4. Moduli Spaces of Elliptic Curves

Suppose that $r$ and $n$ are non-negative integers satisfying $r + n > 0$. Denote the moduli stack over $\text{Spec } \mathbb{Z}$ of smooth elliptic curves with $n$ marked points and $r$ non-zero tangent vectors by $\mathcal{M}_{1,n+r}$. Here the $n$ marked points and the anchor points of the $r$ tangent vectors are distinct. Taking each tangent vector to its anchor point defines a morphism $\mathcal{M}_{1,n+r} \to \mathcal{M}_{1,n+r}$ that is a principal $\mathbb{G}_m^r$-bundle. The Deligne-Mumford compactification [8] of $\mathcal{M}_{1,n}$ will be denoted by $\overline{\mathcal{M}}_{1,n}$. It is also defined over $\text{Spec } \mathbb{Z}$. In this paper, we are primarily concerned with the cases $(n, r) = (1, 0), (0, 1)$ and $(2, 0)$.

For a $\mathbb{Z}$-algebra $A$, we denote by $\mathcal{M}_{1,n+r} \times_A$ the stack $\mathcal{M}_{1,n+r} \times_{\text{Spec } \mathbb{Z}} \text{Spec } A$. Similarly, the pullback of $\overline{\mathcal{M}}_{1,n}$ to $\text{Spec } A$ will be denoted by $\overline{\mathcal{M}}_{1,n} \times_A$.

The universal elliptic curve over $\mathcal{M}_{1,1}$ will be denoted by $\mathcal{E}$. Note that $\mathcal{M}_{1,2}$ is $\mathcal{E}$ with its identity section removed. The extension $\mathcal{E} \to \overline{\mathcal{M}}_{1,1}$ of the universal elliptic curve to $\overline{\mathcal{M}}_{1,1}$ is obtained by gluing in the Tate curve $\mathcal{E}_{\text{Tate}} \to \text{Spec } \mathbb{Z}[[q]]$ (cf. [41, Ch. V]). It is simply $\overline{\mathcal{M}}_{1,2}$.

The unique cusp of $\overline{\mathcal{M}}_{1,1}$ is the moduli point of the nodal cubic. Denote it by $e_o$.

The standard line bundle $\mathcal{L}$ over $\overline{\mathcal{M}}_{1,1}$ is the conormal bundle of the zero section of $\overline{\mathcal{E}} \to \overline{\mathcal{M}}_{1,1}$. Sections of $\mathcal{L}^\otimes n$ over $\overline{\mathcal{M}}_{1,1}$ are modular forms of weight $n$, and those that vanish at the cusp $e_o$ are cusp forms of weight $n$.

The restriction of the relative tangent bundle of $\overline{\mathcal{E}}$ to its identity section is the dual $\mathcal{L}^* \otimes \mathcal{L}$ of $\mathcal{L}$. The moduli space $\mathcal{M}_{1,1}$ is $\mathcal{L}'$, the restriction of $\mathcal{L}$ to $\mathcal{M}_{1,1}$ with its zero-section removed. This is isomorphic to the complement $\mathcal{L}'$ of the zero-section of $\mathcal{L}$.

We will also identify $\overline{\mathcal{M}}_{1,1}$ with the identity section of $\overline{\mathcal{E}}$. With this convention, $e_o$ also denotes the identity of the nodal cubic in $\overline{\mathcal{E}}$ and also the corresponding point of the partial compactification $\mathcal{L}$ of $\mathcal{M}_{1,1}$.

An explicit description of $\mathcal{L}'$ over $\overline{\mathcal{M}}_{1,1}$ can be deduced from the discussion [28, Chapt. 2] and the formulas in [40, Appendix A]. Then $\overline{\mathcal{M}}_{1,1}$ is the quotient stack $\mathbb{G}_m \backslash \mathcal{L}'$.

When 2 and 3 are invertible in $A$, $\mathcal{L}_A'$ is the scheme $\mathcal{L}_A' = \mathbb{A}_A^2 - \{0\} = \text{Spec } A[u,v] - \{0\}$.

