4. Moduli Spaces of Elliptic Curves

Suppose that r and n are non-negative integers satisfying $r+n>0$. Denote the moduli stack over $\operatorname{Spec}\mathbb{Z}$ of smooth elliptic curves with n marked points and r non-zero tangent vectors by $\mathcal{M}_{1,n+\vec{r}}$. Here the n marked points and the anchor points of the r tangent vectors are distinct. Taking each tangent vector to its anchor point defines a morphism $\mathcal{M}_{1,n+\vec{r}} \to \mathcal{M}_{1,n+r}$ that is a principal $\mathbb{G}_m^r$-bundle. The Deligne-Mumford compactification [8] of $\mathcal{M}_{1,n}$ will be denoted by $\overline{\mathcal{M}}_{1,n}$. It is also defined over $\operatorname{Spec}\mathbb{Z}$. In this paper, we are primarily concerned with the cases $(n,r) = (1,0), (0,1)$ and $(2,0)$.

For a $\mathbb{Z}$-algebra A, we denote by $\mathcal{M}_{1,n+\vec{r}/A}$ the stack $\mathcal{M}_{1,n+\vec{r}} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} A$. Similarly, the pullback of $\overline{\mathcal{M}}_{1,n}$ to $\operatorname{Spec} A$ will be denoted by $\overline{\mathcal{M}}_{1,n/A}$.

The universal elliptic curve over $\mathcal{M}_{1,1}$ will be denoted by $\mathcal{E}$. Note that $\mathcal{M}_{1,2}$ is $\mathcal{E}$ with its identity section removed. The extension $\overline{\mathcal{E}} \to \overline{\mathcal{M}}_{1,1}$ of the universal elliptic curve to $\overline{\mathcal{M}}_{1,1}$ is obtained by1 glueing in the Tate curve $\mathcal{E}_{\mathrm{Tate}} \to \operatorname{Spec}\mathbb{Z}[[q]]$ (cf. [41, Ch. V]). It is simply $\overline{\mathcal{M}}_{1,2}$.

The unique cusp of $\overline{\mathcal{M}}_{1,1}$ is the moduli point of the nodal cubic. Denote it by e_o.

The standard line bundle $\mathcal{L}$ over $\overline{\mathcal{M}}_{1,1}$ is the conormal bundle of the zero section of $\overline{\mathcal{E}} \to \overline{\mathcal{M}}_{1,1}$. Sections of $\mathcal{L}^{\otimes n}$ over $\overline{\mathcal{M}}_{1,1}$ are modular forms of weight n, and those that vanish at the cusp e_o are cusp forms of weight n.

The restriction of the relative tangent bundle of $\overline{\mathcal{E}}$ to its identity section is the dual $\check{\mathcal{L}}$ of $\mathcal{L}$. The moduli space $\mathcal{M}_{1,\vec{1}}$ is $\check{\mathcal{L}}'$, the restriction of $\check{\mathcal{L}}$ to $\mathcal{M}_{1,1}$ with its zero-section removed. This is isomorphic to the complement $\mathcal{L}'$ of the zero-section of $\mathcal{L}$.

We will also identify $\overline{\mathcal{M}}_{1,1}$ with the identity section of $\overline{\mathcal{E}}$. With this convention, e_o also denotes the identity of the nodal cubic in $\overline{\mathcal{E}}$ and also the corresponding point of the partial compactification $\check{\mathcal{L}}$ of $\mathcal{M}_{1,\vec{1}}$.

An explicit description of $\mathcal{L}'$ over $\overline{\mathcal{M}}_{1,1}$ can be deduced from the discussion [28, Chapt. 2] and the formulas in [40, Appendix A]. Then $\overline{\mathcal{M}}_{1,1}$ is the quotient stack $\mathbb{G}_m \backslash\!\backslash \mathcal{L}'$.

When 2 and 3 are invertible in A, $\mathcal{L}'_A$ is the scheme

$$\mathcal{L}'_A = \mathbb{A}^2_A - \{0\} = \operatorname{Spec} A[u,v] - \{0\}.$$

The point (u,v) corresponds to the plane cubic $y^2 = 4x^3 - ux - v$ and the abelian differential dx/y. The $\mathbb{G}_m$-action is $\lambda : (u,v) \mapsto (\lambda^{-4}u, \lambda^{-6}v)$. In this case, $\overline{\mathcal{M}}_{1,1}$ and $\mathcal{M}_{1,1}$ are the quotient stacks

$$\overline{\mathcal{M}}_{1,1/A} = \mathbb{G}_m \backslash\!\backslash \mathcal{L}'_A \text{ and } \mathcal{M}_{1,1/A} = \mathbb{G}_m \backslash\!\backslash (\mathbb{A}^2_A - D^{-1}(0)),$$

where $D = u^3 - 27v^2$ is (up to a factor of 4) the discriminant of the cubic.

