May 6, 2014 IHES

LECTURE #1

I HISTORY & CONTEXT

(1) GROTHENDIECK (1960s) — defined the category of pure motives over a field k.

- Objects "generated by" smooth projective varieties over k.
- morphisms: correspondences mod rational equivalence.

(2) BEILINSON (1980) — defined motivic cohomology of a variety X in terms of the algebraic K-theory $K_\bullet(X)$ of X: smooth

Beilinson-Soulé

$$H^j_{\mathcal{M}}(X, \mathbb{Q}(n)) := K_{2n-j}(X)^{(n)}$$

where

Conjectured to vanish when $j<0$

Also equal to Bloch's higher Chow groups

$$K_\bullet(X)\otimes\mathbb{Q} = \bigoplus_{n\geq 0} K_\bullet(X)^{(n)}$$

p^n-eigenspace of the Adams op ψ^p.

If X is defined $/\mathbb{Z}$, then the $H^j_m(X,\mathbb{Q}(n))$ should be finite dimensional, with dims predicted by order of vanishing of suitable L-function.

<u>Examples:</u>

① (Bloch–Gersten–Quillen)

If X is smooth projective, then

$$H^{2n}_m(X,\mathbb{Q}(n)) \cong CH^n(X)\otimes\mathbb{Q}$$

Chow group of cod n-cycles

(Borel–Beilinson)

② If F is a number field, then

(i) $H^j_{\mathcal{M}}(\operatorname{Spec} F, \mathbb{Q}(n)) = 0$

when $n < 0$

(ii) $H^0_{\mathcal{M}}(\operatorname{Spec} F, \mathbb{Q}(n)) = \begin{cases} \mathbb{Q} & n = 0 \\ 0 & n \neq 0 \end{cases}$

(iii) $H^j_{\mathcal{M}}(\operatorname{Spec} F, \mathbb{Q}(n)) = 0$

$j > 1$.

(iv) $H^1(\operatorname{Spec} F, \mathbb{Z}(1)) \cong F^\times$

(v) when $n > 1$

$$\dim H^1(\operatorname{Spec} F, \mathbb{Q}(n)) = \begin{cases} r_1 + r_2 & n \text{ odd} \\ r_2 & n \text{ even.} \end{cases}$$

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It was conjectured (by Beilinson) that there was a tannakian category of mixed motives MM and that

$$H^j_m(X, \mathbb{Q}(n)) = \mathrm{Ext}^j_{MM}(\mathbb{Q}, M(X)(n))$$

Tannakian category: roughly a (neutral) tannakian category $\mathcal{C}$ is a k-linear abelian category with $\otimes$ that admits an exact and faithful functor ω to the category of finite dim k-vector spaces.

Such a category is the category of reps of a pro-algebraic group:

$$\pi_1(\mathcal{C}, \omega) = \mathrm{Aut}^{\otimes}\omega$$

nat isoms $\omega \to \omega$ that resp $\otimes$ product

Voevodsky (Levine) constructed a triangulated tensor category of mixed motives where

$$H^{\cdot}_{m}(X, \mathbb{Q}(n)) = \mathrm{Ext}^{\cdot}_{MM}(\mathbb{Q}(-n), M(X))$$

Here $M(X)$ is a suitable motive associated to X and

"$\mathbb{Q}(1) = H_1(\mathbb{G}_m)$"

"$\mathbb{Q}(-1) = H^1(\mathbb{G}_m)$"

$$\mathbb{Q}(n) = \begin{cases} \mathbb{Q}(1)^{\otimes n} & n \geq 0 \\ \mathbb{Q}(-1)^{\otimes |n|} & n < 0 \end{cases}$$

Tate motives

II Mixed Tate Motives

(Levine, Deligne - Goncharov)

If K is a number field and $S \subset \operatorname{Spec} \mathcal{O}_K$, then there is a tannakian category $MTM(\mathcal{O}_{K,S})$ of "mixed Tate motives over $\mathcal{O}_{K,S}$ such that

$$H^j_m(\operatorname{Spec} \mathcal{O}_{K,S}, \mathbb{Q}(n)) \cong \operatorname{Ext}^j_{MTM(\mathcal{O}_{K,S})}(\mathbb{Q}, \mathbb{Q}(n))$$

$\operatorname{Gr}^M_{odd} V = 0$

Objects V of $MTM(\mathcal{O}_{K,S})$ have a <u>weight filtration</u> in MTM:

Why M. and not W.? Later!

$$0 \subseteq \cdots \subseteq M_r V \subseteq M_{r+1} V \subseteq \cdots \subseteq V$$

Quick review of (mixed) Hodge theory.

