lower central series by

$$\mathfrak{g} = \mathfrak{g}^1 \geq \mathfrak{g}^2 \geq \cdots \quad .$$

Since the Hodge filtration on $\mathfrak{g}^s/\mathfrak{g}^{s+1}$ satisfies

$$\mathfrak{g}^s/\mathfrak{g}^{s+1} = F^{-s} \geq F^{-s+1} \geq \cdots \geq F^0 \geq F^1 = 0 \quad ,$$

it follows that the subalgebra

$$F^0\mathfrak{g} + F^{-1}\mathfrak{g}^2 + F^{-2}\mathfrak{g}^3 + \ldots$$

is an ideal of $\mathfrak{g}$. Denote the quotient Lie algebra $\mathfrak{g}/(F^0\mathfrak{g}^1 + F^{-1}\mathfrak{g}^2 + \ldots)$ by $\mathfrak{h}$ and the corresponding subgroup $\exp\mathfrak{h}$ of G by H. Denote the lower central series of $\mathfrak{h}$ and H by

$$\mathfrak{h} = \mathfrak{h}^1 \geq \mathfrak{h}^2 \geq \ldots \quad \text{and} \quad H = H^1 \geq H^2 \geq \ldots .$$

Set $G_s = G/G^{s+1}$ and $H_s = H/H^{s+1}$. These are simply connected nilpotent Lie groups of class s whose Lie algebras are $\mathfrak{g}_s := \mathfrak{g}/\mathfrak{g}^{s+1}$ and $\mathfrak{h}_s := \mathfrak{h}/\mathfrak{h}^{s+1}$, respectively.

Composing the canonical map $\Theta_s\colon \pi_1(U) \to G_s$ with the Lie group homomorphism $G_s \to H_s$, we obtain, for each $s \geq 1$, a group homomorphism $\theta_s\colon \pi_1(U) \to H_s$.

Recall that a holomorphic connection on the vector bundle $\mathbb{C}^m \times U \to U$ is regular if it can be extended to a meromorphic connection on $\mathbb{C}^m \times X \to X$ whose singularities are contained in D and are at worst logarithmic. By a unipotent representation $\rho\colon \pi_1(U) \to GL(m,\mathbb{C})$ we mean a representation with the property that $(\rho(g)-I)^m = 0$ for all $g \in \pi_1(U)$. A connection on $\mathbb{C}^m \times X \to X$ is nilpotent if the connection form takes values in a nilpotent sub Lie algebra of $gl(m,\mathbb{C})$.

Theorem 12.1. If U is a smooth quasi-projective variety and $\rho\colon \pi_1(U) \to GL(m+1,\mathbb{C})$ is a unipotent representation, then the following statements are equivalent:

(a) There is a Lie group homomorphism $\psi\colon H_m \to GL(m+1,\mathbb{C})$ such that the diagram

$$\begin{tikzcd} \pi_1(U) \arrow[rr, "\theta_m"] \arrow[dr, "\rho"'] & & H_m \arrow[dl, "\psi"] \\ & GL(m+1,\mathbb{C}) & \end{tikzcd}$$

commutes.

(b) There exists a flat, nilpotent, regular, algebraic connection on $\mathbb{C}^{m+1} \times U \to U$ whose monodromy representation is ρ .

(c) There exists a flat, nilpotent, regular holomorphic connection on $\mathbb{C}^{m+1} \times U \to U$ whose monodromy representation is ρ .

Proof. ($a \Rightarrow b$) Choose a smooth projective variety X and a divisor D in X with normal crossings such that $U = X-D$.

Choose a formal connection (ω_m,δ) of order m on $E^\bullet(X \log D)$ that satisfies the conditions of 11.4. Choose a bigraded basis $W_1,\ldots,W_g$, $W_1',\ldots,W_\ell'$ of $s^{-1}H_1(U;\mathbb{C})$, where $W_1',\ldots,W_\ell'$ forms a basis of $s^{-1}F^0H_1(U;\mathbb{C})$. Choose a basis $X_1,\ldots,X_t$, $Y_1,\ldots,Y_n$ of $s^{-1}H_2(U;\mathbb{C})$, where $X_1,\ldots,X_t$ forms a basis of $s^{-1}F^{-1}H_2(U;\mathbb{C})$.

According to 10.2(b),

$$\mathfrak{g}_m = \frac{\mathbb{L}(W_i,W_j')}{(\delta X_k,\delta Y_r) + I^{m+1}} .$$

Since the ideal in $\mathfrak{g}_m$ generated by the images of the W_j' is

$$F^0I + F^{-1}I^2 + \ldots ,$$

it follows that

$$\mathfrak{h}_m = \frac{\mathbb{L}(W_i, W'_j)}{(\delta X_k, \delta Y_r, W'_j) + I^{m+1}}$$

so that

$$\mathfrak{h}_m = \mathbb{L}(W_i)/((\overline{\delta} Y_r) + I^{m+1}) ,$$

where $\overline{\delta} Y_r$ denotes the reduction of δY_r modulo (W'_j) .

Using the fact that $\mathbb{L}(W_i, W'_j, X_k, Y_r)$ has a M.H.S. and that δ is a morphism of M.H.S.'s, one can easily prove the following.

<u>Proposition</u> 12.2.

$$\mathfrak{h}_m = \frac{\mathbb{L}(s^{-1} H_1(U;\mathbb{C})/F^0)}{(\mathrm{im}\overline{\delta}) + I^{m+1}} ,$$

where

$$\overline{\delta}: \; s^{-1} H_2(U)/F^{-1} \to s^{-1} H_1(U)/F^0 \wedge s^{-1} H_1(U)/F^0$$

is the dual of the cup product $F^1 H_1(U) \wedge F^1 H_1(U) \to F^2 H_2(U)$. Note that here we are identifying the vector spaces $V \wedge V$ and $[V,V]$, where $V = s^{-1} H_1(U)/F^0$. □

Denote the part of ω_m that lies in $E^1(X \log D) \otimes \mathbb{L}(W_i, W'_j)$ by ω_m^0 . View it as an $\mathbb{L}(W_i, W'_j)$ valued 1-form on U . Composing ω_m^0 with the canonical map $\mathbb{L}(W_i, W'_j) \to \mathfrak{g}_m$, one obtains a $\mathfrak{g}_m$ valued 1-form ω_c on U that satisfies the integrability condition

$$d\omega_c + \tfrac{1}{2}[\omega_c, \omega_c] = 0 .$$

Recall the following theorem of Chen [10].

<u>Theorem</u> 12.3. The $\mathfrak{g}_m$ valued 1-form ω_c defines a C^∞ flat connection on $G_m \times U \to U$ whose monodromy representation is the canonical group homomorphism $\Theta_m: \pi_1(U) \to G_m$. □

Composing ω_c with the natural surjection $\mathfrak{g}_m \to \mathfrak{h}_m$, we obtain an $\mathfrak{h}_m$ valued 1-form on U . In fact, it is algebraic with, at worst, logarithmic singularities along D . To see this, write

$$\omega_m^0 = \sum w_i W_i + \sum \phi_{ij}[W_i, W_j] + \sum \phi_{ijk}[W_i[W_j W_k]]$$

$$+ \sum w_j' W_j' + \sum \psi_{ij}[W_i, W_j'] + \ldots$$

and note that because (ω_m, δ) satisfies the conditions of 11.4, that $\phi_{ij}, \phi_{ijk} \in F^2 E^1(X\log D)$. But $F^2 E^1(X\log D) = 0$, so that

$$\omega = \sum w_i W_i \quad ,$$

where each $w_i \in \Omega^1(X\log D)$. That is, ω is a holomorphic, and hence algebraic, $\mathfrak{h}_m$ valued 1-form on X whose singularities are contained in D and are, at most, logarithmic. The following proposition is a trivial consequence of Chen's theorem above.

<u>Proposition</u> 12.4. The $\mathfrak{h}_m$ valued algebraic 1-form ω defines a flat connection on $H_m \times U \to U$ whose monodromy representation is $\theta_m: \pi_1(U) \to H_m$. □

Consequently, if we compose ω with the Lie algebra homomorphism $\mathfrak{h}_m \to gl(m+1,\mathbb{C})$ induced by the group homomorphism $\psi\colon H_m \to GL(m+1,\mathbb{C})$, then we obtain an algebraic $gl(m+1,\mathbb{C})$ valued 1-form which defines a flat, regular connection on $\mathbb{C}^{m+1} \times U \to U$ whose monodromy representation is the composite

$$\pi_1(U) \xrightarrow{\theta_m} H_m \longrightarrow GL(m+1,\mathbb{C}) ,$$

which is ρ.

(b $\Rightarrow$ c) is trivial.

