LECTURE #1

I. HISTORY & CONTEXT

1) GROTHENDIECK (1960s) — defined the category of pure motives over a field $k$.
   - Objects “generated by” smooth projective varieties over $k$.
   - Morphisms: correspondences modulo rational equivalence.

2) BEILINSON (1980) — defined motivic cohomology of a variety $X$ in terms of the algebraic $K$-theory $K_0(X)$ of $X$:

$$H^j_{	ext{MW}}(X, \mathbb{Q}(n)) := K_{2n-j}(X)^{(n)}$$

[Beilinson - Soulé]

[Conjectured to vanish when $j < 0$]

[Also equal to Bloch's higher Chow groups]
\[ K^*(X) \otimes \mathbb{Q} = \bigoplus_{n \geq 0} K^*(X)^{(n)} \]

The \( p^n \)-eigenspace of the Adams op \( \Psi^p \).

If \( X \) is defined over \( \mathbb{Z} \), then the \( H^i_m(X, \mathbb{Q}(n)) \) should be finite-dimensional, with dimensions predicted by order-of-vanishing of suitable \( L \)-function.

**Examples:**

1. (Bloch-Gersten-Quillen)

   If \( X \) is smooth projective, then

   \[
   H^i_m(X, \mathbb{Q}(n)) \cong CH^i_m(X) \otimes \mathbb{Q}
   \]

   Chow group of codim \( n \)-cycles.
(Borel–Beilinson)

(2) If \( F \) is a number field, then

(i) \( H^j_m(\text{Spec} \ F, \mathbb{Q}(n)) = 0 \) when \( n < 0 \)

(ii) \( H^0_m(\text{Spec} \ F, \mathbb{Q}(n)) = \begin{cases} \mathbb{Q} & n = 0 \\ 0 & n \neq 0 \end{cases} \)

(iii) \( H^j_m(\text{Spec} \ F, \mathbb{Q}(n)) = 0 \) for \( j > 1 \).

(iv) \( H^1(\text{Spec} \ F, \mathbb{Z}(1)) \cong F^\times \)

(v) When \( n > 1 \)

\[ \dim H^1(\text{Spec} \ F, \mathbb{Q}(n)) = \begin{cases} r_1 + r_2 & n \text{ odd} \\ r_2 & n \text{ even.} \end{cases} \]
It was conjectured (by Beilinson) that there was a tannakian category of mixed motives $\text{MM}$ and that

$$H^j_m(X, \mathbb{Q}(n)) = \text{Ext}^j_{\text{MM}}(\mathbb{Q}, M(X)(n))$$

Tannakian category: roughly a (neutral) tannakian category is a $k$-linear abelian category with $\otimes$ that admits an exact and faithful functor $\otimes$ to the category of finite dim $k$-vector spaces.

Such a category is the category of reps of a pro-algebraic group:

$$\pi_1(G, w) = \text{Aut} \otimes w$$

\[ w \mapsto w \text{ nat isoms} \]
Voevodsky (Levine) constructed a triangulated tensor category of mixed motives where

\[ H^*_m(X, Q(n)) = \text{Ext}^*_{\mathcal{M}}(Q(n), M(X)) \]

Here \( M(X) \) is a suitable motive associated to \( X \) and

\[ Q(1) = H^1(\mathbb{G}_m) \]

\[ Q(-1) = H^1(\mathbb{G}_m) \]

\[ Q(n) = \bigoplus Q(1)^{\otimes n} \quad n \geq 0 \]

\[ \uparrow \begin{cases} Q(-1)^{\otimes |n|} & n < 0 \end{cases} \]

Tate motives
II **Mixed Tate Motives**

(Levine, Deligne - Goncharov)

If $K$ is a number field and $S \subset \text{Spec } \mathcal{O}_K$, then there is a tannakian category $\text{MTM}(\mathcal{O}_K, S)$ of "mixed Tate motives over $\mathcal{O}_K, S"$ such that

$$H^j_m(\text{Spec } \mathcal{O}_K, S, \mathbb{Q}(n))$$

$$\text{Gr}^M_{\text{odd}} V = 0 \quad \Rightarrow \quad \text{Ext}^j_{\text{MTM}(\mathcal{O}_K, S)}(\mathbb{Q}, \mathbb{Q}(n))$$

Objects $V$ of $\text{MTM}(\mathcal{O}_K, S)$ have a weight filtration in $\text{MTM}$:

$$0 \leq \ldots \leq M_r V \leq M_{r+1} V \leq \ldots \leq V$$

Why $M$ and not $W$? Later!
Quick review of (mixed) Hodge theory.