Similarly, when 2 and 3 are invertible in A, $\overline{\mathcal{M}}_{1,2/A}$ is the quotient of the scheme

$$\{(u,v,x,y) \in \mathbb{A}^2_A \times \mathbb{A}^2_A : y^2 = 4x^3 - ux - v,\ (u,v) \neq 0\}$$

by the $\mathbb{G}_m$-action

$$\lambda : (u,v,x,y) \mapsto (\lambda^{-4}u, \lambda^{-6}v, \lambda^{-2}x, \lambda^{-3}y).$$

The point (u,v,x,y) corresponds to the point (x,y) on the cubic $y^2 = 4x^3 - ux - v$ and the abelian differential dx/y. The action of λ multiplies dx/y by λ.

4.1. **Tangent vectors.** The Tate curve $\mathcal{E}_{\vec{\mathsf{t}}} \to \mathrm{fSpec}\,\mathbb{Z}[[q]]$ defines a morphism $\mathrm{fSpec}\,\mathbb{Z}[[q]] \to \overline{\mathcal{M}}_{1,1}$. The parameter q is a formal parameter of $\overline{\mathcal{M}}_{1,1}$ at the cusp e_o. The tangent vector $\partial/\partial q$ of $\overline{\mathcal{M}}_{1,1}$ is integral and its reduction mod p is non-zero for all prime numbers p. Denote it by $\vec{\mathsf{t}}$.

The fiber $\overline{E}_0$ of the Tate curve over $q = 0$ is an integral model of the nodal cubic. Its normalization is $\mathbb{P}^1_{\mathbb{Z}}$. Let E_0 be $\overline{E}_0$ with its double point removed. It is isomorphic to $\mathbb{G}_{m/\mathbb{Z}}$. Let w be a paramerter on $\overline{E}_0$ whose pullback to the normalization of $\overline{E}_0$ takes the values 0 and ∞ on the inverse image of the double point, and the value 1 at the identity. It is unique up to $w \mapsto w^{-1}$. It determines the tangent vector $\vec{\mathsf{w}}_o := \partial/\partial w$ of $\overline{E}_0$ at the identity. It is integrally defined and non-zero at all primes.

We thus have the tangent vector $\vec{\mathsf{v}}_o := \vec{\mathsf{t}} + \vec{\mathsf{w}}_o$ of $\mathcal{M}_{1,\vec{1}}$ (and thus of $\mathcal{M}_{1,2}$ and $\mathcal{E}$ as well) at the identity e_o of E_0 which is non-zero at all primes.

4.2. **Moduli spaces as complex orbifolds.** To fix notation and conventions, we give a quick review of the construction of $\mathcal{M}_{1,1}^{\mathrm{an}}$, $\mathcal{E}^{\mathrm{an}}$ and their Deligne-Mumford compactifications. All will be regarded as complex analytic orbifolds. This material is classical and very well-known. A detailed discussion of these constructions and an explanation of the notation can be found in [17].

4.2.1. *The orbifolds* $\mathcal{M}_{1,1}^{\mathrm{an}}$ *and* $\overline{\mathcal{M}}_{1,1}^{\mathrm{an}}$. The moduli space $\mathcal{M}_{1,1}^{\mathrm{an}}$ is the orbifold quotient $\mathcal{M}_{1,1}^{\mathrm{an}} = \mathrm{SL}_2(\mathbb{Z})\backslash\!\!\backslash\mathfrak{h}$ of the upper half plane $\mathfrak{h}$ by the standard $\mathrm{SL}_2(\mathbb{Z})$ action. For $\tau \in \mathfrak{h}$, set $\Lambda_\tau := \mathbb{Z} \oplus \mathbb{Z}\tau$. The point $\tau \in \mathfrak{h}$ corresponds to the elliptic curve

$$E_\tau := \mathbb{C}/\Lambda_\tau,$$

together with the symplectic basis $\mathbf{a}, \mathbf{b}$ of $H_1(E_\tau, \mathbb{Z})$ that corresponds to the generators $1, \tau$ of Λ_τ via the canonical isomorphism $\Lambda_\tau \cong H_1(E_\tau, \mathbb{Z})$. The element

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of $\mathrm{SL}_2(\mathbb{Z})$ takes the basis $\mathbf{a}, \mathbf{b}$ of $H_1(E_\tau, \mathbb{Z})$ to the basis

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \end{pmatrix} \tag{4.1}$$

of $H_1(E_{\gamma\tau}, \mathbb{Z})$.