(1) a $\mathbb{Q}$-<u>Hodge structure</u> V of weight m consists of

(a) a fte dim vector space $V_{\mathbb{Q}}$,

(b) a decreasing filtration

$$V_{\mathbb{C}} = F^a V \supseteq \cdots \supseteq F^p V \supseteq F^{p+1} \supseteq \cdots \supseteq F^b V = 0$$

where $V_{\mathbb{C}} = V_{\mathbb{Q}} \otimes \mathbb{C}$

"The Hodge filtration"

These satisfy

$$V_{\mathbb{C}} = \bigoplus_{p+q=m} V^{p,q}$$

where $V^{p,q} := F^p V \cap \overline{F^q V}$

(2) a $\mathbb{Q}$-mixed Hodge structure (MHS) V consists of

(1) a finite dimensional $\mathbb{Q}$ vector space $V_{\mathbb{Q}}$

(2) an increasing filtration

"the weight filtration"

$$0 = W_n V_{\mathbb{Q}} \subseteq \dots \subseteq W_r V_{\mathbb{Q}} \subseteq W_{r+1} V_{\mathbb{Q}} \subseteq \dots \subseteq W_N V_{\mathbb{Q}} = V_{\mathbb{Q}}$$

(3) a decreasing <u>Hodge filtration</u> of $V_{\mathbb{C}}$:

$$V_{\mathbb{C}} = F^a V \supseteq \dots \supseteq F^p V \supseteq F^{p+1} \supseteq \dots \supseteq F^b V = 0$$

These satisfy:

Each $(Gr^W_m V, F^\bullet)$

is a Hodge structure of weight m. Here

$$(Gr^W_m V)_{\mathbb{Q}} := W_m V_{\mathbb{Q}} / W_{m-1} V_{\mathbb{Q}}.$$

$$\boxed{\text{2 A MHS is more than the sum of its graded quotients}}$$

A HS V of weight m is a MHS. Set $W_{m-1} V = 0$, $W_m V = V$

<u>Important facts</u>:

① The functor $Gr^W_\bullet$ is exact.

② (Deligne) the $\mathbb{Q}$-cohomology of every variety $/\mathbb{C}$ of fte type has a natural MHS.

Example:

① X smooth & projective $/\mathbb{C}$

$$V = H^m(X;\mathbb{Q})$$

has a HS of weight m.

$$F^p H^m(X) = \bigoplus_{s \geq p} H^{s,m-s}(X).$$

Realizations of MTM$(\mathbb{Z})$

Ex: $V = \mathbb{Q}(1) = H_1(\mathbb{G}_{m/\mathbb{Z}})$

Betti $\quad H_1(\mathbb{G}_m^{an}, \mathbb{Q}) = H_1(\mathbb{C}^*, \mathbb{Q})$

$$= \mathbb{Q}\sigma \qquad \sigma$$

Q-DR: $\quad \mathrm{Hom}(H^1_{DR}(\mathbb{G}_{m/\mathbb{Q}}), \mathbb{Q})$

$$V^{DR} = \mathbb{Q}\frac{dz}{z} \qquad \mathbb{Q}(2\pi i \check{\sigma})$$

$$V^{DR} = F^{-1} = \mathbb{Q}\left(\frac{dz}{z}\right)^{\vee} \supseteq F^{0} = 0$$

<u>Hodge:</u> $\quad (\text{Betti}) \otimes \mathbb{C} \cong (\text{Q-DR}) \otimes \mathbb{C}$

$F^{\bullet}$

HS type $(-1,-1)$

<u>ℓ-adic étale:</u>

$V^{B} \otimes \mathbb{Q}_{\ell} \circlearrowleft G_{\mathbb{Q}}$, unramified outside ℓ.

crystalline @ ℓ.