1. A \(\mathbb{Q}\)-Hodge structure \(V\) of weight \(m\) consists of:
   - (a) a finite-dimensional vector space \(V_\mathbb{Q}\),
   - (b) a decreasing filtration

\[
V = F^a V \supseteq \ldots \supseteq F^b V \supseteq F^{b+1} V \supseteq \ldots \supseteq F^b V = 0
\]

where \(V_\mathbb{Q} = V_\mathbb{Q} \otimes \mathbb{C}\)

These satisfy

\[
V_\mathbb{Q} = \bigoplus_{p+q = m} V^{p,q}
\]

where \(V^{p,q} := F^p V \cap \overline{F^q V}\)
(2) a \( \mathbb{Q} \)-mixed Hodge structure (MHS) \( V \) consists of

(1) a finite dimensional \( \mathbb{Q} \) vector space \( V_{\mathbb{Q}} \)

(2) an increasing filtration

\[
0 = W_0 V_{\mathbb{Q}} \subseteq \cdots \subseteq W_r V_{\mathbb{Q}} \subseteq W_{r+1} V_{\mathbb{Q}} \\
\subseteq \cdots \subseteq W_N V_{\mathbb{Q}} = V_{\mathbb{Q}}
\]

(3) a decreasing Hodge filtration of \( V_{\mathbb{Q}} : \)

\[
V_{\mathbb{Q}} = F^a V \supseteq \cdots \supseteq F^{-b} V \supseteq F^{-b+2} V \\
\cdots \supseteq F^{-b} V = 0
\]

These satisfy:

Each \( (\text{Gr}_m W V, F^\cdot) \)
is a Hodge structure of weight $m$. Here

$$(\text{Gr}_m^w V)_q = \frac{W_m V_q}{W_{m-1} V_q}.$$
Example:

1. $X$ smooth a projective curve

$$V = H^m(X; Q)$$

has a HS of weight $m$.

$$F^p H^m(X) = \bigoplus H^{s + m - s}(X), \quad s \geq p$$

Realizations of $\text{MTM}(\mathbb{Z})$

Ex: $V = Q(1) = H_1(G_m/\mathbb{Z})$

Betti:

$$H_1(G_m^q, Q) = H_1(C^*, Q)$$

$$= Q \circ \phi$$

Q-DK:

$$\text{Hom}(H^1_{DR}(G_m/Q), Q)$$

$$V_{DR} = Q \frac{d^2}{dz^2} \cdot Q(2\pi i \delta)$$
\[ V^\text{DR} = F^{-1} \subset \Omega \left( \frac{dt}{t} \right)^* \geq F^0 = 0 \]

#### Hodge:
\[(\text{Betti}) \otimes \mathbb{C} \cong (\Omega - \text{DR}) \otimes \mathbb{C} \]

HS type (1, -1)

#### \(l\)-adic étale:
\[ V^B \otimes \mathbb{Q}_l \cong \mathbb{Q}_l \text{, unramified outside } l. \]

Crystalline at \(l\).

given by the \(l\)-adic cyclotomic character
\[ \kappa_l : \mathbb{G}_m \to \mathbb{Z}_l, \quad \mathbb{G}_m \cong \text{Gal}(\overline{\mathbb{Q}} / \mathbb{Q}) \]

Similarly for \( \mathbb{Q}(n) \)

Betti - \( \mathbb{Q} \)
Hodge - HS type (-n, -n)
\(l\)-adic \(\mathbb{Q}_l \)
General $V \in \text{MTM}(\mathbb{Z})$,

$$(V, M.) - \text{Betti filtered } \mathbb{Q} \text{- vect sp}$$

$$\mathbb{Q} - \text{DR}$$

$$(V^{\text{DR}}, M., F.)$$

all $\mathbb{Q} \text{- vect sp}$

Hodge $- V^{\text{DR}} \otimes \mathbb{C} \cong V^{B} \otimes \mathbb{C}$

$\text{MHS} \begin{cases}
\text{Gr}^M_{\text{odd}} = 0 \\
\text{Gr}^M_{-2m} \cong \mathbb{Q}(m)^N
\end{cases}$

$l$-adic $: \begin{cases}
V_e \cong V^B \otimes \Omega_e \\
\uparrow \text{unramified coextension } l \\
\text{"crystalline @ } \mathbb{C}"
\end{cases}$