(c $\Rightarrow$ a) Suppose that X is a smooth projective variety and that D is a divisor in X with normal crossings such that $U = X-D$. Suppose that $\mathfrak{n}$ is a nilpotent sub Lie algebra of $gl(m+1,\mathbb{C})$ and that $\hat{\omega}$ is an $\mathfrak{n}$-valued meromorphic 1-form on X that defines a regular flat connection on $\mathbb{C}^{m+1} \times U \to U$ and whose monodromy representation is ρ. That is, $\hat{\omega} \in \Omega^1(X\log D) \otimes gl(m+1,\mathbb{C})$. Since $\hat{\omega}$ is holomorphic, $d\hat{\omega} = 0$ and since it defines a flat connection, $[\hat{\omega},\hat{\omega}] = 0$. Recall that the natural map $\Omega^p(X\log D) \to F^pH^p(U;\mathbb{C})$ is an isomorphism (c.f. [12], 3.2.14; [21], (6.5)). Thus, the coefficients $w_1,\ldots,w_g$ of $W_1,\ldots,W_g$ in ω_m form a basis of $\Omega^1(X\log D)$. It follows that

$$\hat{\omega} = \sum_j w_j A_j ,$$

where each $A_j \in \mathfrak{n}$.

Choose a basis $u_1,\ldots,u_n$ of $\Omega^2(X\log D)$ dual to the basis $Y_1,\ldots,Y_n$ of $s^{-1}H_2(U;\mathbb{C})/F^{-1}$ used earlier. Define complex constants a_{ij}^k by

$$w_i \wedge w_j + \sum a_{ij}^k u_k = 0 .$$

Since $\overline{\delta} Y_k = \sum_{i<j} a_{ij}^k [W_i, W_j]$ (c.f. 12.2 and ([24], (2.5))),

$$\mathfrak{h}_m = \frac{\mathbb{L}(W_1,\ldots,W_g)}{(\sum_{i<j} a_{ij}^k [W_i,W_j]) + I^{m+1}} .$$

The integrability condition $[\hat{\omega},\hat{\omega}] = 0$ implies that

$$\sum_k u_k \sum_{i<j} a_{ij}^k [A_i, A_j] = \sum_{i<j} w_i \wedge w_j [A_i, A_j] = 0 .$$

Since the u_k are linearly independent, we must have

$$\sum_{i<j} a_{ij}^k [A_i, A_j] = 0 .$$

It follows from Engel's theorem [49] that the Lie algebra homomorphism $\mathbb{L}(W_1,\ldots,W_g) \to \mathfrak{n}$, defined by $W_i \longrightarrow A_i$, induces a Lie algebra homomorphism

$$d\psi\colon \mathfrak{h}_m = \mathbb{L}(W_1,\ldots,W_g)/(\sum_{i<j} a_{ij}^k [W_i,W_j]) + I^{m+1}) \to \mathfrak{n} .$$

Since H_m is simply connected, $d\psi$ induces a Lie group homomorphism $\psi\colon H_m \to GL(m+1,\mathbb{C})$. Finally it follows from 12.4 that the monodromy map of $\hat{\omega}$ is the composite

$$\pi_1(U) \xrightarrow{\theta_m} H_m \xrightarrow{\psi} GL(m+1,\mathbb{C}) .$$

That is, ρ factors through $\pi_1(U) \to H_m$. □

Corollary 12.5. For a quasi projective variety U the following two statements are equivalent:

(a) Every unipotent representation $\rho: \pi_1(U) \to GL(m,\mathbb{C})$ is the monodromy representation of a regular, flat, algebraic connection on $\mathbb{C}^m \times U \to U$.

(b) $W_1H^1(U;\mathbb{C}) = 0$. □

Since $H^1(\mathbb{P}^n) = 0$ and since $W_1H^1(U;\mathbb{C})$ is the image of $H^1(\overline{U};\mathbb{C}) \to H^1(U;\mathbb{C})$, where $\overline{U}$ is any projective variety containing U as a Zariski open set ([12], 3.2.17), we have:

Corollary 12.6. If U is a Zariski open subset of $\mathbb{P}^n$, then every unipotent representation $\pi_1(U) \to GL(m,\mathbb{C})$ arises as the monodromy representation of a regular, flat, algebraic connection on $\mathbb{C}^m \times U \to U$. □

Remarks 12.7. Corollary 12.6 is originally due to Aomoto [1]. When U is compact (i.e. projective) it is interesting to note that Hwang-Ma [28] has shown that the following statements are equivalent

(a) the normalized period matrix of U lies in a purely quadratic extension $\mathbb{Q}(\sqrt{-\alpha})$, $\alpha > 0$, of $\mathbb{Q}$.

(b) The image of $\pi_1(U) \to H_2$ is a cocompact discrete subgroup of H_2.

(c) The image of $\pi_1(U) \to H_m$ is a cocompact discrete subgroup for all $m \geq 1$.

2. The homotopy type of varieties

Recall that a formal space is one whose rational cohomology ring determines all of its rational algebro-topological invariants. All Kaehler manifolds, and consequently all smooth projective varieties, are formal [13]. Here we give new examples of formal spaces.

Theorem 12.8. If V is an irreducible projective variety whose underlying topological space is a rational homoyogy manifold, then V is formal.

Proof. It is sufficient to show that the complex Lie algebra model $\mathbb{L}\hat{_V}$ of V has generators $U_1,\ldots,U_n$ such that each δU_i is a linear combination of the brackets $[U_j,U_k]$ (c.f. [23] and [54]).

Suppose that $\dim_{\mathbb{C}} V = n$. Since V is compact, $H^{2n}(V;\mathbb{C})$ has a pure H.S. of weight n and each $H^m(V;\mathbb{C})$ has a M.H.S. whose weights ℓ satisfy $0 \leq \ell \leq m$ ([12], 8.2.4). But V is a rational homology manifold, so that the cup product

$$H^{n-k}(V;\mathbb{C}) \otimes H^{n+k}(V;\mathbb{C}) \to H^{2n}(V;\mathbb{C})$$

is a non singular pairing of M.H.S.'s. This forces $W_{m-1}H^m(V;\mathbb{C}) = 0$ for all m . That is, $H^m(V;\mathbb{C})$ has a H.S. of weight m .

The canonical projection

$$\mathbb{L}\hat{_V} \to Q\,\mathbb{L}\hat{_V} \approx s^{-1}\,\tilde{H}_*(V;\mathbb{C})$$

is a morphism of M.H.S.'s. Choose a splitting $\phi\colon s^{-1}\,\tilde{H}_*(V;\mathbb{C}) \to \mathbb{L}\hat{_V}$. Choose a bigraded basis $U_1,\ldots,U_N$ of the image of ϕ . Because the differential δ is a morphism of M.H.S.'s, each δU_i has to be a linear combination of the $[U_j, U_k]$. □

Example 12.9. Since the link of the complex singularity

$$x_0^k + x_1^2 + \ldots + x_n^2 = 0$$

is a rational homology sphere, provided that $n+1$ and k are not both even, a projective variety all of whose singularities are of one of these forms is a rational homology manifold and hence formal.

The following theorem shows that there are more restrictions on the rational homotopy type of a projective variety than those imposed by its rational cohomology ring.

Theorem 12.10. There exists a simply connected, finite CW-complex whose integral cohomology ring is isomorphic to that of a projective variety, but does not have the rational homotopy type of a projective variety.

Proof. Consider the d.g. Lie algebra

$$\mathbb{L} = (\mathbb{L}_{\mathbb{Q}}(W_j, X_j, Y_j, Z, V\colon 1 \le j \le 4), \delta) ,$$

where

a) $\deg W_j = \deg X_j = \deg Y_j = 2j - 1$, $\deg Z = 4$ and $\deg V = 9$,

b) $\delta W_1 = \delta X_1 = \delta Y_1 = 0$, $\delta W_2 = 1/2 [W_1, W_1]$,

$\delta W_3 = 2[W_1, W_2]$, $\delta W_4 = [W_1, W_3] + [W_2, W_2]$,

$\delta Z = [W_1 [X_1, Y_1]]$,

$\delta X_k = 1/2 \sum_{i+j=k} [X_i, X_j]$, $\delta Y_k = 1/2 \sum_{i+j=k} [Y_i, Y_j]$, $k = 2,3,4$,

$\delta V = 1/2 \sum_{i+j=5} ([W_i, W_j] + [X_i, X_j] + [Y_i, Y_j])$.

The reduced cohomology ring $H^{\bullet}$ of the corresponding rational homotopy type is the suspension of $\mathrm{Hom}(Q\mathbb{L}, \mathbb{Q})$. That is, the reduced cohomology ring has basis w_j, x_j, y_j, z, v $(1 \le j \le 4)$, where

c) $\deg w_j = \deg x_j = \deg y_j = 2j$, $\deg z = 5$, $\deg v = 10$,

d) $x_1^j = x_j$, $y_1^j = y_j$, $w_1^2 = w_2$, $w_1^3 = 2w_3$, $w_1^4 = 2w_4$ and

$v = 1/2\, w_1^5 = x_1^5 = y_1^5$.