The orbifold $\overline{\mathcal{M}}_{1,1}^{\mathrm{an}}$ underlying the Deligne-Mumford compactification $\overline{\mathcal{M}}_{1,1}$ of $\mathcal{M}_{1,1}$ is obtained by glueing in the quotient $C_2\backslash\!\!\backslash\mathbb{D}$ of a disk $\mathbb{D}$ of radius $e^{-2\pi}$ by a trivial action of the cyclic group $C_2 := \{\pm 1\}$. These are glued together by the diagram

$$C_2\backslash\!\!\backslash\mathbb{D} \longleftarrow \begin{pmatrix} \pm 1 & \mathbb{Z} \\ 0 & \pm 1 \end{pmatrix}\backslash\!\!\backslash\{\tau \in \mathfrak{h} : \mathrm{Im}(\tau) > 1\} \longrightarrow \mathrm{SL}_2(\mathbb{Z})\backslash\!\!\backslash\mathfrak{h}$$

where the left-hand map takes τ to $q := \exp(2\pi i \tau)$ and where C_2 is included in $\begin{pmatrix} \pm 1 & \mathbb{Z} \\ 0 & \pm 1 \end{pmatrix}$ as the scalar matrices.

4.2.2. *The line bundle* $\mathcal{L}^{\mathrm{an}}$. The restriction of $\mathcal{L}^{\mathrm{an}}$ to $\mathcal{M}_{1,1}^{\mathrm{an}}$ is the orbifold quotient of the trivial line bundle $\mathbb{C}\times\mathfrak{h}\to\mathfrak{h}$ by the $\mathrm{SL}_2(\mathbb{Z})$ action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (z,\tau) \mapsto \big((c\tau+d)z, (a\tau+b)/(c\tau+d)\big)$$

It is an orbifold line bundle over $\mathcal{M}_{1,1}^{\mathrm{an}}$. Its restriction to the punctured q-disk $\mathbb{D}^*$ is naturally isomorphic to the trivial bundle $\mathbb{C}\times\mathbb{D}^*\to\mathbb{D}^*$, and therefore extends naturally to a (trivial) line bundle over $\mathbb{D}$. The line bundle $\mathcal{L}^{\mathrm{an}}$ over $\overline{\mathcal{M}}_{1,1}$ is the unique extension of the line bundle above to $\overline{\mathcal{M}}_{1,1}^{\mathrm{an}}$ that restricts to this trivial bundle over the q-disk.

4.2.3. *Eisenstein series.* To fix notation and normalizations we recall some basic facts from the the analytic theory of elliptic curves.

Suppose that $k\geq 1$. The (normalized) Eisenstein series of weight $2k$ is defined by the series

$$G_{2k}(\tau) = \frac{1}{2}\frac{(2k-1)!}{(2\pi i)^{2k}} \sum_{\substack{\lambda\in\mathbb{Z}\oplus\mathbb{Z}\tau \\ \lambda\neq 0}} \frac{1}{\lambda^{2k}} = -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n,$$

where $\sigma_k(n)=\sum_{d|n} d^k$. When $k>1$ it converges absolutely to a modular form of weight $2k$. When properly summed, it also converges when $k=1$. In this case the logarithmic 1-form

$$\frac{d\xi}{\xi} - 2\cdot 2\pi i\, G_2(\tau)\, d\tau$$

is $\mathrm{SL}_2(\mathbb{Z})$-invariant, where $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{C}^*\times\mathfrak{h}$ by $\gamma : (\xi,\tau)\mapsto \big(\xi/(c\tau+d), \gamma\tau\big)$.

The ring of modular forms of $\mathrm{SL}_2(\mathbb{Z})$ is the polynomial ring $\mathbb{C}[G_4, G_6]$.