The point $(u, v)$ corresponds to the plane cubic $y^2 = 4x^3 - ux - v$ and the abelian differential $dx/y$. The $\mathbb{G}_m$-action is $\lambda : (u, v) \mapsto (\lambda^{-4}u, \lambda^{-6}v)$. In this case, $\overline{\mathcal{M}}_{1,1}$ and $\mathcal{M}_{1,1}$ are the quotient stacks

$$\overline{\mathcal{M}}_{1,1} = \mathbb{G}_m \backslash \mathcal{L}_A' \text{ and } \mathcal{M}_{1,1} = \mathbb{G}_m \backslash (\mathbb{A}^2_A - D^{-1}(0)),$$

where $D = u^3 - 27v^2$ is (up to a factor of 4) the discriminant of the cubic.

Similarly, when 2 and 3 are invertible in $A$, $\overline{\mathcal{M}}_{1,2}/A$ is the quotient of the scheme

$$\{(u,v,x,y) \in \mathbb{A}_A^2 \times \mathbb{A}_A^2 : y^2 = 4x^3 - ux - v, \ (u,v) \neq 0\}$$

by the $\mathbb{G}_m$-action

$$\lambda : (u,v,x,y) \mapsto (\lambda^{-4}u, \lambda^{-6}v, \lambda^{-2}x, \lambda^{-3}y).$$

The point $(u, v, x, y)$ corresponds to the point $(x, y)$ on the cubic $y^2 = 4x^3 - ux - v$ and the abelian differential $dx/y$. The action of $\lambda$ multiplies $dx/y$ by $\lambda$. 
4.1. **Tangent vectors.** The Tate curve \( E_t \rightarrow \text{fSpec } \mathbb{Z}[q] \) defines a morphism \( \text{fSpec } \mathbb{Z}[q] \rightarrow \mathcal{M}_{1,1} \). The parameter \( q \) is a formal parameter of \( \mathcal{M}_{1,1} \) at the cusp \( \epsilon_0 \). The tangent vector \( \partial/\partial q \) of \( \mathcal{M}_{1,1} \) is integral and its reduction mod \( p \) is non-zero for all prime numbers \( p \). Denote it by \( \vec{t} \).

The fiber \( E_0 \) of the Tate curve over \( q = 0 \) is an integral model of the nodal cubic. Its normalization is \( \mathbb{P}^1_2 \). Let \( E_0 \) be \( E_0 \) with its double point removed. It is isomorphic to \( \mathbb{G}_m/\mathbb{Z} \). Let \( w \) be a parameter on \( E_0 \) whose pullback to the normalization of \( E_0 \) takes the values 0 and 1 on the inverse image of the double point, and the value 1 at the identity. It is unique up to \( w \sim w \). It determines the tangent vector \( \vec{w}_0 := \partial/\partial w \) of \( E_0 \) at the identity. It is integrally defined and non-zero at all primes.

We thus have the tangent vector \( \vec{v}_0 := \vec{t} + \vec{w}_0 \) of \( \mathcal{M}_{1,1} \) (and thus of \( \mathcal{M}_{1,2} \) and \( \mathcal{E} \) as well) at the identity \( \epsilon_0 \) of \( E_0 \) which is non-zero at all primes.

4.2. **Moduli spaces as complex orbifolds.** To fix notation and conventions, we give a quick review of the construction of \( \mathcal{M}^\an_{1,1} \) and \( \mathcal{E}^\an \) and their Deligne-Mumford compactifications. All will be regarded as complex analytic orbifolds. This material is classical and very well-known. A detailed discussion of these constructions and an explanation of the notation can be found in [17].