If $\mathbb{L}$ were the Lie algebra model of a projective variety, then v would have weight 10. Since $W_2H^2 = H^2$ and w_1^5, x_1^5, y_1^5 are all nonzero in H^{10}, it follows that H^2 would have a H.S. of weight 2. Since $\dim H^5 = 1$, H^5 would necessarily have a pure H.S. of weight ≤ 4. Consequently, each of W_j, X_j, Y_j would have weright $-2j$ in $Q\mathbb{L}$ and Z would have weight ≥ -4. There would be a bigraded section

$$\sigma\colon Q\mathbb{L}_{\mathbb{C}} \to \mathbb{L}_{\mathbb{C}}$$

of the canonical map $\mathbb{L}_{\mathbb{C}} \to Q\mathbb{L}_{\mathbb{C}}$ with the property that the composite

$$Q\mathbb{L}_{\mathbb{C}} \xrightarrow{\sigma} \mathbb{L}_{\mathbb{C}} \xrightarrow{\delta} I^2\mathbb{L}_{\mathbb{C}}$$

preserved the bigrading. In particular, $\delta\sigma Z$ would be an element of $I^2\mathbb{L}$ of degree 3 and weight ≥ -4. But all elements of $I^2\mathbb{L}$ of degree 3 have weight ≤ -6. Consequently, $\delta\sigma Z$ would vanish. If $\sigma Z = Z + U$, where $U \in I^2\mathbb{L}$, then $\delta\sigma Z = 0$ would imply that $[W_1[X_1, X_2]]$ was a boundary in $(\mathbb{L}(W_j, X_j, Y_j), \delta)$, a contradiction. Therefore, $\mathbb{L}$ is not the Lie algebra model of a projective variety.

Next we show that there is a finite complex X having Lie algebra model $\mathbb{L}$ and that there is a complex projective variety V whose integral cohomology ring is isomorphic to that of X.

Let $M = \{M_t\} \to \Delta$ be a family of smooth conics of dimension 5 degenerating to a conic with one ordinary double point. Choose two sections $s_1, s_2 \colon \Delta \to M$ such that $s_1(t) \neq s_2(t)$ for all $t \in \Delta$. Let $\tilde{M}$ be the blow-up of M along s_1 and s_2. That is, $\tilde{M} = \{V_t\}$, where

$$V_t = B|_{s_1(t), s_2(t)} M_t .$$

Set $V = V_0$. The integral cohomology ring of V has basis w_j, x_j, y_j, z, v satisfying c) and d) above.

A finite CW-complex having Lie algebra model $\mathbb{L}$ and integral cohomology ring isomorphic to that of V can be constructed as follows: When $t \neq 0$, the vanishing cycles are imbedded 5-spheres $g_j \colon S^5 \to V_t$, $j = 1,2$ and their homology classes form a basis of $H_5(V_t;\mathbb{Z})$. Since V_t is simply connected,

$$\pi_2(V_t) = H_2(V_t;\mathbb{Z}) .$$

Choose maps $f_j \colon S^2 \to V_t$ whose homology classes are dual to w_1, x_1, y_1, respectively. Set

$$X = (V_t \cup_{g_1} e^6 \cup_{g_2} e^6) \cup_f e^5 ,$$

where $f \colon S^4 \to V_t$ is a representative of the Whitehead product $[f_1[f_2, f_3]]$.

Since V_t is formal (12.8), it follows from standard properties of the Lie algebra model that $\mathbb{L}$ is a Lie algebra model of X. It is an elementary exercise to verify that X and V_0 have isomorphic integral cohomology rings. □

3. A Torelli theorem for pointed curves

In modern parlance, the classical Torelli theorem [55] asserts that if X is a smooth projective curve, then the polarized H.S. on $H^1(X)$ determines X up to isomorphism. We have already seen, in 9.4, that the M.H.S. on $\pi_1(X,*)$ depends non trivially on the base point $*$. It is natural to ask whether the M.H.S. on $\pi_1(X,*)$ determines the pointed curve $(X,*)$. Theorem 12.12 says that this is true for generic $(X,*)$.

For a commutative ring R and a pointed topological space (Z,z), denote the augmentation ideal of $R\pi_1(Z,z)$ by $J_R(Z,z)$. We will usually write $J(Z,z)$ instead of $J_{\mathbb{Z}}(Z,z)$.

Proposition 12.11. If (Z,z) is a pointed, smooth, projective curve, then

(a) the natural map

$$\pi_1(Z,z) \to J(Z,z)$$

$$g \to g - 1$$

induces an isomorphism

$$H_1(Z) \longrightarrow J/J^2 ,$$

where J denotes $J(Z,z)$.

(b) the sequence

$$0 \to H_2(Z) \xrightarrow{\Delta} H_1(Z) \otimes H_1(Z) \to J/J^3 \to H_1(Z) \to 0$$

is exact, where Δ is the dual of the cup product

$$H^1(Z) \otimes H^1(Z) \to H^2(Z)$$

and where μ is the map that corresponds to the product

$$J/J^2 \otimes J/J^2 \to J^2/J^3$$

via the isomorphism in (a).

Proof. That the map $\pi_1(Z) \to J$ induces a map $H_1(Z) \to J/J^2$ follows from the identity

$$gh - 1 = (g-1) + (h-1) + (g-1)(h-1)$$

Choose a standard set of crosscuts $\alpha_1,\ldots,\alpha_g$, $\beta_1,\ldots,\beta_g$, where each α_i,β_j is a loop based at z . Denote the free associative algebra over $\mathbb{Z}$ generated by the indeterminates $A_1,\ldots,A_g$, $B_1,\ldots,B_g$ by $\mathbb{Z}\langle A_1,\ldots,A_g , B_1,\ldots,B_g\rangle$ and its augmentation ideal by I . Using the fact that $\Pi[\alpha_j,\beta_j] = 1$, one can show that the algebra homomorphism

$$\mathbb{Z} < A_1, \ldots, B_g > \to \mathbb{Z}\pi_1(Z,z) \quad ,$$

that takes A_i to $\alpha_i - 1$ and B_j to $\beta_j - 1$, induces an isomorphism

$$I/(\sum_j [A_j, B_j]) + I^3 \to J/J^3 \; .$$

From this it follows that J/J^2 is isomorphic with $H_1(Z)$.

Since the diagonal

$$\Delta\colon H_2(Z) \to H_1(Z) \otimes H_1(Z)$$

takes the fundamental class of Z to $\sum(A_j \otimes B_j - B_j \otimes A_j)$, it follows from the above presentation of J/J^3 that the sequence

$$0 \to H_2(Z) \xrightarrow{\Delta} H_1(Z) \otimes H_1(Z) \to J/J^3 \to H_1(Z) \to 0$$

is exact. □

<u>Theorem</u> 12.12. Suppose that (X,x) and (Y,y) are pointed, smooth projective curves. There is a finite set $\{x = x_1, x_2, \ldots, x_N\}$ of points in X such that if the M.H.S.'s on $J(X,x)/J^3(X,x)$ and $J(Y,y)/J^3(Y,y)$ are isomorphic as rings, then there is an isomorphism $\phi\colon Y \to X$ with $\phi(y) \in \{x_1, \ldots, x_N\}$. Moreover, for generic (X,x) , $N = 1$.[1]

---

[1]Mike Pulte has recently shown that $N \le 2$ for all (X,x) and that, for all curves X and generic $x \in X$, $N = 1$.

Proof. First we use the classical Torelli theorem to show that X and Y are isomorphic. To do this, note that if (Z,z) is a pointed, smooth curve, then $J(Z,z)/J^3(Z,z)$ has a M.H.S. with weights $-1,-2$ and that

$$W_{-2}\, J/J^3 = J^2/J^3 .$$

From 12.11, it follows that

$$(J/J^3)/W_{-2} \approx H_1(Z)$$

and, consequently, the M.H.S. on J/J^3 determines the H.S. on $H^1(Z)$. According to 12.11(b), the ring structure of J/J^3 determines, up to a sign, the cup product

$$H^1(Z) \otimes H^1(Z) \to \mathbb{Z} .$$

Of these two pairings, only one will yield a normalized period matrix of the H.S. on $H^1(Z)$ with positive definite imaginary part. Therefore the M.H.S. on $J(Z,z)/J^3(Z,z)$ determines the polarized H.S. on $H^1(Z)$ and thus, by the Torelli theorem, Z.

Now, 12.11(b) implies that the sequence

$$(*) \qquad 0 \to H^1(Z) \to \mathrm{Hom}(J/J^3,\mathbb{Z}) \to K_{\mathbb{Z}}(Z) \to 0$$

is exact, where $K_R(Z)$ denotes the kernel of the cup product

$$H^1(Z;R) \otimes H^1(Z;R) \to H^2(Z;R) .$$

Since the cup product is a morphism of H.S.'s, $K_{\mathbb{Z}}$ has a H.S. of weight 2 and (*) is an extension of M.H.S.'s, separated in the sense of [6].