$G_\mathbb{Q}$

$\text{Gr}^M_{-2m} V_e \cong \bigoplus \mathbb{Q}_l(m).$
EXAMPLES OF MTM(\(K\))

\[ V = H^1(\mathcal{G}_m, \{1, \lambda\}), \lambda \in K^x - \{1\}. \]

\[ 0 \rightarrow \tilde{H}^0(\{1, \lambda\}) \rightarrow V \rightarrow H^1(\mathcal{G}_m) \rightarrow 0 \]

\[ \gamma \in \mathbb{Z}(0) \]

\[ \mathbb{Z}(-1) \]

\[ F^{-1} V_{DR} = Q \frac{d^2}{dz} \]

\[ \frac{d^2}{dz} = 2\pi i \delta \gamma + \log \lambda \]

MHS det by \( \log \lambda \) mod \( 2\pi i \mathbb{Z} \)

\( \in \) by \( \lambda \in \mathbb{C}^x \)
(2) (details later)

(Deligne - Gorchinski)

\[ \pi_{un} = \pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, x)^{un} \uparrow \ x \in \mathbb{C}^* \setminus \{1\} \]

unipotent completion
(to be explained later)

\[ \text{Lie } (\pi_{un}) \text{ is a (pro) object of } \text{MTM}(K). \]

(b) \[ K = \mathbb{Q}, \quad x = \overrightarrow{01} = \tfrac{2}{3}w \]
\[ e \in T_0 \mathbb{P}^1 \]

\[ \text{Lie } (\pi_1^{uni}) \text{ is a pro object of } \text{MTM}(\mathbb{Z}). \]

Point \( \mathbb{P}^1 - \{0, 1, \infty\}, \overrightarrow{01} \) has everywhere good reduction
MHS \text{ has periods multi-zeta values}

\[ \delta(n_1, \ldots, n_r) = \sum_{0 < k_1 < \ldots < k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}} \]

\[ \pi_1(\text{M+M}(z)) \]

\[ \begin{align*}
(1) & \quad \pi_1(\text{split M+M}) \cong G_m \\
& \quad G_m \subseteq \mathbb{Q}(n) \\
& \quad \uparrow \\
& \quad \text{n-th power of std char} \\
(2) & \quad 1 \to K \to \pi_1(\text{M+M}) \to G_m \to 1 \\
& \quad \uparrow \\
& \quad \text{pro-rep/potent.} \\
\text{Set } k = \text{Lie}(K) \\
\text{So } k \text{ has a } G_m \text{-action.}
\]
\[ \text{Ext}^j_{\text{MTM}}(\mathbb{Q}, \mathbb{Q}(n)) < \bigoplus_{j=1}^n \mathbb{Q}, \text{odd} \geq 3. \]

\[ = H^j(\Pi_1(\text{MTM}), \mathbb{Q}(n)) \]

\[ = \left[ H^j_k(\mathbb{K}) \otimes \mathbb{Q}(n) \right]^G_m \]

\[ \Rightarrow \quad 0 \neq \mathbb{K} \quad \overset{\text{os}}{\leftarrow} H^2_k(\mathbb{K}) = 0 \]

1. \( \mathbb{K} \) free

2. \( \text{Gr}_k \mathbb{K} \cong \mathbb{U}(z_3, z_5, z_7, \ldots) \)

\( G_m \) acts on \( \mathbb{Z}_{2n-1} \) by the \( (2n-1)^{st} \) power of std char.

3. \( \mathbb{Z}_{2n-1} \) spans a copy of \( \mathbb{Q}(2n-1) \) in \( \text{Gr}_k \mathbb{K} \)

\text{Conclusion:}

\[ \pi^1_1^{\text{MTM}} \cong G_m \times \exp \mathbb{U}(z_3, z_5, \ldots)^\wedge \]
THE FACT THAT

\[ \text{Lie}_{\mathcal{V}}(\mathbb{P}^1 - \{0,1,\infty\}, \overline{\mathcal{O}_1})^{un} \]

is in \( \text{MTM}(2) \) implies that we have

\[ \pi_1(\text{MTM}) \to \text{Aut}_{\mathcal{V}}(\mathbb{P}^1 - \{0,1,\infty\})^{un} \]

\( \text{THM (BROWN)} \) This is 1-1.

**COR:** Periods of all \( \text{MTM}/2 \) are MZVs.

The story of \( \text{MTM}/2 \) is reasonably well understood.