4.2.1. **The orbifolds \( \mathcal{M}^\an_{1,1} \) and \( \overline{\mathcal{M}}^\an_{1,1} \).** The moduli space \( \mathcal{M}^\an_{1,1} \) is the orbifold quotient \( \mathcal{M}^\an_{1,1} = \text{SL}_2(\mathbb{Z})/\mathfrak{h} \) of the upper half plane \( \mathfrak{h} \) by the standard \( \text{SL}_2(\mathbb{Z}) \) action. The point \( \tau \in \mathfrak{h} \), set \( \Lambda_\tau := \mathbb{Z} \oplus \mathbb{Z} \tau \). The point \( \tau \in \mathfrak{h} \) corresponds to the elliptic curve

\[
E_\tau := \mathbb{C}/\Lambda_\tau,
\]

together with the symplectic basis \( a, b \) of \( H_1(E_\tau, \mathbb{Z}) \) that corresponds to the generators 1, \( \tau \) of \( \Lambda_\tau \) via the canonical isomorphism \( \Lambda_\tau \cong H_1(E_\tau, \mathbb{Z}) \). The element

\[
\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

takes the basis \( a, b \) of \( H_1(E_\tau, \mathbb{Z}) \) to the basis

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix}
\]

of \( H_1(E_{\gamma \tau}, \mathbb{Z}) \).

The orbifold \( \overline{\mathcal{M}}^\an_{1,1} \) underlying the Deligne-Mumford compactification \( \overline{\mathcal{M}}_{1,1} \) of \( \mathcal{M}_{1,1} \) is obtained by gluing in the quotient \( C_2 \backslash \mathbb{D} \) of a disk \( \mathbb{D} \) of radius \( e^{-2\pi} \) by a trivial action of the cyclic group \( C_2 := \{ \pm 1 \} \). These are glued together by the diagram

\[
\begin{array}{ccc}
C_2 \backslash \mathbb{D} & \leftarrow & \left( \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right) \backslash \{ \tau \in \mathfrak{h} : \text{Im}(\tau) > 1 \} \\
& \rightarrow & \text{SL}_2(\mathbb{Z}) \backslash \mathfrak{h}
\end{array}
\]

where the left-hand map takes \( \tau \) to \( q := \exp(2\pi i \tau) \) and where \( C_2 \) is included in \( \left( \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right) \) as the scalar matrices.
4.2.2. The line bundle $L^\an$. The restriction of $L^\an$ to $M^\an_{1,1}$ is the orbifold quotient of the trivial line bundle $\mathbb{C} \times \mathfrak{h} \to \mathfrak{h}$ by the $\text{SL}_2(\mathbb{Z})$ action

$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} : (z, \tau) \mapsto ((c\tau + d)z, (a\tau + b)/(c\tau + d))
$$

It is an orbifold line bundle over $M^\an_{1,1}$. Its restriction to the punctured $q$-disk $D^*$ is naturally isomorphic to the trivial bundle $\mathbb{C} \to D^*$, and therefore extends naturally to a (trivial) line bundle over $D$. The line bundle $L^\an$ over $M_{1,1}$ is the unique extension of the line bundle above to $M^\an_{1,1}$ that restricts to this trivial bundle over the $q$-disk.

4.2.3. Eisenstein series. To fix notation and normalizations we recall some basic facts from the analytic theory of elliptic curves.

Suppose that $k \geq 1$. The (normalized) Eisenstein series of weight $2k$ is defined by the series

$$
G_{2k}(\tau) = \frac{1}{2} \frac{(2k-1)!}{(2\pi i)^{2k}} \sum_{\lambda \in \mathbb{Z} \setminus 2\pi \mathbb{Z}} \frac{1}{\lambda^{2k}} = -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n)q^n,
$$

where $\sigma_k(n) = \sum_{d|n} d^k$. When $k > 1$ it converges absolutely to a modular form of weight $2k$. When properly summed, it also converges when $k = 1$. In this case the logarithmic 1-form

$$
d\xi - 2 \cdot 2\pi i G_2(\tau) d\tau
$$

is $\text{SL}_2(\mathbb{Z})$-invariant, where $\text{SL}_2(\mathbb{Z})$ acts on $\mathbb{C}^* \times \mathfrak{h}$ by $\gamma : (\xi, \tau) \mapsto (\xi/(c\tau + d), \gamma \tau)$.

The ring of modular forms of $\text{SL}_2(\mathbb{Z})$ is the polynomial ring $\mathbb{C}[G_4, G_6]$.