given by the ℓ-adic cyclotomic char

$$\chi_{\ell} : G_{\mathbb{Q}} \to \mathbb{Z}_{\ell}^{\times}.$$

$G_{\mathbb{Q}} := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

Similarly for $\mathbb{Q}(n)$

Betti — $\mathbb{Q}$

Hodge — HS type $(-n,-n)$

ℓ-adic ét — $\chi_{\ell}^{\otimes n}$

General $V \in MTM(\mathbb{Z})$

$(V, M_\cdot)$ – Betti filtered $\mathbb{Q}$-vect sp
V^B

– $\mathbb{Q}$-DR

$$\underbrace{(V^{DR}, M_\cdot, F^\cdot)}_{\text{all } \mathbb{Q}\text{-vect sp}}$$

Hodge – $\underbrace{V^{DR} \otimes \mathbb{C} \cong V^B \otimes \mathbb{C}}_{\text{MHS}}$ $\begin{cases} Gr^M_{odd} = 0 \\ Gr^M_{-2m} \cong \mathbb{Q}(m)^N \end{cases}$

ℓ-"adic":

$$V_\ell \cong V^B \otimes \mathbb{Q}_\ell$$

$\circlearrowleft$ $G_{\mathbb{Q}}$ unramified outside ℓ
"crystalline @ ℓ"

$$Gr^M_{-2m} V_\ell \cong \bigoplus \mathbb{Q}_\ell(m).$$

EXAMPLES OF MTM(K)

① $V = H^1(\mathbb{G}_m, \{1, \lambda\})$, $\lambda \in K^\times - \{1\}$.

$\sigma \longmapsto$ Betti generator

$$0 \to \tilde{H}^0(\{1,\lambda\}) \to V \to H^1(\mathbb{G}_m) \to 0$$

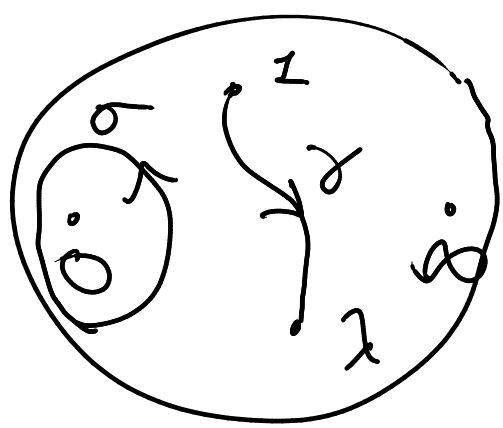
generator γ

$\tilde{H}^0(\{1,\lambda\}) \cong \mathbb{Z}(0)$, $\quad H^1(\mathbb{G}_m) \cong \mathbb{Z}(-1)$

Element of $\mathrm{Ext}^1_{MHS}(\mathbb{Z}(-1), \mathbb{Z}(0))$

$\cong \mathrm{Ext}^1_{MHS}(\mathbb{Z}, \mathbb{Z}(1))$

$\cong \mathbb{C}^*$



$$F^1 V_{DR} = \mathbb{C} \frac{dz}{z}$$

$$\frac{dz}{z} = 2\pi i \check{\sigma} + \log \lambda$$

MHS det by $\log \lambda \bmod 2\pi i \mathbb{Z}$

$\underline{\text{or}}$ by $\lambda \in \mathbb{C}^\times$

② (details later)

(Deligne - Goncharov)

(a)

$$\pi^{un} = \pi_1(\mathbb{P}^1 - \{0,1,\infty\}, x)^{un} \quad x \in K^\times - \{1\}$$

unipotent completion
(to be explained later)

$\mathrm{Lie}(\pi^{un})$ is a (pro) object of $MTM(K)$.

(b) $K = \mathbb{Q}$, $x = \vec{01} := \partial/\partial w \in T_0 \mathbb{P}^1$

$\mathrm{Lie}(\pi_1^{un})$ is a pro object of $MTM(\mathbb{Z})$.

