LIPSCHITZ GEOMETRIES OF ANALYTIC GERMS

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ABSTRACT. Any germ of a real or complex analytic space in $\mathbb{C}^n$ or $\mathbb{R}^n$ is equipped with two natural metrics: the outer metric induced by the Euclidean metric of the ambient space, and the inner metric, which is the associated Riemannian metric on the germ. The two metrics are natural in the sense that up to bilipschitz equivalence they do not depend on the embedding.

I will describe applications of these metrics, and in particular the implications of when these two metrics are equivalent to each other ("Lipschitz normal embedding" or "LNE"), which has seen a surge of interest involving work of many people, in particular Birbrair, Fernandes, Kerner, Mendes, Pichon, Pedersen, Ruas, Sampaio and others.

In my talk I will speak mostly about work with Anne Pichon and my former student Helge Møller Pedersen, but first give some background.

Let us consider trying to classify all germs of real or complex analytic spaces in $\mathbb{C}^n$ or $\mathbb{R}^n$. Zariski worked into the 1980's refining his approach to manage this uncountable collection of objects. *Zariski equisingularity* creates analytic families of analytic objects with essentially identical properties in order to give more manageable classifications.

Another way to classify comes from a conjecture of Siebenman and Sullivan ([14] 1977) based on metric properties. An analytic germ is equipped with two "natural" metrics: the outer metric induced by the Euclidean metric of the ambient space, and the inner metric, which is the associated Riemannian metric on the germ. Each of the two metrics is "unique" in the sense that up to bilipschitz equivalence it does not depend on the embedding. They conjectured that a classification just in terms of the metric structure would be discrete, and therefore reduce the number of objects from uncountable to countable. This was then proved by Mostowski ([9] 1985) in the complex case and later by Parusiński ([11, 12] 1988 and 1994) in general. Since outer metric determines inner metric, we have a sequence of coarser classifications:

$$\text{Analytic structure} \Rightarrow \text{outer metric} \Rightarrow \text{inner metric} \Rightarrow \text{topology}\,.$$

For complex curve germs, i.e., complex dimension one, classification by Zariski equisingularity is equivalent to classification via outer metric (e.g., Pham-Teissier [13] 1969). In a paper with Pichon, still work in progress, we claim the same for dimension two (an early version restricting to normal complex surface singularities, is on the arXiv). But it is still unknown if it is true in higher dimensions.

Explicit classifications for dimension one (curves) are classical and well understood, starting from the early 1900's with Brauner, Burau, Zariski, later Schubert and others (see also [10]). An explicit classification for normal surface germs, but for the weaker inner metric, is in (Birbrair, N., Pichon [3] 2014).

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As I said in the abstract, there has been a surge of interest concerning "Lipschitz normal embedding" or "LNE", which is defined to mean that the inner and outer metrics are bilipschitz equivalent. This surge was in some part instigated by Asuf Shachar asking (http://mathoverflow.net/questions/222162) if $GL_n^+(\mathbb{R})$ is LNE. This was proved by Katz, Katz, Kerner and Liokumovich ([5]), and then much simplified and broadly extended to various determinantal singularities by Kerner, Møller Pedersen and Ruas ([6]).

However, LNE did not just start then, and part of what I will talk about is a preprint of Anne Pichon, Helge Møller Pedersen and myself ([8]). LNE was first named "normal embedding" by Birbrair and Mostowski ([1] 2000) although the metrics do *not* depend on the embeddings. So we like "LNE".

In general LNE is rare, and the main theorem of the mentioned paper states that rational surface singularities are LNE if and only if they are *minimal singularities*.

Minimal surface singularities play a key role in resolution theory of normal complex surfaces since they appear as central objects in the two main resolution algorithms: the resolution obtained as a finite sequence of normalized Nash transformations ([15]), and the one obtained by a sequence of normalized blow-up of points ([16], [4]). The question of the existence of a duality between these two algorithms, asserted by D. T. Lê in [7, Section 4.3] (see also [4, Section 8]) remains open, and the fact that minimal singularities seem to be the common denominator between them suggests the need of a better understanding of this class of surface germs.

Proving LNE can be difficult. In a very recent paper, Birbrair and Mendes ([2] 2018) give a very general criterion for LNE, by requiring that all "pairs of real arcs" based at the origin have the same bilipschitz inner and outer geometry. To have a general criterion for LNE for normal surface singularities we now start from their theorem and apply various tools to reduce this from uncountably many real arc pairs to a small finite number of pairs to check.

We apply it to prove LNE for minimal singularities. Pichon and Filip Misev are applying it to show LNE for a class of superisolated singularities.

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