RANDOM GROUPS FROM GENERATORS AND RELATIONS

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The class group is a finite abelian group attached to a number field, a finite extension K of the rational numbers $\mathbb{Q}$. It plays a central role in the number theory of K. In 1984, Cohen and Lenstra [CL84] asked, as the number field K varies, what is the distribution of these finite abelian groups. Equivalently, in the language of probability, for a (suitable notion of) random number field K, can we identify the random group that is the class group of K?

Cohen and Lenstra made a conjecture to the answer to this question for imaginary quadratic fields (i.e. $K = \mathbb{Q}(\sqrt{-D})$ for a positive, square-free integer D), based on the following philosophy. Class groups are completely mysterious and we know (essentially) nothing about them, so we should conjecture that they follow the most natural possible distribution on finite abelian groups, which they thought was a distribution such that a group G appears with frequency inversely proportional to its number of automorphisms.

In 1986, Friedman and Washington [FW89] were trying to understand a function field analog of these questions, i.e. an analog in which $\mathbb{Q}$ is replaced by $\mathbb{F}_q(t)$, the rational functions over a finite field in a formal variable t. In this case the finite extensions and their class groups have interpretations as curves and their Jacobians over the finite field and this is an additional perspective that can be brought to bear on the question. That perspective led them to considering the free abelian group $\mathbb{Z}^n$ on n generators, modulo n nice random relations, and letting n go to infinity. (More precisely, they looked at $\mathbb{Z}_p^n$, where $\mathbb{Z}_p$ is the p-adic completion of the integers and is a compact group, and took independent relations from Haar measure on $\mathbb{Z}_p^n$.) They found that this construction of a free group modulo the nicest possible random relations gave the same random group that Cohen and Lenstra had constructed.

In my work [Woo15], I have found that this Cohen-Lenstra random group is universal in the following sense. If you start with a free abelian group $\mathbb{Z}^n$ on n generators, and take the quotient by almost any kind of independent random relations, no matter how you take the relations, as n goes to infinity you obtain the Cohen-Lenstra random group. This is the kind of phenomenon that we see in the Central Limit Theorem, in which no matter what kind of random variable you start with, many independent copies have a normalized average that approaches a universal distribution. If you are building a random abelian group from n generators and n independent relations, even if those relations are drawn from a strange distribution, as n increases the resulting group approaches a universal group—the Cohen-Lenstra random group.

I am interested in understanding the distribution of non-abelian analogs of class groups. One interpretation of the class group of K is, via class field theory, the Galois group of the maximal, unramified abelian extension of K. Via this understanding, one naturally asks to understand the distribution of the Galois group of the maximal unramified extension (without the abelian restriction) of a varying number field K. Geometrically, this Galois group can be viewed as a fundamental group (of the ring of integers in K, or in the function field analog, of the associated curve). There are conjectures for what the maximal profinite quotient of this random group is (for imaginary and real quadratic fields) due to Boston, Bush, and Hajir [BBH16, BBH18], and some results in the function field analog proving moments of this random group [BW17, Woo17]. Yet to even conjecture what this random group should be for a random number field, we need candidate random groups that are (at least sometimes) non-abelian and infinite.

Joint with Yuan Liu [LW17], we study the random group constructed from the free group F_n modulo n random relations (from the most nice/uniform distribution) as n goes to infinity. While there is lot of work on the Gromov model of random groups [Oll05], in which one fixes a number of generators, and takes a growing number of relations, our desired random groups appear to require something more like balanced presentations (the same number of generators and relations). In [LW17], we prove that the random group constructed from the free group F_n modulo n random relations approaches a limiting random group, and we determine explicitly the probabilities for this limiting random group (e.g. to be a certain group or to be in a nice subset of groups).

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