HIGHER DIMENSIONAL ANALOGUES OF K3 SURFACES

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ABSTRACT. After an introduction to the theory of compact hyper-Kähler manifolds I will explain, focusing on recent developments, how this class of manifolds should be considered as the higher dimensional analogues of K3.

The Beauville–Bogomolov decomposition theorem [Bea83] states that, up to a finite cover, any compact Kähler manifold with trivial first Chern class can be decomposed as the product of complex tori, irreducible (or strict) Calabi–Yau manifolds, and irreducible hyper-Kähler manifolds. Thus, these three classes of manifolds constitute the building blocks of compact Kähler manifold with trivial first Chern class. This note is a short introduction to the last class.

The theorem above is a consequence of Yau's solution to the Calabi conjecture, and the division into the three classes above is based on the classification of the holonomy group (with respect to the unique Ricci flat metric in a given Kähler class). According to the holonomy principle, a specific holonomy group gives rise to certain invariant tensor on the underlying manifold. For strict Calabi–Yau manifolds (holonomy group $SU(k)$), this means that there is no holomorphic form except in top dimension. In the case of irreducible hyper-Kähler manifolds (holonomy group $Sp(r)$) this amounts to the existence of a holomorphic symplectic form, which is unique up to scalar. For this reason, irreducible compact hyper-Kähler manifolds are characterized as those simply connected compact Kähler manifolds which have a unique, up to scalar, holomorphic symplectic form. The existence of this holomorphic symplectic form has surprising consequences on the geometry of these manifolds, which we will refer to interchangably as irreducible compact hyper-Kähler manifolds or as irreducible holomorphic symplectic (IHS).

There are many examples of non–compact hyper–Kähler manifolds, most of them coming up naturally in representation theory and mathematical physics (quiver varieties, Springer resolutions, Hitchin system). By contrast, up to deformation only very few examples of IHS manifolds are known. This marks a difference also with the class of strict Calabi–Yau manifolds that appears in the decomposition above.

In complex dimension 2, the only examples of IHS are K3 surfaces. For every $n \geq 2$, there are two $2n$-dimensional deformation classes of IHSs, each fitting into a series defined by the same kind of construction [Bea83]. In addition to these, there are two more examples of IHS in dimension 6 and 10 [O'G99, O'G03], which are referred to as the exceptional or sporadic examples.

The first series is that of the Hilbert schemes of n points on a K3 surface. Smooth deformations of these are still IHS manifolds and are called of K3$^{[n]}$–type. More generally, thanks to seminal work of Mukai [Muk84], any moduli space of stable sheaves on a K3 surface has a holomorphic symplectic form and inherits many of the geometric properties of the underlying K3 surface. It turns out that these moduli spaces, when compact, are IHS of K3$^{[n]}$–type. A crucial tool in proving that any such moduli space can be realized as a deformation of a Hilbert scheme of points on a K3 is a result by Huybrechts [Huy99] which states that two birational IHS are deformation equivalent. Then one uses the fact that, thanks to work of Yoshioka and O'Grady, it is known that up to freely deforming the underlying K3 surface, one can relate any moduli space of sheaves on a K3 surface to a Hilbert scheme of points through a sequence of birational transformations. The second series of examples is obtained by a similar construction, starting instead with an abelian surface A. Hilbert schemes of points on A have a holomorphic symplectic form, but are not simply connected. The Beauville–Bogomolov decomposition of the Hilbert scheme of $n+1$ points on A is a product of two irreducible factors: the abelian surface A itself and an IHS of dimension $2n$ called the n–th generalized Kummer variety of A. As in the case of K3 surfaces, one gets deformations of generalized Kummer varieties when considering the Beauville–Bogomolov factors of a moduli space of sheaves on an abelian surface. As for the two exceptional examples, there were produced by resolving symplectically a singular moduli space of sheaves on an Abelian surface (dimension 6) and on a K3 surface (dimension 10) [O'G99, O'G03]. We refer to them as OG_6 and OG_{10}. These examples have been less studied and are not as well understood. For example, the Hodge numbers of OG_6 were computed only recently in our joint paper [MRS18], and only the second Betti number and the Euler characteristic of OG_{10} are known [Moz06].

For an introduction to the topic, the reader may look at [Bea83], [Huy99], or at [O'G12], which inspired the title of this talk.