4.3. The Weierstrass $\wp$-function. For a lattice $\Lambda$ in $\mathbb{C}$, the Weierstrass $\wp$-function is defined by

$$
\wp_\Lambda(z) := \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus 0} \left[ \frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right].
$$

For $\tau \in \mathfrak{h}$, set $\wp_\tau(z) = \wp_\Lambda(z, \tau)$. One has the expansion

$$
\wp_\tau(z) = (2\pi i)^2 \left( \frac{1}{(2\pi iz)^2} + \sum_{m=1}^{\infty} \frac{2}{(2m)!} G_{2m+2}(\tau)(2\pi iz)^{2m} \right).
$$

The function $\mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ defined by $z \mapsto [(2\pi i)^{-2}\wp_\tau(z), (2\pi i)^{-3}\wp'_\tau(z), 1]$ induces an embedding of $\mathbb{C}/\Lambda_\tau$ into $\mathbb{P}^2(\mathbb{C})$. The image has affine equation

$$
y^2 = 4x^3 - g_2(\tau)x - g_3(\tau),
$$

where

$$
g_2(\tau) = 20G_4(\tau) \quad \text{and} \quad g_3(\tau) = \frac{7}{3}G_6(\tau).
$$

This has discriminant the normalized cusp form of weight 12:

$$
\Delta(\tau) : = g_2(\tau)^3 - 27g_3(\tau)^2 = q \prod_{n \geq 1} (1 - q^n)^{24}.
$$
The abelian differential $dx/y$ corresponds to $2\pi i dz$ and the differential $xdx/y$ of the second kind corresponds to $(2\pi i)^{-1} \varphi_r(z)dz$. These differentials form a symplectic basis of $H^1_{\text{DR}}(E_r)$ as
\[
\int_{E_r} \frac{dx}{y} \sim \frac{xdx}{y} = 2\pi i.
\]
Cf. [18, Prop. 19.1].

4.4. The analytic space $\mathcal{M}^\text{an}_{1,1}$. Recall that $D(u,v) = u^3 - 27v^2$. The map
\[
\mathbb{C}^* \times \mathfrak{h} \to \mathbb{C}^2 - D^{-1}(0), \quad (\xi, \tau) \mapsto (\xi^{-2}g_2(\tau), \xi^{-3}g_3(\tau))
\]
induces an biholomorphism $\mathcal{L}^\text{an}' \to \mathcal{M}^\text{an}_{1,1} = \mathbb{C}^2 - D^{-1}(0)$. The point $(\xi, \tau)$ of $\mathbb{C}^* \times \mathfrak{h}$ corresponds to the point
\[
(\mathbb{C}/\xi\Lambda_r, 2\pi i dz) \cong (\mathbb{C}/\Lambda_r, 2\pi i \xi dz)
\]
of $\mathcal{M}^\text{an}_{1,1}$, which also corresponds to the curve $y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)$ with the abelian differential $\xi dx/y$.

4.4.1. The universal elliptic curve $\mathcal{E}^\text{an}$. Define $\Gamma$ to be the subgroup of $\text{GL}_3(\mathbb{Z})$ that consists of the matrices
\[
\gamma = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ m & n & 1 \end{pmatrix}
\]
where $a, b, c, d, m, n \in \mathbb{Z}$ and $ad - bc = 1$. It is isomorphic to the semi-direct product of $\text{SL}_2(\mathbb{Z})$ and $\mathbb{Z}^2$ and acts on $X := \mathbb{C} \times \mathfrak{h}$ on the left via the formula
\[
\gamma : (z, \tau) \mapsto (z', \tau'), \quad \text{where}
\]
\[
\begin{pmatrix} \tau' \\ 1 \end{pmatrix} = (ct + d)^{-1} \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ m & n & 1 \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix}.
\]
The universal elliptic curve $\pi^\text{an} : \mathcal{E}^\text{an} \to \mathcal{M}^\text{an}_{1,1}$ is the orbifold quotient of the projection $\mathbb{C} \times \mathfrak{h} \to \mathfrak{h}$ by $\Gamma$, which acts on $\mathfrak{h}$ via the quotient map $\Gamma \to \text{SL}_2(\mathbb{Z})$. The fiber of $\pi^\text{an}$ over the orbit of $\tau$ is $E_r$.