Suppose that

$$\Phi\colon J(Y,y)/J^3 \to J(X,x)/J^3$$

is a ring isomorphism that induces an isomorphism of M.H.S.'s. Since Φ is a ring homomorphism, the diagram

$$\begin{array}{ccccccc} 0 \to \mathbb{Z} & \to & H_1(Y) \otimes H_1(Y) & \to & J(Y,y)/J^3 \\ \Vert & & \downarrow \Phi_* \otimes \Phi_* & & \downarrow \Phi \\ 0 \to \mathbb{Z} & \to & H_1(X) \otimes H_1(X) & \to & J(X,x)/J^3 \end{array}$$

commutes. Consequently, the induced map

$$\Phi_*\colon H_1(Y) \to H_1(X)$$

is an isomorphism of H.S.'s that either preserves or reverses the polarization. However, the Riemann bilinear relations imply that an isomorphism of Jacobians of curves cannot reverse the polarization. Thus, Φ_* preserves the polarization and, by the classical Torelli theorem [55], there is an isomorphism $\phi\colon Y \to X$ that induces either plus or minus Φ_*. We can therefore assume that $X = Y$ and that the diagram

$$\begin{array}{ccccccccc} 0 \to & H^1(X) & \to & \mathrm{Hom}(J(X,x)/J^3, \mathbb{Z}) & \to & K(X) & \to 0 \\ \pm\,\mathrm{id} & \downarrow & & \Phi \downarrow & & \Vert & \\ 0 \to & H^1(X) & \to & \mathrm{Hom}(J(X,y)/J^3, \mathbb{Z}) & \to & K(X) & \to 0 \end{array}$$

commutes. That is, the M.H.S. on $J(X,x)/J^3$ is congruent to plus or minus the M.H.S. on $J(X,y)/J^3$.

Recall from [6] that the set of congruence classes of extensions of $K(X)$ by $H^1(X)$ is isomorphic to the complex torus

$$J^0\mathrm{Hom}(K_{\mathbb{C}}(X), H^1(X;\mathbb{C})) .$$

where J^0 denotes the 0th Jacobian of the natural H.S. on $\mathrm{Hom}(K, H^1(X))$ (see [6]).

Define a map

$$\Psi: X \to J^0\mathrm{Hom}(K, H^1(X))$$

by taking $x \in X$ to the extension class of the extension

$$0 \to H^1(X) \to \mathrm{Hom}(J(X,x)/J^3, \mathbb{Z}) \to K \to 0 .$$

The next step is to prove that Ψ is holomorphic and that $d\Psi$ is nowhere vanishing.

Recall from [6] that an extension

$$0 \to H^1(X) \to E \to K(X) \to 0$$

of M.H.S.'s corresponds to an element Ψ_E of

$$J^0\mathrm{Hom}(K_{\mathbb{C}}, H^1_{\mathbb{C}}) = \frac{\mathrm{Hom}(K_{\mathbb{C}}, H^1_{\mathbb{C}})}{\mathrm{Hom}(K_{\mathbb{Z}}, H^1_{\mathbb{Z}}) + F^0} ,$$

where F^0 is the space of Hodge filtration preserving maps as follows: The extension class $\Psi_{\mathbb{E}}$ is obtained as $r_{\mathbb{Z}} \circ s_F$, where $s_F: K_{\mathbb{C}} \to E_{\mathbb{C}}$ is a Hodge filtration preserving section of $E_{\mathbb{C}} \to K_{\mathbb{C}}$ and $r_{\mathbb{Z}}: E_{\mathbb{Z}} \to H^1_{\mathbb{Z}}$ is an integral retraction of $H^1_{\mathbb{Z}} \to E_{\mathbb{Z}}$.

An integral retraction

$$r_{\mathbb{Z}}(z): \mathrm{Hom}(J(X,z)/J^3, \mathbb{Z}) \to H^1(X;\mathbb{Z})$$

can be defined as follows: Choose a basis $A_1,\ldots,A_{2g}$ of $H_1(X;\mathbb{Z})$ and loops $\alpha_1,\ldots,\alpha_{2g}$, based at x, whose homology classes are $A_1,\ldots,A_{2g}$. Choose a smooth path γ from x to z. Set

$$\alpha_j' = \gamma \alpha_j \gamma^{-1} ,$$

which is a loop, based at z, whose homology class is A_j. For $\phi \in \mathrm{Hom}(J^3(X,z)/J^3, \mathbb{Z})$, define

$$r_{\mathbb{Z}}(\phi)(A_j) = \phi(\alpha_j - 1) .$$

To define $s_F(z)$, first note that it follows from 8.3(b) that integration induces an isomorphism

$$\mathrm{Hom}(J(X,z)/J^3, \mathbb{C}) \simeq N^{-2} IH^0(\Omega_z X; \mathbb{C}) .$$

Choose a basis $w_1,\ldots,w_g$ of the abelian differentials on X. The section s_F can be defined by

$$s_F(w_i \otimes w_j) = \int w_i w_j , \quad s_F(\overline{w}_i \otimes \overline{w}_j) = \int \overline{w}_i \overline{w}_j ,$$

$$s_F(\sum a_{ij} w_i \otimes \overline{w}_j + \sum b_{ij} \overline{w}_i \otimes w_j) = \int (\sum a_{ij} w_i \overline{w}_j + \sum b_{ij} \overline{w}_i w_j + \psi) ,$$

where ψ is a 1,0 form for which

$$d\psi + \sum a_{ij} w_i \wedge \overline{w}_j + \sum b_{ij} \overline{w}_i \wedge w_j = 0 .$$

Note that s_F does not depend on the base point z .

The following proposition is easily verified.

Proposition 12.13. If β is a loop in X based at x , if γ is a path with $\gamma(1) = x$, and if ϕ_1 and ϕ_2 are 1-forms on X , then

$$\int_{\gamma\beta\gamma^{-1}} \phi_1\phi_2 = \int_\beta \phi_1\phi_2 + \begin{vmatrix} \int_\gamma \phi_1 & \int_\gamma \phi_2 \\ \int_\beta \phi_1 & \int_\beta \phi_2 \end{vmatrix} . \qquad \square$$

A short calculation, using 12.13, shows that

$$r_{\mathbb{Z}}(z) \circ s_F = \Psi(z) + \Gamma(z) ,$$

where

$$\Psi(z)(w_i \otimes w_j)A_k = \int_{\alpha_k} w_i w_j + \int_{\alpha_k} w_j \int_x^z w_i - \int_{\alpha_k} w_i \int_x^z w_j ,$$

$$\Psi(z)(\sum a_{ij} w_i \otimes \overline{w}_j + \sum b_{ij} \overline{w}_i \otimes w_j)A_k$$

$$= \int_{\alpha_k} (\sum a_{ij} w_i \overline{w}_j + \sum b_{ij} \overline{w}_i w_j + \psi) + \sum a_{ij} \int_{\alpha_k} \overline{w}_j \int_x^z w_i$$

$$- \sum b_{ij} \int_{\alpha_k} \overline{w}_i \int_x^z w_j ,$$

$$\Psi(\overline{w}_i \otimes \overline{w}_j)A_k = \int_{\alpha_k} \overline{w}_i\overline{w}_j ,$$

and $\Gamma(z) \in F^0\mathrm{Hom}(K_{\mathbb{C}}, H^1_{\mathbb{C}})$ is defined by

$$\Gamma(z)(w_i \otimes w_j) = 0 ,$$

$$\Gamma(z)(\sum a_{ij}w_i \otimes \overline{w}_j + \sum b_{ij}\overline{w}_i \otimes w_j)A_k = -\sum a_{ij} \int_{\alpha_k} w_i \int_x^z \overline{w}_j$$

$$+ \sum b_{ij} \int_{\alpha_k} w_j \int_x^z \overline{w}_i ,$$

$$\Gamma(z)(\overline{w}_i \otimes \overline{w}_j)(A_k) = \int_{\alpha_k} \overline{w}_j \int_x^z \overline{w}_i + \int_{\alpha_k} \overline{w}_i \int_x^z \overline{w}_j .$$

Differentiating the first formula we obtain

$$d\Psi(w_i \otimes w_j)(A_k) = (\int_{\alpha_k} w_j)w_i - (\int_{\alpha_k} w_i)w_j ,$$

$$d\Psi(\sum a_{ij}w_i \otimes \overline{w}_j + \sum b_{ij}\overline{w}_i \otimes w_j)A_k \equiv \sum a_{ij}(\int_{\alpha_k} \overline{w}_j)w_i - \sum b_{ij}(\int_{\alpha_k} \overline{w}_i)w_j , \bmod F^0 ,$$

$$d\Psi(\overline{w}_i \otimes \overline{w}_j) = 0 .$$

Thus $d\Psi$ is a holomorphic 1-form and ψ is holomorphic. Since we can always find a holomorphic 1-form that does not vanish at z, $d\Psi(z) \neq 0$ and it follows that Ψ is an immersion.[2] In particular, Ψ is not constant.

Set

$$\{x_1, \ldots, x_N\} = \Psi^{-1}(\pm \overline{\Psi}(x))$$

[2] M. Pulte has shown that $\overline{\Psi}$ is injective.

Denote the normalization of the image of $\overline{\Psi}/\pm$ by Y. The map $\Psi: X \to \Psi(X)$ lifts to a holomorphic map $\tilde{\Psi}: X \to Y$. Since Ψ is an immersion, $\tilde{\Psi}$ is an unramified covering. If X is a generic curve, then there is no finite map $X \to Y$ of degree > 1. Consequently, $\tilde{\Psi}$ is an isomorphism. Since the map $Y \to \operatorname{im}\Psi$ is generically one-to-one, Ψ is generically one-to-one, and so for generic (X,x), $N = 1$. $\square$

13. Variation of the Hodge Filtration

Suppose that X and T are smooth varieties and that $\pi: X \to T$ is a proper regular map of X onto T. Suppose that $b: T \to X$ is a (possibly multivalued) section of π. This allows us to view π as a family of pointed varieties: the fiber $X_t = \pi^{-1}(t)$ has base point $b(t)$.