Since A. Weil rediscovered K3 surfaces in the 1950s, these surfaces have occupied a central role in algebraic and arithmetic geometry, as well as in topology and differential geometry. An appealing feature of higher dimensional HK manifolds is that they enjoy many of the nice properties of K3 surfaces. This arguably makes them the natural higher dimensional analogues of K3 surfaces. Next, we describe some of the ways in which the geometry of higher dimensional HK manifolds behaves like that of K3s.

Roughly speaking, the Torelli theorem for K3 surfaces guarantees that the Hodge structure on the second cohomology group of a K3 surface determines the complex structure of the underlying topological manifold. A theorem of Verbistky and Markman states that a similar property holds true for higher dimensional irreducible hyper–Kähler manifolds [Mar11]. While there are other classes of higher dimensional varieties for which Torelli type theorems hold, the remarkable fact in the case of IHSs is that the cohomology group that plays the crucial role is the second cohomology group and not the middle cohomology one, as one would expect when studying higher dimensional varieties. More generally, much of the geometry of an IHS is encoded in its second cohomology group [Bea83, Huy03, KLSV17].

Another surprising similarity between K3s and IHSs holds at the level of the possible morphisms originating from them. It is not hard to see that if S is a K3 surface, then there are only two types of morphisms with connected fibers having S as a domain: either S maps birationally onto a singular K3, or S fibers over $\mathbb{P}^1$, in which case the general fibers of the morphism are elliptic curves. The first types of morphism are examples of symplectic resolutions, while the second of Lagrangian fibrations. As for morphisms from IHS manifolds, a remarkable theorem of Matsushita [Mat99] states that if M is a $2n$–dimensional HK manifolds, $f : M \to Y$ is a morphism with positive dimensional connected fibers, and Y is not a point, then f is a *Lagrangian fibration*. This means that $\dim Y = n$ and the general fiber is a complex torus, which is Lagrangian with respect to the holomorphic symplectic form. Rephrasing this, we can say that any morphism with connected fibers from an IHS is either a Lagrangian fibration or a symplectic resolution. These two classes of morphisms are crucial tools for both constructing and studying IHSs (cf. for example [DM96, O'G99, O'G03, BM14, ASF15, Saw16, AS18]).

Donagi–Markman [DM96] have given a necessary and sufficient condition on a family of complex tori to carry a holomorphic symplectic form. Using this one finds many examples of open, Lagrangian-fibered IHS manifolds. However, it is in general hard to determine whether or not a given family admits a HK compactification. A method for checking whether fibrations belonging to a certain class admit a HK compactification was introduced in our joint work [ASF15] and developed further in [LSV17]. In this last paper, we solve a longstanding conjecture of Donagi–Markman (see also [MT03, KM09] by constructing a holomorphic symplectic compactification of the intermediation Jacobian fibration of the hyperplane sections of a cubic fourfold. Our compactification gives the first examples of IHS that are fibered in principally polarized abelian varieties that are not Jacobian of curves. We also show that such a compactified fibration is an IHS manifold that is deformation equivalent to OG_{10}.

This result adds to previous work [BD85, LLSvS17, AL17, AT14] where other geometric ties between HK manifolds and cubic fourfolds were established. Theses relations are in line with the connections between K3s and cubic fourfolds that appear at the level of Hodge theory [Has00], derived categories [Kuz10, AT14], and the Grothendieck group of varieties [GS14]. The connection between cubic fourfolds and IHS manifolds constitutes another for studying this class of varieties.

Finally, another way in which IHS can be thought of as higher dimensional K3 surfaces appears when one studies degenerations. Degenerations of K3 surfaces have been extensively studied, and understood thanks to the celebrated work of Kulikov and Persson–Pinkham [Kul77, PP81]. Degenerations of K3s over the unit disk are divided into three cases, depending on the nilpotence index of the monodromy action on H^2. In each of these cases it is understood what the central fiber should look like, once some natural conditions on the total space of the degeneration are imposed. In our recent joint preprint [KLSV17], some of these results are generalized to HK manifolds of arbitrary dimension. Surprisingly, the relevant cohomology group is again the second one and not the middle one. We also prove a generalization of the aforementioned theorem of Huybrechts, which allows us to give many new and quick proofs that certain examples of IHS belong to a given deformation class.

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