Point $(\mathbb{P}^1 - \{0,1,\infty\}, \vec{01})$ has everywhere good reduction

MHS (on π^{un}) has periods multizeta values

$$\zeta(n_1, \dots, n_r) = \sum_{0<k_1<\dots<k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}}$$

$\pi_1(MTM(\mathbb{Z}))$

(1) $\pi_1(\text{split } MTM) \cong \mathbb{G}_m$

$$\mathbb{G}_m \circlearrowright \mathbb{Q}(n)$$

n^{th} power of std char

(2) $1 \to K \to \pi_1(MTM) \to \mathbb{G}_m \to 1$

pro unipotent.

naturally split by $\mathbb{Q}$-DR str

Set $\underline{k} = \mathrm{Lie}(K)$

So $\underline{k}$ has a $\mathbb{G}_m$-action.

$$\mathrm{Ext}^j_{MTM}(\mathbb{Q}, \mathbb{Q}(n)) \begin{cases} 0 & j>1 \\ \mathbb{Q} & j=1,\ n \text{ odd} \geq 3. \end{cases}$$

$$= H^j(\pi_1(MTM), \mathbb{Q}(n))$$

$$= \left[H^j(\underline{\mathfrak{k}}) \otimes \mathbb{Q}(n)\right]^{\mathbb{G}_m}$$

$\Longrightarrow$ ① $\underline{\underline{\mathfrak{k}}}$ free ← as $H^2(\underline{\mathfrak{k}})=0$

② $\mathrm{Gr}^M_\bullet \underline{\underline{\mathfrak{k}}} \cong \mathbb{L}(z_3, z_5, z_7, \dots)$

$\mathbb{G}_m$ acts on z_{2n-1} by $(2n-1)^{st}$ power of std char.

③ z_{2n-1} spans a copy of $\mathbb{Q}(2n-1)$ in $\mathrm{Gr}^M_\bullet \underline{\mathfrak{k}}$

CONCLUSION:

$$\pi_1(MTM) \cong \mathbb{G}_m \ltimes \exp \mathbb{L}(z_3, z_5, \dots)^\wedge$$

THE FACT THAT

$$\mathrm{Lie}\,\pi_1(\mathbb{P}^1-\{0,1,\infty\},\vec{01})^{un}$$

is in MTM($\mathbb{Z}$) implies that we have

$$\pi_1(\mathrm{MTM}) \longrightarrow \mathrm{Aut}\,\pi_1(\mathbb{P}^1-\{0,1,\infty\})^{un}$$

<u>THM</u> (BROWN) This is 1-1.

<u>COR:</u> Periods of all MTM/$\mathbb{Z}$ are MZVs.

The story of MTM/$\mathbb{Z}$ is reasonably well understood.

$$\mathrm{MTM} \longleftrightarrow \text{genus } 0$$

$$\mathbb{P}^1-\{0,1,\infty\} = \mathcal{M}_{0,4}$$

next step — genus 1.

III ELLIPTIC MOTIVES

(1) CLASSICAL: $\left\{ \begin{array}{l} \text{eg: Work of} \\ \text{Goncharov} \end{array} \right\}$

E = elliptic curve $/\mathbb{Q}$

have equation

$$\left[\begin{array}{l} y^2 = 4x^3 - ux - v \\ D := u^3 - 27v^2 \neq 0 \end{array} \right]$$

"$h^1(E) = H^1(E)$"

diff'l of the 2nd kind.

Betti: $H_B = H^1(E^{an}, \mathbb{Q})$

$\mathbb{Q}$-DR: $H_{DR} = H^1_{DR}(E_{/\mathbb{Q}}) = \mathbb{Q}\frac{dx}{y} \oplus \mathbb{Q}\frac{x\,dx}{y}$

$$F^1 H^1_{DR} = \mathbb{Q}\frac{dx}{y}.$$

$H_B \otimes \mathbb{C} \cong H_{DR} \otimes \mathbb{C}$

$\rightsquigarrow$ HS on $H_B \otimes \mathbb{C}$

ℓ-adic étale:

$$G_{\mathbb{Q}} \circlearrowright E[\ell^n](\overline{\mathbb{Q}}) = E[\ell^n](\mathbb{C})$$
$$= H_1(E^{an}, \mathbb{Z}/\ell^n)$$

$$\pi_1^{ét}(E_{/\overline{\mathbb{Q}}}, 0) \cong \varprojlim E[\ell^n](\overline{\mathbb{Q}})$$
$$\cong H_1(E^{an}, \mathbb{Z}_\ell)$$

$$H_\ell := \pi_1^{ét}(E_{/\overline{\mathbb{Q}}}, 0) \otimes \mathbb{Q}_\ell.$$

Mixed elliptic motives:

iterated extensions of

$$S^n H(r) := S^n H \otimes \mathbb{Q}(r).$$

Category of $MEM(E)$ should depend heavily on E.