The set of critical values of π is a proper subvariety C of T. Let $U = T - C$ and denote the pullback of π to U by $\pi: Y \to U$.

Two natural questions arise. Loosely speaking, they can be stated as follows. The vector bundle $\{\pi_k(X_t,b(t)) \otimes \mathbb{C}\}_{t \in U}$ over U has a canonical flat connection that we shall call the Gauss-Manin connection. The first question asks whether this connection is regular along C. The second question is local: Suppose that Δ is a polydisk in U. Since the restriction of the bundle of homotopy groups to Δ is trivial, we can identify $\pi_k(X_t,b(t))$ with $\pi_k(X_0,b(0))$ and consequently we can view $F^p(t) = F^p\pi_k(X_t,b(t)) \otimes \mathbb{C}$ as a subspace of $\pi_k(X_0,b(0)) \otimes \mathbb{C}$. Does $F^p(t)$ vary holomorphically with t? Or, in Griffith's language, is the period mapping

$$U \to \text{M.H.S.'s on } \pi_k(\text{fiber})$$

holomorphic?

In this section we establish the regularity of the Gauss-Manin connection and prove that the period mapping is holomorphic. We remark that in the course of proving 12.12, we verified that the period mapping is holomorphic in the special case of a fixed curve with a variable base point.

So that we do not have to separate the simply connected and non-simply connected cases, we work with the loop space cohomology. The corresponding theorems for homotopy groups can then be obtained from 8.2 and 8.3.

Suppose that $(V,*)$ is a smooth projective manifold. Recall that the canonical filtration

$$\mathbb{C} \subseteq B^{-1}(E^{\bullet}(V)) \subseteq B^{-2}(E^{\bullet}(V)) \subseteq \cdots$$

of the bar construction on its deRham complex gives a sequence

$$\mathbb{C} \subseteq N^{-1}H^{*}(\Omega_{*}V;\mathbb{C}) \to N^{-2}H^{*}(\Omega_{*}V;\mathbb{C}) \to \cdots$$

of cohomology groups. Note that the weight filtration on $N^{-s}H^{*}(\Omega_{*}V)$ is defined by

$$W_{\ell}N^{-s}H^{*}(\Omega_{*}V) = \operatorname{im}\{N^{-\ell}H^{*}(\Omega_{*}V) \to N^{-s}H^{*}(\Omega_{*}V)\} .$$

<u>Proposition</u> 13.1. For each $\ell > 1$, there is a long exact sequence of M.H.S.'s

$$\cdots \to N^{-\ell+1}H^{k}(\Omega_{*}V) \xrightarrow{i_*} N^{-\ell}H^{k}(\Omega_{*}V) \xrightarrow{p_*} [\otimes^{\ell} s^{-1} \tilde{H}^{*}(V)]^{k} \xrightarrow{d_*} N^{-\ell+1}H^{k+1}(\Omega_{*}V) \to \cdots .$$

<u>Proof</u>. From 9.1 it follows that

$$0 \to B^{-\ell+1}(E^{\bullet}(V)) \xrightarrow{i} B^{-\ell}(E^{\bullet}(V)) \xrightarrow{p} \otimes^{\ell} s^{-1} IE^{\bullet}(V) \to 0$$

is an exact sequence of M.H.C.'s. This gives the desired long exact sequence. To prove that the sequence is a long exact sequence of M.H.S.'s, we have to show that each map in the sequence is defined over $\mathbb{Z}$ and preserves the Hodge and weight filtrations. Clearly, i_* and p_* are morphisms of M.H.S.'s, while d_* preserves the Hodge filtration. Recall that the differential d of $\mathcal{B}(E^\bullet(V))$ is of the form $d_E + d_I$. It is easy to check that $d_* = (d_I)_*$, where $d_I\colon \mathcal{B}^{-\ell} \to \mathcal{B}^{-\ell+1}$ is the internal differential. It follows from this that d_* preserves the weight filtration.

Finally, the canonical filtration

$$\mathbb{Z} \subseteq \mathcal{B}^{-1}(S^\bullet(V)) \subseteq \mathcal{B}^{-2}(S^\bullet(V)) \subseteq \ldots$$

of the bar construction on its singular cochain complex gives a sequence of cohomology groups

$$\mathbb{Z} \subseteq N^{-1}H^*(\Omega_* V;\mathbb{Z}) \to N^{-2}H^*(\Omega_* V;\mathbb{Z}) \to \ldots$$

with the property that

$$N^{-\ell}H^*(\Omega_* V;\mathbb{Z}) \otimes \mathbb{C} = N^{-\ell}H^*(\Omega_* V;\mathbb{C}) \quad .$$

It follows that all maps in the exact sequence are defined over $\mathbb{Z}$. □

Denote the bundle

$$\{N^{-\ell}H^k(\Omega_{b(t)}X_t;\mathbb{C})\}_{t \in U} \to U$$

by

$$N^k_\ell \to U \quad .$$

This bundle has a canonical flat holomorphic connection, that we shall call the <u>Gauss-Manin connection</u>.

<u>Theorem</u> 13.2. The Gauss-Manin connection on $\mathcal{N}_\ell^k \to U$ is regular along C and has quasi-unipotent monodromy.

<u>Proof</u>. The proof is by induction on ℓ . When $\ell = 1$,

$$N^{-1}H^k(\Omega_* V) = \mathbb{C} \oplus H^{k+1}(V;\mathbb{C}) \quad ,$$

so that in this case the result follows directly from the corresponding well known result for cohomology (7.9, [56]).

Denote the flat bundle whose fiber over t is $[\otimes^\ell \tilde{H}^*(X_t;\mathbb{C})]^k$ by $\mathcal{H}_\ell^k$ and its canonical extension (II. 5.4, [56]) to U by $\tilde{\mathcal{H}}_\ell^k$. Denote the canonical extension of $\mathcal{N}_\ell^k$ to U by $\tilde{\mathcal{N}}_\ell^k$. From 13.1 and the exactness properties of the canonical extension (c.f. II.5.4, [56]), there is an exact sequence of flat bundles

$$\cdots \to \tilde{\mathcal{H}}_\ell^{k-1} \to \tilde{\mathcal{N}}_{\ell-1}^k \to \tilde{\mathcal{N}}_\ell^k \to \tilde{\mathcal{H}}_\ell^k \to \tilde{\mathcal{N}}_{\ell-1}^{k+1} \to \cdots$$

over U . From (III.2.3, [56]) and induction on ℓ, it follows that the monodromy representation of $\tilde{\mathcal{N}}_\ell^k \to U$ is quasi-unipotent. □

Suppose that Δ is a polydisk imbedded in U . Since the restriction of the family $Y \to U$ to Δ is a trivial C^∞ fiber bundle, there is a canonical isomorphism

$$N^{-s}H^k(\Omega_{\sigma(t)}X_t) \to N^{-s}H^k(\Omega_{\sigma(0)}X_0) \quad .$$

Via this isomorphism, we can view

$$F^p(t) = F^pN^{-s}H^k(\Omega_{\sigma(t)};\mathbb{C})$$

as a subspace of $N^{-s}H^k(\Omega_{\sigma(0)}X_0;\mathbb{C})$.

Theorem 13.3. The subspace $F^p(t)$ of $N^{-s}H^k(\Omega_{\sigma(0)}X_0;\mathbb{C})$ depends holomorphically on t.

Proof. There are two main steps in the proof. The first is to show that $F^p(t)$ depends smoothly on t. The second is to show that if

$$s\colon \Delta \to N^{-s}H^k(\Omega_{\sigma(0)}X_0;\mathbb{C})$$

is a smooth function with $s(t) \in F^p(t)$ for all t, then

$$\frac{\partial s}{\partial \bar{t}}(t) \in F^p(t)$$

for all t. This implies, via (4.27, [20]), that $F^p(t)$ depends holomorphically on t. The proof is long and we only sketch many of the details.

It suffices to consider the case when $\dim\Delta = 1$. Abusing notation, we shall denote the restriction of $Y \to U$ to Δ by $Y \to \Delta$.

Recall the following facts from Griffiths ([20], pp. 811-813): Y is a Kaehler manifold and each $X_t = \pi^{-1}(t)$ is a compact Kaehler manifold with metric that varies holomorphically with t. There is a C^∞ trivialization $\phi\colon X_0 \times \Delta \to X$, a covering $\{V_\alpha\}$ of X_0 by coordinate charts V_α with holomorphic coordinates $(z_\alpha^1,\dots,z_\alpha^n)$, and linearly independent 1 forms $w_\alpha^1(t),\dots,w_\alpha^n(t)$ on V_α such that

(i) $\xi_\alpha(z_\alpha,\bar{z}_\alpha,t) = \phi^*(z_\alpha)$ is holomorphic in t and that the transition functions $h_{\alpha\beta}(\xi_\beta,t)$, defined by

$$\xi_\alpha(z,t) = h_{\alpha\beta}(\xi_\beta(z,t),t) ,$$

are holomorphic in ξ_β and t.

(ii) $w_\alpha^j(t)$ depends holomorphically on t. In fact,

$$w_j^\alpha(t) = dz_\alpha^j + \sum_k \phi_{\alpha k}^j(t)\, d\bar{z}_\alpha^k ,$$

where $\phi_{\alpha k}^j(t)$ depends holomorphically on t.