\( \text{MTM} \leftrightarrow \text{genus 0} \quad \mathbb{P}^1 - \{0,1,\infty\} = \text{M}_0,4 \)

next step - genus 1.
III ELLIPTIC MOTIVES

(1) CLASSICAL: \{ eg! work of Goncharov \}

\[ E = \text{elliptic curve } / \mathbb{Q} \]

have equation

\[
\begin{bmatrix}
  y^2 = 4x^3 - ux - v \\
  D := u^3 - 27v^2 \neq 0
\end{bmatrix}
\]

\[ "h'(E) = H'(E)" \]

Betti: \( H_b = H^1(\mathbb{E}^a, \mathbb{Q}) \)

\( \mathbb{Q} \)-DR: \( H^{1, 0}_{dR} = H^1_{dR}(\mathbb{E}/\mathbb{Q}) = \mathbb{Q} \frac{dx}{y} \oplus \mathbb{Q} \frac{x dx}{y} \)

\[ F^1 H^{1, 0}_{dR} = \mathbb{Q} \frac{dx}{y} \]

\[ H_b \otimes \mathbb{C} \cong H^{dR} \otimes \mathbb{C} \]

\( \sim \) HS on \( H_b \otimes \mathbb{C} \)
$\mathfrak{C}_Q \subseteq E[l^n](\overline{Q}) = E[l^n](\mathbb{C})$

$= H_1(E^o, \mathbb{Z}/l^n)$

$\pi_1^{\text{ét}}(E/\overline{Q}, o) \cong \varprojlim E[l^n](\overline{Q})$

$\cong H_1(E^o, \mathbb{Q})$

$H_l := \pi_1^{\text{ét}}(E/\overline{Q}, o) \otimes \mathbb{Q}_l$.

Mixed elliptic motives:

Iterated extensions of

$S^n H(r) := S^n H \otimes \mathbb{Q}(r)$.

Category of MEM(E) should depend heavily on $E$. 

(2) UNIVERSAL MIXED ELLIPTIC MOTIVES (Overview)
(Joint with Makoto Matsumoto)

Idea: These are "motives" associated (uniformly) to all elliptic curves. More precisely, to the universal elliptic curve

\[ \mathcal{E} \leftarrow \text{universal elliptic curve} \]

\[ f \downarrow \]

\[ M_{1,1} \leftarrow \text{moduli stack of elliptic curves} / \mathbb{Z} \]

Set \[ H = R^2 f^*_x \mathbb{Q} \]

(suitably interpreted - compatible realizations). This is the local system whose fiber over \[ \mathbb{Z} \] = moduli point of \( \mathcal{E} \)
is $H^1(E, \mathbb{Q})$.

Realizations of $H$

Betti: This is the rank 2 local system that corresponds to the defining representation $\sigma_1 (M_{1,1}, *) \simeq \text{SL}_2(\mathbb{Z})$.

Hodge: This is a polarized variation of Hodge structure of weight 1.

$\ell$-adic étale: This is the lisse sheaf whose fiber over $[E]$ is $\pi_1(E/\mathbb{Q}, \circ) \otimes \mathbb{Q}_\ell$.

$\mathbb{Q} - \text{DR} : (\text{later})$
Base point: we need to use a tangential base point as every elliptic curve $\mathbb{Q}$ has bad reduction at primes dividing its discriminant.

\[ E_{\text{Tate}} \quad \xrightarrow{\text{t}} \quad \mathbb{Z}[E] \]

Smooth at every prime $p$.

\[ t = \frac{e}{2g} \quad \text{tangential base point of the modular curve} \]
Fiber of $H$ over $t = 2q$:

Given by limit mixed Hodge Theory (etc). Fiber is

\[(\star) \quad H := H_t = \Bbb Q \oplus \Bbb Q(-1)\]

in all theories (Hodge, étale).

\[\bigotimes\quad\text{This is the realization of an object of MTM}(\Bbb Z).\]

\[\text{PAUSE:}\]

1. $H$ is a PVHS of weight 1. So $0 = W_0 H \subseteq W_1 H = H$

2. This restricts to a weight filtration on every fiber. In particular, to
one on $H$:

$$0 = W_0H \subseteq W_1H = H$$

3. The monodromy logarithm acts on $H$

$$N: H \rightarrow H$$

$\rightarrow$ monodromy weight filtration

$$0 = M_{-1}H \subseteq M_0H = M_1H \subseteq M_2H = H$$

This is the weight filtration of the limit Mixed Hodge Structure (MHS) $(X)$. The Hodge filtration on $H$ is the limit Hodge filtration.
Likewise, the $G_\mathbb{Q}$ action on the Tate module of the restriction of the Tate curve to $\mathfrak{m}_g \leftrightarrow \overline{\mathbb{Q}}((q^n : n \geq 1))$ is $\mathbb{Q}_\ell \oplus \mathbb{Q}_\ell(1)$. 

---