(2) UNIVERSAL MIXED ELLIPTIC MOTIVES (Overview)

(Joint with Makoto Matsumoto)

details later } <u>Idea</u>: These are "motives" associated (uniformly) to all elliptic curves – More precisely, to the universal elliptic curve

$$\begin{array}{c} \mathcal{E} \\ f \downarrow \\ \mathcal{M}_{1,1} \end{array}$$

← universal elliptic curve

← moduli stack of elliptic curves $/\mathbb{Z}$

Set $\mathbb{H} = R^1 f_* \mathbb{Q}$

(suitably interpreted – compatible realizations). This is the local system whose fiber over $[E]$:= moduli point of E

is $H^1(E, \mathbb{Q})$.

Realizations of $\mathbb{H}$

<u>Betti</u>: This is the rank 2 local system that corresponds to the defining representation of $\pi_1(\mathcal{M}_{1,1}^{an}, *) \cong SL_2(\mathbb{Z})$.

<u>Hodge</u>: This is a polarized variation of Hodge structure of weight 1.

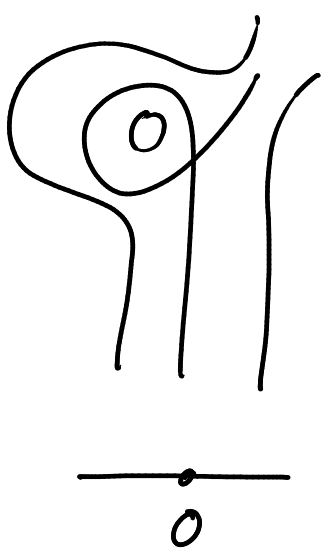
<u>ℓ-adic étale</u>: This is the lisse sheaf whose fiber over $[E]$ is $\pi_1(E_{/\bar{\mathbb{Q}}}, 0) \otimes \mathbb{Q}_\ell$.

<u>$\mathbb{Q}$-DR</u>: (later)

<u>Base point</u>: we need to use a tangential base point as every elliptic curve $/\mathbb{Z}$ has bad reduction at primes dividing its discriminant.

<u>Tate curve</u>

$$\begin{array}{c} \mathcal{E}_{Tate} \\ | \\ \mathbb{Z}[[q]] \end{array}$$



smooth at every prime p.

$\underline{t} := \partial/\partial q$. | tangential base point of the modular curve.

Fiber of $\mathbb{H}$ over $\underline{t} = \partial/\partial q$:

Given by limit mixed Hodge Theory (etc). Fiber is

(*) $H := H_{\underline{t}} = \mathbb{Q} \oplus \mathbb{Q}(-1)$

in <u>all</u> theories (Hodge, étale)

$$\boxed{\text{(*) This is the realization of an object of } MTM(\mathbb{Z}).}$$

PAUSE:

1. $\mathbb{H}$ is a PVHS of weight 1.

So $$0 = W_0\mathbb{H} \subseteq W_1\mathbb{H} = \mathbb{H}$$

2. This restricts to a weight filtration on every fiber. In particular, to

one on H:

$$0 = W_0 H \subseteq W_1 H = H$$

3. The monodromy logarithm acts on H

$$N : H \to H$$

$\rightsquigarrow$ monodromy weight filtration

$$0 = M_{-1} H \subseteq M_0 H = M_1 H$$
$$\subseteq M_2 H = H$$

This is the weight filtration of the limit MHS $(\mathcal{X})$.

The Hodge filtration on H is the limit Hodge filtration.

Likewise, the $G_{\mathbb{Q}}$ action on the Tate module of the restriction of the Tate curve to $\partial/\partial q \longleftrightarrow \overline{\mathbb{Q}}((q^{1/n} : n \geq 1))$ is $\mathbb{Q}_\ell \oplus \mathbb{Q}_\ell(1)$.