(iii) The $w_\alpha^j(t)$ define the (almost) complex structure on X_t.

Now, according to Kodaira-Spencer, via Griffiths, given a harmonic (p,q) form

$$\phi = \sum \phi_{I,\bar{J}}\, dz_I \wedge d\bar{z}_J$$

on X_0, there is a (p,q) form

$$\phi(t) = \sum \phi_{I,\bar{J}}\, w_I(t) \wedge \bar{w}_J(t) ,$$

which is C^∞ in t and harmonic with respect to the metric on X_t. Taylor expanding $\phi(t)$, we get

$$\phi(t) = \phi_1 + t(\phi'+\phi_2) + \bar{t}(\phi''+\phi_3) + o(|t|) ,$$

where ϕ_1, ϕ' and ϕ'' are forms of type (p,q) on X_0, and ϕ_2 and ϕ_3 are forms on X_0 of types $(p-1,q+1)$ and $(p+1,q-1)$, respectively.

This first step in proving that the Hodge filtration varies smoothly with t is the following lemma.

Lemma 13.4. Suppose that $w_1(t),\ldots,w_m(t)$ is a basis of the harmonic (p,q) forms on X_t that depends smoothly on t. If $w(t)$ is a closed (p,q) form on X_t that varies smoothly with t, then the unique complex valued functions $a_i(t)$, defined by

$$\{w(t)\} = \sum_i a_i(t)\{w_i(t)\} \quad \text{in} \quad H^{p,q}(X_t) \quad ,$$

are smooth. Moreover, if $\psi(t)$ is the unique $\bar{\partial}$ coexact $(p-1,q)$ form satisfying

$$\bar{\partial}\psi(t) + w(t) = \sum a_i(t)w_i(t) \ ,$$

then $\psi(t)$ depends smoothly on t .

Proof. The proof of the first assertion is straightforward and is left to the reader. To prove the second assertion, it suffices to prove that if $u(t)$ is an exact (p,q) form depending smoothly on t , and if $\psi(t)$ is the unique $\bar{\partial}$-coexact form such that

$$\bar{\partial}\psi(t) = u(t) \ ,$$

then $\psi(t)$ depends smoothly on t .

Denote the Green's operator associated to the Laplacian of $\bar{\partial}_t$ on X_t by G_t . A result of Kodaira and Spencer (c.f. [57]) asserts that is $\phi(t)$ is a smoothly varying form, then so is $G_t\ \phi(t)$.

Observe that since $du = 0$ and u has pure type, $\bar{\partial}_t u(t) = 0$. Recall that

$$\bar{\partial}\colon \bar{\partial}^*\bar{\partial}\ \mathcal{E}^{p,q}(V) \to \mathcal{E}^{p,q}(V)$$

is injective, where V is a compact Kaehler manifold and $\bar{\partial}^*$ denotes the adjoint of $\bar{\partial}$.

Let $\phi(t) = G_t\, u(t)$. That is

$$(\bar\partial_t \bar\partial_t^* + \bar\partial_t^* \bar\partial_t)\phi(t) = u(t) \quad .$$

Since $\bar\partial_t u(t) = 0$, it follows from the preceeding remark that $\bar\partial_t^* \bar\partial_t\, \phi(t) = 0$. Consequently,

$$u(t) = \bar\partial_t\, \bar\partial_t^*\, \phi(t) \quad .$$

Let $\psi(t) = \bar\partial_t^*\, G_t\, u(t)$, so that

$$\begin{aligned} \bar\partial_t \psi(t) &= \bar\partial_t \bar\partial_t^*\, \phi(t) \\ &= u(t) \; . \end{aligned}$$

Note that $\psi(t)$ depends smoothly on t .

The proof will be complete if we can show that $d_t \psi(t) = u(t)$, or equivalently, that $\partial_t \psi(t) = 0$.

Denote the adjoint of cupping with the Kaehler form by

$$\Lambda\colon E^{p,q} \to E^{p-1,q-1} \quad .$$

The following basic identity can be found in ([57], p. 112):

$$i\bar\partial^* = \partial\Lambda - \Lambda\partial \; .$$

From this we get

$$i(\partial\bar\partial^* + \bar\partial^*\partial) = \partial(\partial\Lambda - \Lambda\partial) + (\partial\Lambda - \Lambda\partial)\partial = 0 \quad .$$

Therefore $\partial\,\bar{\partial}^* = -\bar{\partial}^*\partial$. Using this, one can check that ∂ commutes with the Laplacian $\bar{\partial}\,\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$, from which it follows that $\partial G = G\partial$. Consequently,

$$\partial(\bar{\partial}^* G) = -(\bar{\partial}^* G)\partial \quad ,$$

and, finally,

$$\partial_t \phi(t) = \partial_t(\bar{\partial}_t^* G_t)u(t) = -(\bar{\partial}_t^* G_t)(\partial_t u) = 0 \quad . \qquad \square$$

Suppose that $w_1(t),\ldots,w_N(t)$ are harmonic forms on X_t which depend smoothly on t and whose cohomology classes form a bigraded basis of $\tilde{H}^*(X_t;\mathbb{C})$. Denote the dual basis of $s^{-1}\tilde{H}_*(X_t;\mathbb{C})$ by $W_1,\ldots,W_N$. It may be extended to a bigraded basis (W_I) of $\mathbb{L}(X_t) = \mathbb{L}(s^{-1}\tilde{H}_*(X_t;\mathbb{C}))$. Let $\mathbb{L}(X_t) = \bigoplus \mathbb{L}^{-p,q}$ be the canonical bigrading. Denote the underlying C^∞ manifold of X_0 by V . Via the C^∞ trivialization $\phi\colon V\times\Delta \to X$ of the family, identify $E^\bullet(X_t)$ with $E^\bullet(V)$.

Corollary 13.5. If

$$(\omega(t) = \sum w_i(t)W_i + \sum_{|I|\geq 2} w_I(t)W_I \ , \ \delta(t))$$

is the unique formal connection on $E^\bullet(X_t)$ with each $w_I(t)$ $\bar{\partial}_t$ coclosed when $|I|\geq 2$, then $\omega(t)$ and $\delta(t)$ depend smoothly on t . Moreover, if $W_I \in \mathbb{L}^{-p,q}$, then

(i) $w_I(t) \in F^p E^\bullet(X_t)$,

(ii) $\left.\dfrac{\partial w_I}{\partial \bar{t}}\right|_{t=0} \in F^p E^\bullet(X_t)$,

(iii) $\left.\dfrac{\partial w_I}{\partial t}\right|_{t=0} \in F^{p-1} E^\bullet(X_t)$. $\square$

Recalling that V denotes the underlying manifold of X_0 and that (V_α) is an open cover of V, we can assume that the base point $\sigma(0)$ lies in some $V_\alpha \cap V_\beta$. Since the function $(V_\alpha \cap V_\beta) \times \Delta \to \mathbb{C}^n \times \Delta$: $(\xi,t) \to (h_{\alpha\beta}(\xi,t),t)$ defines the local analytic structure on the family, the base point section $b: t \to (\sigma(0),t)$ is holomorphic. We will show that, with respect to this "constant" base point section, $F^p(t)$ varies holomorphically. Later we will show that $F^p(t)$ varies holomorphically with the base point. This constant base point section defines augmentations $E^\bullet(X_t) \to \mathbb{C}$ that are compatible with the identifications $E^\bullet(X_t) = E^\bullet(V)$ induced by ϕ. Thus, in this case, we can identify $B(E^\bullet(X_t))$ with $B(E^\bullet(V))$.

Extend (W_I) to a bigraded basis (U_I) of $U\mathbb{L}(X_t)$, the enveloping algebra of $\mathbb{L}(X_t)$. Denote its canonical bigrading by $\bigoplus A^{-p,q}$. Combining 13.5 with the fact that the transport $T(t)$ of $(\omega(t),\delta(t))$ is defined by

$$T(t) \quad = \quad 1 + [\omega(t)] + [\omega(t)|\omega(t)] + \dots \quad ,$$

one can prove the following result.