4.3. **The Weierstrass $\wp$-function.** For a lattice Λ in $\mathbb{C}$, the Weierstrass $\wp$-function is defined by

$$\wp_\Lambda(z) := \frac{1}{z^2} + \sum_{\substack{\lambda\in\Lambda \\ \lambda\neq 0}} \left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right].$$

For $\tau\in\mathfrak{h}$, set $\wp_\tau(z)=\wp_{\Lambda_\tau}(z)$. One has the expansion

$$\wp_\tau(z) = (2\pi i)^2\left(\frac{1}{(2\pi i z)^2} + \sum_{m=1}^{\infty}\frac{2}{(2m)!}G_{2m+2}(\tau)(2\pi i z)^{2m}\right),$$

The function $\mathbb{C}\to\mathbb{P}^2(\mathbb{C})$ defined by $z\mapsto [(2\pi i)^{-2}\wp_\tau(z), (2\pi i)^{-3}\wp_\tau'(z), 1]$ induces an embedding of $\mathbb{C}/\Lambda_\tau$ into $\mathbb{P}^2(\mathbb{C})$. The image has affine equation

$$y^2 = 4x^3 - g_2(\tau)x - g_3(\tau),$$

where

$$g_2(\tau) = 20G_4(\tau) \text{ and } g_3(\tau) = \frac{7}{3}G_6(\tau).$$

This has discriminant the normalized cusp form of weight 12:

$$\Delta(\tau) := g_2(\tau)^3 - 27g_3(\tau)^2 = q\prod_{n\geq 1}(1-q^n)^{24}.$$

The abelian differential dx/y corresponds to $2\pi i dz$ and the differential xdx/y of the second kind corresponds to $(2\pi i)^{-1}\wp_\tau(z)dz$. These differentials form a symplectic basis of $H^1_{\mathrm{DR}}(E_\tau)$ as

$$\int_{E_\tau} \frac{dx}{y} \smile \frac{xdx}{y} = 2\pi i.$$

Cf. [18, Prop. 19.1].

4.4. **The analytic space $\mathcal{M}^{\mathrm{an}}_{1,\vec{1}}$.** Recall that $D(u,v) = u^3 - 27v^2$. The map

$$\mathbb{C}^* \times \mathfrak{h} \to \mathbb{C}^2 - D^{-1}(0), \quad (\xi,\tau) \mapsto \big(\xi^{-2}g_2(\tau), \xi^{-3}g_3(\tau)\big)$$

induces an biholomorphism $\mathcal{L}^{\mathrm{an}\prime} \to \mathcal{M}^{\mathrm{an}}_{1,\vec{1}} = \mathbb{C}^2 - D^{-1}(0)$. The point (ξ,τ) of $\mathbb{C}^*\times\mathfrak{h}$ corresponds to the point

$$\big(\mathbb{C}/\xi\Lambda_\tau, 2\pi i dz\big) \cong \big(\mathbb{C}/\Lambda_\tau, 2\pi i \xi\, dz\big)$$

of $\mathcal{M}^{\mathrm{an}}_{1,\vec{1}}$, which also corresponds to the curve $y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)$ with the abelian differential $\xi dx/y$.

4.4.1. *The universal elliptic curve $\mathcal{E}^{\mathrm{an}}$.* Define Γ to be the subgroup of $\mathrm{GL}_3(\mathbb{Z})$ that consists of the matrices

$$\gamma = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ m & n & 1 \end{pmatrix}$$

where $a,b,c,d,m,n \in \mathbb{Z}$ and $ad - bc = 1$. It is isomorphic to the semi-direct product of $\mathrm{SL}_2(\mathbb{Z})$ and $\mathbb{Z}^2$ and acts on $X := \mathbb{C}\times\mathfrak{h}$ on the left via the formula $\gamma : (z,\tau) \mapsto (z',\tau')$, where

$$\begin{pmatrix} \tau' \\ 1 \\ z' \end{pmatrix} = (c\tau+d)^{-1} \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ m & n & 1 \end{pmatrix} \begin{pmatrix} \tau \\ 1 \\ z \end{pmatrix}.$$

The universal elliptic curve $\pi^{\mathrm{an}} : \mathcal{E}^{\mathrm{an}} \to \mathcal{M}^{\mathrm{an}}_{1,1}$ is the orbifold quotient of the projection $\mathbb{C}\times\mathfrak{h} \to \mathfrak{h}$ by Γ, which acts on $\mathfrak{h}$ via the quotient map $\Gamma \to \mathrm{SL}_2(\mathbb{Z})$. The fiber of π^{an} over the orbit of τ is E_τ.

