HITCHIN TYPE MODULI SPACES IN AUTOMORPHIC REPRESENTATION THEORY

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In an influential 1987 Duke paper [3], Hitchin introduced the famous integrable system, the moduli space of Higgs bundles. Let X be a smooth proper and geometrically connected curve over a field k. Let G be a connected reductive group over k. A G-Higgs bundle over X is a pair $(\mathcal{E}, \varphi)$ where $\mathcal{E}$ is a principal G-bundle over X and φ is a global section of the vector bundle $\mathrm{Ad}(\mathcal{E}) \otimes \omega_X$ over X. Here, $\mathrm{Ad}(\mathcal{E})$ is the vector bundle associated to $\mathcal{E}$ and the adjoint representation of G. Hitchin studied the moduli space $\mathcal{M}^s_G$ of stable G-Higgs bundles over X via the Hitchin map

$$f : \mathcal{M}^s_G \to \mathcal{A}_G \tag{0.1}$$

to some affine space $\mathcal{A}_G$ parametrizing invariants of φ such as its trace and determinant in the case $G = \mathrm{GL}_n$. Hitchin showed that f exhibited $\mathcal{M}^s_G$ as an algebraically completely integrable system, and gave concrete descriptions of the fibers of f in terms of spectral curves.

In the same Duke volume, Laumon [5], generalizing ideas of Drinfeld, formulated a geometric version of the Langlands correspondence for GL_n over function fields. This is a conjectural bijection between certain perverse sheaves on the moduli stack $\mathrm{Bun}_{\mathrm{GL}_n}$ of rank n vector bundles over an algebraic curve X (over a finite field k) and local systems of rank n over X. Without knowing Hitchin's work at the time (despite publishing in the same Duke volume), Laumon was able to make insightful predictions about these perverse sheaves using the symplectic geometry of $\mathcal{M}_G$ (the stack version of $\mathcal{M}^s_G$). The reason is that the Hitchin moduli stack $\mathcal{M}_G$ is essentially the total space of the cotangent bundle of Bun_G, therefore perverse sheaves on Bun_G are related to the symplectic geometry of $\mathcal{M}_G$ via the characteristic cycle construction. This was probably the first time the Hitchin moduli space appeared in automorphic representation theory. Subsequently, the Hitchin moduli space has been playing more and more important roles in the development of geometric representation theory. For example, Beilinson and Drinfeld [1] studied quantizations of Hitchin's integrable system and used them to realize part of the geometric Langlands correspondence. The work of Kapustin and Witten [4] puts geometric Langlands correspondence in the context of mirror symmetry between $\mathcal{M}_G$ and $\mathcal{M}_{G^\vee}$.

We now focus on the applications of the Hitchin moduli space (and its variants) to the classical Langlands conjectures for groups over function fields concerning automorphic forms. A key tool in the study of automorphic forms is the Arthur-Selberg trace formula, which expressed spectral quantities (traces on representations) in terms of geometric quantities (orbital integrals). B.C.Ngô [6] made the fundamental observation that point-counting on the fibers of the Hitchin map f are closely related to orbital integrals that appear in the trace formula. This observation, along with ingenious technical work, allowed him to prove the Lie algebra version of the Fundamental Lemma conjectured by Langlands and Shelstad in the function field case [7]. In the first half of the talk, I will explain, in heuristic terms, why Hitchin-type moduli spaces naturally show up in the study of the trace formula. A good reference to this topic is [2].

The notion of automorphic forms has a highly nontrivial yet totally natural generalization. Drinfeld introduced the moduli stack of Shtukas as an analogue of Shimura varieties for function fields, and cohomology classes of the moduli of Shtukas generalize the notion of automorphic forms. In this context, traces of operators on the space of automorphic forms are generalized to intersection numbers of algebraic cycles on the moduli of Shtukas. In the second half of the talk, I will talk about a new direction initiated in the joint work with Wei Zhang concerning the study of these generalized traces (i.e., intersection numbers). In the simplest nontrivial situation, our work [8] and [9] shows that such intersection numbers can be used to express higher derivatives of automorphic L-functions, and we call such an expression a "higher

Gross-Zagier formula". The geometry of Hitchin type moduli spaces and variants of the Hitchin map again play a key role in the proof of the formula.

References

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