<u>Corollary</u> 13.6. If

$$T(t) \quad = \quad \sum u_I(t)\, U_I$$

is the transport of $\omega(t)$, then $T(t)$ depends smoothly on t and, if $U_I \in A^{-p,q}$, then

(i) $u_I(t) \in F^p\mathcal{B}(\mathcal{E}^\bullet(X_t))$,

(ii) $\left.\dfrac{\partial u_I}{\partial \bar{t}}\right|_{t=0} \in F^p\mathcal{B}(\mathcal{E}^\bullet(X_t))$,

(iii) $\left.\dfrac{\partial u_I}{\partial t}\right|_{t=0} \in F^{p-1}\mathcal{B}(\mathcal{E}^\bullet(X_t))$. □

Now, $T(t)$ defines a d.g. coalgebra homomorphism

$$T(t)\colon \mathrm{Hom}(U\mathbb{L}(X_t)(s)\ ,\ \mathbb{C}) \to \mathcal{B}^{-s}(\mathcal{E}^\bullet(X_t))$$

$$\phi \mapsto \sum u_I\ \phi(U_I)$$

that preserves the Hodge filtration, while the integration map

$$\mathcal{E}^\bullet(X_t) = \mathcal{E}^\bullet(V) \to S^\bullet(V;\mathbb{C})$$

defines a d.g. colagebra homomorphism

$$\mathcal{B}^{-s}(\mathcal{E}^\bullet(X_t)) \to \mathcal{B}^{-s}(S^\bullet(V;\mathbb{C}))$$

that is independent of t . We view this as a topological marking of $\mathcal{B}^{-s}(\mathcal{E}^\bullet(X_t))$. From 13.6 it follows that their composite

$$\tilde{T}(t)\colon \mathrm{Hom}(U\mathbb{L}(X_t)(s)\ ,\ \mathbb{C}) \to \mathcal{B}^{-s}(V;\mathbb{C})$$

is a d.g. coalgebra homomorphism that depends smoothly on t . For $p \geq 0$, define

$$T^p(t) = \sum_{U_I \in F^{-p+1}U\mathbb{L}} u_I(t)\ U_I \ .$$

From the fact that

$$F^pHom(UIL(X_t)(s), \mathbb{C}) = Hom(UIL(X_t)(s)/F^{-p+1}, \mathbb{C}),$$

it follows that $T^p(t)$ defines a chain map

$$T^p(t)\colon F^pHom(UIL(X_t)(s), \mathbb{C}) \to B^{-s}(S^\bullet(V))$$

that depends smoothly on t. A direct consequence of 11.5 is that the image of

$$T^p(t)_*\colon H^*(F^pHom(UIL(X_t)(s), \mathbb{C})) \to N^{-s}H^*(\Omega_{\sigma(t)}X_t;\mathbb{C})$$

is $F^p(t)$. This proves that $F^p(t)$ depends smoothly on t.

Next, it follows from 13.6 that if $\phi \in F^pHom(UIL(X_t)(s), \mathbb{C})$, then

(a) $$\left.\frac{\partial}{\partial \bar{t}}\right|_{t=0} T^p(t)(\phi) \in F^pB^{-s}(E^\bullet(X_t)),$$

(b) $$\left.\frac{\partial}{\partial t}\right|_{t=0} T^p(t)(\phi) \in F^{p-1}B^{-s}(E^\bullet(X_t)).$$

From (a) we conclude that $F^p(t)$ varies holomorphically with t and from (b) that the Gauss-Manin connection ∇ satisfies Griffith's transversality:

$$\nabla\colon \mathcal{O}(F^p) \to \mathcal{O}(F^{p-1}) \otimes \Omega^1(\Delta).$$

Since the M.H.S. on $H^*(B^{-s}(E^\bullet(X_t)))$ does not depend upon the base point when the fibers are simply connected (11.8), we have completed the proof of 13.3 in this case, and we have also established Griffith's transversality.

Our final task is to prove that the Hodge filtration depends holomorphically on t for an arbitrary base point section and to establish Griffith's transversality in this case. Since the space of infinitessiaml deformations of the pointed variety $(X_0, \sigma(0))$ is the direct sum of the space of infinitessiaml deformations of X_0 with the holomorphic tangent space $T'_{\sigma(0)}X_0$ of X_0 at $\sigma(0)$, we need only show that if X is a Kaehler manifold and $\sigma:\Delta \to X$ a holomorphic arc in X, then the pth part of the Hodge filtration $F^p(t)$ of $H^*(\Omega_{\sigma(t)}X;\mathbb{C})$ varies holomorphically when viewed as a subspace of $H^*(\Omega_{\sigma(0)}X;\mathbb{C})$ via the isomorphism

$$\Phi_t^*: H^*(\Omega_{\sigma(t)}X) \to H^*(\Omega_{\sigma(0)}X)$$

induced by the smooth map

$$\Phi_t:\Omega_{\sigma(0)}X \to \Omega_{\sigma(t)}X$$

$$\alpha \mapsto \sigma_t^{-1}\alpha\,\sigma_t,$$

where σ_t is the image under σ of the line segment in Δ that joins 0 to t.

Now, Φ_t^* is induced by the map

$$\int w_1 \cdots w_r \mapsto \sum_{0 \le i \le j \le r} \left(\int_{\sigma_t^{-1}} w_1 \cdots w_i\right) \int w_{i+1} \cdots w_j \left(\int_{\sigma_t} w_{j+1} \cdots w_r\right)$$

of iterated integrals. From this it follows that $F^p(t)$ depends smoothly on t. Moreover, if $t = u + iv$, then

$$\left.\frac{\partial}{\partial u}\right|_{t=0} \Phi_t^* \int w_1 \cdots w_r = \langle w_r, \sigma_* \frac{\partial}{\partial u}\rangle \int w_1 \cdots w_{r-1} - \langle w_1, \sigma_* \frac{\partial}{\partial u}\rangle \int w_2 \cdots w_{r-1}$$

A similar formula, obtained by replacing u by v in the previous formula, also holds. It follows that

$$\left.\frac{\partial}{\partial \bar{t}}\right|_{t=0} \Phi_t^*: H^0(\Omega_{\sigma(0)}X) \to H^0(\Omega_{\sigma(0)}X)$$

is induced by the map

$$(*) \qquad \int w_1 \cdots w_r \to \langle w_r, \sigma_* \frac{\partial}{\partial \bar{t}}\rangle \int w_1 \cdots w_{r-1} - \langle w_1, \sigma_* \frac{\partial}{\partial \bar{t}}\rangle \int w_2 \cdots w_r$$

of iterated integrals and that

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \Phi_t^*: H^0(\Omega_{\sigma(0)}X) \to H^0(\Omega_{\sigma(0)}X)$$

is induced by

$$(\dagger) \qquad \int w_1 \cdots w_r \to \langle w_r, \sigma_* \frac{\partial}{\partial t}\rangle \int w_1 \cdots w_{r-1} - \langle w_1, \sigma_* \frac{\partial}{\partial t}\rangle \int w_2 \cdots w_r .$$

Since X is a Kaehler manifold, we can assume that each w_j is of pure type. It follows from * and the fact that $\langle w, \sigma_* \partial/\partial\bar{t} \rangle = 0$ if w is a $1,0$ form that

$$\left.\frac{\partial}{\partial \bar{t}}\right|_{t=0} F^p(t) \subseteq F^p(0),$$

and from † that

$$\left.\frac{\partial}{\partial t}\right|_{t=0} F^p(t) \subseteq F^{p-1}(0).$$

This completes the proof of 13.3 and establishes Griffith's transversality for homotopy.

<u>Theorem</u> 13.7. The Gauss-Manin connection ∇ on $N_s^k \to \Delta$ satisfies

$$\nabla : \mathcal{O}(F^p) \to \mathcal{O}(F^{p-1}) \otimes \Omega^1(\Delta). \quad \square$$

Appendix

In this appendix we prove a dual version of the Poincare-Birkhoff-Witt theorem for the bar construction $\mathcal{B}(E^\bullet)$ on a commutative d.g. algebra $E^\bullet$. The case when the algebra $E^\bullet$ is simply connected is proved in ([24], p.56).

Theorem A.1. If $E^\bullet$ is a commutative d.g. algebra over the field k of characteristic zero, then there is a natural splitting $s : Q\mathcal{B}(E^\bullet) \to I\mathcal{B}(E^\bullet)$ of the natural projection $\pi : I\mathcal{B}(E^\bullet) \to Q\mathcal{B}(E^\bullet)$. The splitting s commutes with the differentials. Moreover, the map $\Lambda(Q\mathcal{B}(E^\bullet)) \to \mathcal{B}(E^\bullet)$ induced by s, from the free commutative d.g. algebra generated by $Q\mathcal{B}(E^\bullet)$ into $\mathcal{B}(E^\bullet)$, is a d.g. algebra isomorphism.

In fact, the idempotent $\gamma = s\circ\pi$ is given by the formula

$$\gamma[a_1|\dots|a_n] = \sum_{m=1}^{n} \sum_{r_1+\dots+r_m=n} \sum_{\sigma\in sh(r_1,\dots,r_m)} (-1)^{m-1} \frac{\varepsilon(\sigma)}{m}[a_{\sigma(1)}|\dots|a_{\sigma(n)}],$$

where $sh(r_1,\dots,r_m)$ denotes the shuffles of $\{1,\dots,m\}$ of type $(r_1,\dots,r_m)$ and where $\varepsilon(\sigma)\in\{1,-1\}$ is the permutation symbol defined in paragraph (5.2) of [24].

Corollary A.2. The natural map

$$QH^*(\mathcal{B}(E^\bullet)) \to H^*(Q\mathcal{B}(E^\bullet))$$

is an isomorphism. □

Proof. First consider the case where the underlying algebra of $E^\bullet$ is the free graded commutative algebra $\Lambda(a_1,\dots,a_n)$, where $\deg a_j \geq 2$ for each j. The bar construction $\mathcal{B}(E^\bullet)$ is of finite type and so

$$A = \mathrm{Hom}(\mathcal{B}(E^\bullet), k)$$

is a cocommutative d.g. Hopf algebra. From the Poincare-Birkhoff-Witt theorem and the Milnor-Moore theorem [37], we know that A is isomorphic, as a d.g. coalgebra, with the free symmetric coalgebra $S(PA)$ on the set of primitives PA of A. The dual of the natural projection $p : \overline{S(PA)} \to PA$ is the map $s : Q\mathcal{B}(E^{\bullet}) \to I\mathcal{B}(E^{\bullet})$ that we seek, and the d.g. algebra map

$$\Lambda s : \Lambda(Q\mathcal{B}(E^{\bullet})) \to \mathcal{B}(E^{\bullet})$$

induced by s is dual to the d.g. coalgebra isomorphism $A \to S(PA)$. Consequently, Λs is an isomorphism of d.g. algebras.