The universal elliptic curve $\mathcal{E}^\text{an} \to \mathcal{M}^\text{an}_{1,1}$ is obtained by gluing in the Tate curve as described in [17, §5]. The restriction to the $q$-disk $\mathbb{D}$ of $\mathcal{E}^\text{an}$ minus the double point of $E_0$ is the quotient of $\mathbb{C}^* \times \mathbb{D}$ by the $\mathbb{Z}$-action
\[
n : (w, q) \mapsto \begin{cases} (q^n w, q) & q \neq 0 \\ (w, 0) & q = 0. \end{cases}
\]

To relate this to the algebraic construction, note that $\mathcal{M}^\text{an}_{1,2+\mathfrak{f}}$ is the analytic variety $\Gamma \backslash (\mathbb{C}^* \times \mathbb{C} \times \mathfrak{h})$, where $\gamma(\xi, z, \tau) = ((ct + d)\xi, \gamma(z, \tau))$. The function
\[
(\xi, z, \tau) \mapsto (g_2(\tau), g_3(\tau), [(2\pi i)^{-2} \varphi_r(z), (2\pi i)^{-3} \varphi'_r(z), 1])
\]
from $\mathbb{C}^* \times \mathbb{C} \times \mathfrak{h}$ to $\mathbb{C}^2 - D^{-1}(0) \times \mathbb{P}^2$ induces a biholomorphism
\[
\Gamma \backslash (\mathbb{C}^* \times \mathbb{C} \times \mathfrak{h}) \to \mathcal{M}^\text{an}_{1,2+\mathfrak{f}}.
\]
It is invariant with respect to the $\mathbb{C}^*$ action $\lambda(\xi, z, \tau) = (\lambda \xi, z, \tau)$ on $\Gamma \backslash (\mathbb{C}^* \times \mathbb{C} \times \mathfrak{h})$ and the $\mathbb{C}^*$-action on $\mathcal{M}^\text{an}_{1,2+\mathfrak{f}}$ that multiplies the abelian differential by $\lambda$. 


4.4.2. Orbifold fundamental groups. Since \( M_{1,1}^{\text{an}} \) is the orbifold quotient of \( h \) by \( \text{SL}_2(\mathbb{Z}) \), there is a natural isomorphism

\[
\pi_1^{\text{top}}(M_{1,1}, p) \cong \text{Aut}(h \to \text{SL}_2(\mathbb{Z}) \backslash h) \cong \text{SL}_2(\mathbb{Z}),
\]

where \( p \) denotes the projection \( h \to M_{1,1}^{\text{an}} \). The inclusion of the imaginary axis in \( h \) induces an isomorphism

\[
\pi_1^{\text{top}}(M_{1,1}, \vec{t}) \cong \pi_1(M_{1,1}, p) \cong \text{SL}_2(\mathbb{Z}).
\]

The orbifold fundamental group of \( M_{1,1}^{\text{an}} \) is a central extension of \( \text{SL}_2(\mathbb{Z}) \) by \( \mathbb{Z} \).

There are natural isomorphisms

\[
\pi_1(M_{1,1}^{\text{an}}, \vec{v}_o) \cong B_3 \cong \tilde{\text{SL}}_2(\mathbb{Z}),
\]

where \( B_3 \) denotes the braid group on 3 strings and \( \tilde{\text{SL}}_2(\mathbb{Z}) \) denotes the inverse image of \( \text{SL}_2(\mathbb{Z}) \) in the universal covering group \( \tilde{\text{SL}}_2(\mathbb{R}) \) of \( \text{SL}_2(\mathbb{R}) \). Again, this is well-know; details can be found in [17, §8].

Similarly, there are natural isomorphisms

\[
\pi_1(\mathcal{E}_{\text{an}}, \vec{v}_o) \cong \pi_1(\mathcal{E}_{\text{an}}, p') \cong \text{Aut}(\mathbb{C} \times h \to \mathcal{E}_{\text{an}}) \cong \Gamma \cong \text{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2,
\]

where \( p' : \mathbb{C} \times h \to \mathcal{E}_{\text{an}} \) is the projection.