The universal elliptic curve $\overline{\mathcal{E}}^{\mathrm{an}} \to \overline{\mathcal{M}}^{\mathrm{an}}_{1,1}$ is obtained by glueing in the Tate curve as described in [17, §5]. The restriction to the q-disk $\mathbb{D}$ of $\overline{\mathcal{E}}^{\mathrm{an}}$ minus the double point of $\overline{E}_0$ is the quotient of $\mathbb{C}^*\times\mathbb{D}$ by the $\mathbb{Z}$-action

$$n : (w,q) \mapsto \begin{cases} (q^n w, q) & q \neq 0 \\ (w,0) & q = 0. \end{cases}$$

To relate this to the algebraic construction, note that $\mathcal{M}^{\mathrm{an}}_{1,2+\vec{1}}$ is the analytic variety $\Gamma\backslash\big(\mathbb{C}^*\times\mathbb{C}\times\mathfrak{h}\big)$, where $\gamma(\xi,z,\tau) = \big((c\tau+d)\xi, \gamma(z,\tau)\big)$. The function

$$(\xi,z,\tau) \mapsto \big(g_2(\tau), g_3(\tau), [(2\pi i)^{-2}\wp_\tau(z), (2\pi i)^{-3}\wp'_\tau(z), 1]\big)$$

from $\mathbb{C}^*\times\mathbb{C}\times\mathfrak{h}$ to $\mathbb{C}^2 - D^{-1}(0)\times\mathbb{P}^2$ induces a biholomorphism

$$\Gamma\backslash\big(\mathbb{C}^*\times\mathbb{C}\times\mathfrak{h}\big) \to \mathcal{M}^{\mathrm{an}}_{1,2+\vec{1}}.$$

It is invariant with respect to the-$\mathbb{C}^*$ action $\lambda\cdot(\xi,z,\tau) = (\lambda\xi,z,\tau)$ on $\Gamma\backslash(\mathbb{C}^*\times\mathbb{C}\times\mathfrak{h})$ and the $\mathbb{C}^*$-action on $\mathcal{M}_{1,2+\vec{1}}$ that multiplies the abelian differential by λ.

4.4.2. *Orbifold fundamental groups.* Since $\mathcal{M}_{1,1}^{\mathrm{an}}$ is the orbifold quotient of $\mathfrak{h}$ by $\mathrm{SL}_2(\mathbb{Z})$, there is a natural isomorphism

$$\pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, p) \cong \mathrm{Aut}(\mathfrak{h} \to \mathrm{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{h}) \cong \mathrm{SL}_2(\mathbb{Z}),$$

where p denotes the projection $\mathfrak{h} \to \mathcal{M}_{1,1}^{\mathrm{an}}$. The inclusion of the imaginary axis in $\mathfrak{h}$ induces an isomorphism

$$\pi_1^{\mathrm{top}}(\mathcal{M}_{1,1}, \vec{\mathsf{t}}) \cong \pi_1(\mathcal{M}_{1,1}, p) \cong \mathrm{SL}_2(\mathbb{Z}).$$

The orbifold fundamental group of $\mathcal{M}_{1,\vec{1}}^{\mathrm{an}}$ is a central extension of $\mathrm{SL}_2(\mathbb{Z})$ by $\mathbb{Z}$. There are natural isomorphisms

$$\pi_1(\mathcal{M}_{1,\vec{1}}^{\mathrm{an}}, \vec{\mathsf{v}}_o) \cong B_3 \cong \widetilde{\mathrm{SL}}_2(\mathbb{Z}),$$

where B_3 denotes the braid group on 3 strings and $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$ denotes the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ in the universal covering group $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$. Again, this is well-know; details can be found in [17, §8].

Similarly, there are natural isomorphisms

$$\pi_1(\mathcal{E}^{\mathrm{an}}, \vec{\mathsf{v}}_o) \cong \pi_1(\mathcal{E}^{\mathrm{an}}, p') \cong \mathrm{Aut}(\mathbb{C} \times \mathfrak{h} \to \mathcal{E}^{\mathrm{an}}) \cong \Gamma \cong \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2,$$

where $p' : \mathbb{C} \times \mathfrak{h} \to \mathcal{E}^{\mathrm{an}}$ is the projection.