The next step is to derive the formula for the associated idempotent $\gamma : I\mathcal{B}(E^{\bullet}) \to I\mathcal{B}(E^{\bullet})$ in this case. For the time being, we will forget about differentials. Denote the standard basis $a_1, a_2, \ldots, a_n, a_1 \wedge a_2, a_1 \wedge a_3, \ldots, a_1 \wedge \cdots \wedge a_n$ of $IE^{\bullet}$ by $b_1, b_2, \ldots, b_N$, respectively. Denote the dual basis of $\mathrm{Hom}(IE^{\bullet}, k)$ by $X_1, X_2, \ldots, X_N$. Desuspend (i.e., reduce the degree of) each X_j by 1. Denote the free graded associative ring generated by $X_1, \ldots, X_n$ by $k\langle X_1, \ldots, X_N\rangle$ and the ring of formla power series in the indeterminates $X_1, \ldots, X_N$ with coefficients in $\mathcal{B}(E^{\bullet})$ by

$$\mathcal{B}(E^{\bullet})\langle\langle X_1, \ldots, X_N \rangle\rangle .$$

By defining each X_j to be primitive, these algebras become (complete) Hopf algebras.

The key to establishing the formula for γ is to view a certain element of $\mathcal{B}(E^{\bullet})\langle\langle X_1, \ldots, X_N\rangle\rangle$ as a Hopf algebra isomorphism between $k\langle X_1, \ldots, X_N\rangle$ and A. Viewing the isomorphism in this way allows one to take its logarithm, which turns out to be the correct thing to do.

Consider the element

$$T = 1 + \sum [b_j]X_j + \sum [b_i|b_j]X_iX_j + \sum [b_i|b_j|b_k]X_iX_jX_k + \cdots$$

of $\mathcal{B}(E^\bullet)\langle\langle X_1,\ldots,X_N\rangle\rangle$. An element φ of $A = \mathrm{Hom}(\mathcal{B}(E^\bullet), k)$ can be evaluated on T to give an element $\langle T,\varphi\rangle$ of $k\langle X_1,\ldots,X_N\rangle$. That is, T defines a linear map

$$\theta : A \to k\langle X_1,\ldots,X_N\rangle$$
$$\varphi \mapsto \langle T,\varphi\rangle.$$

In fact, θ is a graded Hopf algebra isomorphism (c.f. (6.17), [2]).

Since $\mathcal{B}(E^\bullet)$ is a commutative algebra,

$$T = \exp(\log T)$$

in $\mathcal{B}(E^\bullet)\langle\langle X_1,\ldots,X_N\rangle\rangle$. It follows from formula (b) ([24], p.58) that T is a group-like element of the complete Hopf algebra $\mathcal{B}(E^\bullet)\langle\langle X_1,\ldots,X_N\rangle\rangle$. That is, $\Delta T = T\hat{\otimes}T$. It follows that log T is primitive. That is,

$$\Delta\log T = \log T\hat{\otimes}1 + 1\hat{\otimes}\log T.$$

Consequently, the linear map $A \to k\langle X_1,\ldots,X_N\rangle : \varphi \mapsto \langle \log T,\varphi\rangle$ lands in the set of primitive elements P of $k\langle X_1,\ldots,X_N\rangle$.

The isomorphism $k\langle X_1,\ldots,X_N\rangle \to S(P)$ induces a direct sum decomposition

$$k\langle X_1,\ldots,X_N\rangle = k\oplus P\oplus S^2(P)\oplus S^3(P)\oplus\ldots,$$

where $S^m(P)$ denotes the symmetric tensors of rank m on P. It is not hard to check, because $\mathcal{B}(E^\bullet)$ is commutative, that the map $A \to k\langle X_1,\ldots,X_n\rangle : \varphi \mapsto \langle(\log T)^m,\varphi\rangle$ lands in $S^m(P)$. Since $T = \exp(\log T)$, It follows that the composite

$$IA \to Ik\langle X_1,\ldots,X_N\rangle \xrightarrow{p} P$$

takes φ to $\langle \log T,\varphi\rangle$. Since the diagram

$$\begin{array}{ccc} IA & \longrightarrow & Ik\langle X_1\ldots,X_N\rangle \\ p\downarrow & & \downarrow p \\ PA & \longrightarrow & P \end{array}$$

commutes, and since γ is dual to $p : IA \to PA$, it follows that

$$\log T = \sum \gamma[b_j]X_j + \sum \gamma[b_i|b_j]X_iX_j + \sum \gamma[b_i|b_j|b_k]X_iX_jX_k + \ldots .$$

But,

$$\log T = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m}(T-1)^m$$

$$= \ldots$$

$$= \sum_{n=1}^{\infty} \sum_{(j_1,\ldots,j_n)} \sum_{m=1}^{n} \sum_{r_1+\ldots+r_m=n} \sum_{\sigma\in sh(r_1,\ldots,r_m)} (-1)^{m-1}\frac{\varepsilon(\sigma)}{m}[b_{j_{\sigma(1)}}|\ldots|b_{j_{\sigma(n)}}]X_{j_1}X_{j_2}\ldots X_{j_n}.$$

Comparing the coefficients of $X_1X_2\ldots X_n$ we see that

$$(*) \quad \gamma[a_1|\ldots|a_n] = \sum_{m=1}^{n} \sum_{r_1+\ldots+r_m=n} \frac{(-1)^{m-1}}{m}[a_1|\ldots|a_{r_1}]\wedge[a_{r_1+1}|\ldots|a_{r_2}] \ldots\wedge[a_{n-r_{m+1}}|\ldots|a_n]$$

$$= \sum_{m=1} \sum_{r_1+\ldots+r_m=n} \sum_{\sigma\in sh(r_1,\ldots,r_m)} (-1)^{m-1}\frac{\varepsilon(\sigma)}{m}[a_{\sigma(1)}|\ldots|a_{\sigma(n)}] .$$

Now suppose that E^* is an arbitrary commutative d.g. algebra. Define $\gamma : IB(E^*) \to IB(E^*)$ using the formula above. It is immediate that $\gamma = \gamma^2$, that γ commutes with the differential and that γ is a graded map of degree 0.

If $a_1,\ldots,a_n \in IE^{\bullet}$, then we can find a free graded commutative algebra

$$A^{\bullet} = \Lambda(\bar{a}_1,\ldots,\bar{a}_n)$$

where

(a) $\deg \bar{a}_j \geq 2 \quad j = 1,\ldots,n$,

(b) $\deg \bar{a}_j \equiv \deg a_j \mod 2 \quad j = 1,\ldots,n.$

The natural map $A^{\bullet} \to E^{\bullet}$ defined by taking $\bar{a}_j$ to a_j is $\mathbb{Z}/2$ graded and thus induces a $\mathbb{Z}/2$ graded Hopf algebra map $\mathcal{B}(A^{\bullet}) \to \mathcal{B}(E^{\bullet})$ that commutes with γ. The assertions of the theorem now follow. For example, we will prove that $\ker\gamma = I^2\mathcal{B}(E^{\bullet})$.

First, if $u \in I^2\mathcal{B}(E^{\bullet})$, then there exists a sub-algebra $F^{\bullet}$ of $E^{\bullet}$ generated by $a_1,\ldots,a_n$ say, such that $u \in I^2\mathcal{B}(F^{\bullet})$. Pick a free graded commutative algebra $A^{\bullet} = \Lambda(\bar{a}_1,\ldots,\bar{a}_n)$ as above and a map $A^{\bullet} \to F^{\bullet}$. Denote by $\rho : \mathcal{B}(A^{\bullet}) \to \mathcal{B}(E^{\bullet})$ the induced map. There exists $\hat{u} \in I^2\mathcal{B}(A^{\bullet})$ such that $\rho(\hat{u}) = u$. Since $\gamma(\hat{u}) = 0$ and $\rho\circ\gamma = \gamma\circ\rho$, it follows that $\gamma(u) = 0$. That is, $I^2\mathcal{B}(E^{\bullet}) \subseteq \ker\gamma$. One the other hand, it follows directly from (*) that $\operatorname{im}(\gamma-1) \subseteq I^2\mathcal{B}(E^{\bullet})$. Since γ is an idempotent, $\ker\gamma = \operatorname{im}(1-\gamma)$. Thus $\ker\gamma \subseteq I^2\mathcal{B}(E^{\bullet})$.

The assertion that the map

$$\Lambda s : \Lambda(Q\mathcal{B}(E^{\bullet})) \to \mathcal{B}(E^{\bullet})$$

is a d.g. algebra isomorphism is proved similarly. □

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