18. Relative Weight Filtrations

Universal mixed elliptic motives are mixed Tate motives with additional structure. This extra structure includes a second weight filtration $W_\bullet$ and a nilpotent endomorphism N of V that preserves $W_\bullet$. Every universal mixed elliptic motive V will thus have two weight filtrations: its weight filtration $M_\bullet$ as an object of MTM and the second weight filtration $W_\bullet$. One of axioms of a universal mixed elliptic motive is that $M_\bullet$ be the relative weight filtration of the nilpotent endomorphism N of the filtered vector space $(V, W_\bullet)$. In this section we review Deligne's definition [4] of the relative weight filtration of a nilpotent endomorphism of a filtered vector space.

Although this material is technical, it is essential. Just as a pure motive (pure Hodge structure, etc) acquires a monodromy weight filtration when it degenerates, a mixed motive (mixed Hodge structure, etc) will acquire a second weight filtration when it degenerates. The monodromy weight filtration is associated to the original weight filtration and the logarithm of the local monodromy transformation (assumed to be unipotent). In the current situation, the degeneration occurs when a smooth elliptic curve degenerates to the nodal cubic — that is, as one approaches the cusp of $\mathcal{M}_{1,1}$.

More information about relative weight filtrations can be found in [43]. A concise expsosition is given in Section 7 of [23].

18.1. **The weight filtration of a nilpotent endomorphism.** There is a natural weight filtration of a vector space associated to a nilpotent endomorphism N of it.

Proposition 18.1. *If N is a nilpotent endomorphism of a finite dimensional vector space V over a field of characteristic zero, then there is a unique filtration*

$$0 = W(N)_{-m-1} \subseteq W(N)_{-m} \subseteq W(N)_{-m+1} \subseteq \cdots \cdots \subseteq W(N)_{m-1} \subseteq W(N)_m = V$$

of V such that

(i) *for all $n \in \mathbb{Z}$, $NW(N)_n \subseteq W(N)_{n-2}$;*
(ii) *for each $k \in \mathbb{Z}$, $N^k : \operatorname{Gr}^{W(N)}_k V \to \operatorname{Gr}^{W(N)}_{-k} V$ is an isomorphism.*

The filtration $W(N)_\bullet$ of V is called the *weight filtration of N*.

Note that $W(N)_\bullet$ is centered at 0. When V is a a motive of weight m, it is natural to reindex the weight filtration of a nilpotent endomorphism N of V so that it is centered at m. The shifted filtration

$$M_k V := W(N)_{k-m}$$

is centered at m. The reindexed filtration $M_\bullet$ satisfies $NM_k \subseteq M_{k-2}$ and

$$N^k : \operatorname{Gr}^M_{m+k} V \xrightarrow{\simeq} \operatorname{Gr}^M_{m-k} V$$

is an isomorphism for all $k \in \mathbb{Z}$. We will call the shifted weight filtration $M_\bullet$ the monodromy weight filtration of $N : V \to V$.

Example 18.2. Let $H = \mathbb{C}\mathbf{w} \oplus \mathbb{C}\mathbf{a}$, regarded as a vector space of weight 1. Let N be the nilpotent endomorphism $\mathbf{a}\frac{\partial}{\partial \mathbf{w}}$ of H. It induces a nilpotent endomorphism of $V = S^n H$, the space of homogeneous polynomials in $\mathbf{a}$ and $\mathbf{w}$ of degree n, which we regard as a vector space of weight n. The shifted monodromy weight filtration

$M_\bullet$ of V is obtained by giving $\mathbf{a}$ weight 0 and $\mathbf{w}$ weight 2. The monomial $\mathbf{a}^{n-j}\mathbf{w}^j$ has weight $2j$. Then

$$M_k V = \text{span of the monomials } \mathbf{a}^{n-j}\mathbf{w}^j \text{ of weight} \leq k.$$

18.2. **The weight filtration of a nilpotent endomorphism of a filtered vector space.** Now suppose that N is a nilpotent endomorphism of a *filtered* finite dimensional vector space V over a field of characteristic zero. That is, V has a filtration

$$0 \subseteq \cdots \subseteq W_{m-1}V \subseteq W_m V \subseteq W_{m+1}V \subseteq \cdots \subseteq V$$

which is stable under N.

Since N preserves the weight filtration, it induces a nilpotent endomorphism

$$N_m := \operatorname{Gr}^W_m N : \operatorname{Gr}^W_m V \to \operatorname{Gr}^W_m V.$$

of the mth weight graded quotient of V. Proposition 18.1 implies that each graded quotient has a weight filtration $W(N_m)$. The reindexed filtration $W(N_m)[m]_\bullet$ is centered at m. Denote it by $M^{(m)}_\bullet$.

Definition 18.3. A filtration $M_\bullet$ of V is called a *relative weight filtration* of $N : (V, W_\bullet) \to (V, W_\bullet)$ if

(i) for each $k \in \mathbb{Z}$, $NM_k \subseteq M_{k-2}$;
(ii) the filtration induced by $M_\bullet$ on $\operatorname{Gr}^W_m V$ is the reindexed weight filtration $M^{(m)}_\bullet$.

Relative weight filtrations, if they exist, are unique. (Cf. [43]).

Example 18.4. If $N : (V, W_\bullet) \to (V, W_\bullet)$ satisfies $N(W_m V) \subseteq W_{m-2}V$ for all $m \in \mathbb{Z}$, then each $N_m = 0$ and the relative weight flirtation $M_\bullet$ of N exists and equals the original weight filtration $W_\bullet$.

Even though the weight filtration of a nilpotent endomorphism of a finite dimensional vector space always exists, the relative weight filtration of a nilpotent endomorphism of a *filtered* vector space $(V, W_\bullet)$ does not. Necessary and sufficient conditions for the existence of a relative weight filtration are given in [43]. They imply that the generic nilpoent endomorphisms of $(V, W_\bullet)$ does not have a relative weight filtration.

Example 18.5. Let E be a compact Riemann surface of genus 1 and P, Q two distinct points of E. Let $V = H_1(E, \{P, Q\}; \mathbb{Q})$. Then one has the exact sequence

$$0 \to H_1(E; \mathbb{Q}) \to V \to \widetilde{H}_0(\{P, Q\})$$

Define a filtration $W_\bullet$ on V by $W_{-2}V = 0$, $W_{-1}V = H_1(E)$, and $W_0 V = V$. Choose any path γ from P to Q. It determines a class $[\gamma]$ in V. Let u be any non-trivial element of $H_1(E; \mathbb{Q})$. Define a nilpotent automorphism N of $(V, W_\bullet)$ by insisting that N be trivial on $\operatorname{Gr}^W_\bullet V$ and that $N[\gamma] = u$. Since N is trivial on $\operatorname{Gr}^W_\bullet V$, the relative weight filtration $M_\bullet$, should it exist would equal $W_\bullet$. But $W_\bullet$ is not a relative weight filtration because $NW_0 V$ is not contained in $W_{-2}V$.