REPRESENTATIONS OF MAPPING CLASS GROUPS ASSOCIATED TO SURFACES WITH A GIVEN FINITE AUTOMORPHISM GROUP

A classical theorem of Hurwitz (1886) asserts that for a *very general* Riemann surface of genus $g \geq 2$, the endomorphism ring of its Jacobian is as small as possible, namely $\mathbb{Z}$. Lefschetz showed in 1928 that this is still true if we restrict ourselves to hyperelliptic Riemann surfaces. It turns out that a similar minimality property holds for Riemann surfaces coming with an action of a given finite group: if S is a closed oriented surface S of endowed with a finite group G of orientation preserving diffeomorphisms, then for almost all G-invariant complex structures on S, the resulting Riemann surface C has the property that the natural map $\mathbb{Z}G \to \mathrm{End}(J(C))$ becomes an isomorphism after tensoring with $\mathbb{Q}$, provided that the quotient surface S_G has genus at least 3. The proof (which we will probably not discuss) involves the deformation theory of a canonically embedded smooth projective curve inside a quadric that contains it [1]. By combining this with Deligne's semisimplicity theorem for variations of polarized Hodge structure [3], we obtain some consequences for certain representations of the mapping class group $\mathrm{Mod}(S_G)$ of S_G considered as an orbifold (these are by definition the mapping classes of the underlying surface which take orbifold points to orbifold points of the same type).

To be a bit more specific, the centralizer $\mathrm{Mod}(S)^G$ of G in $\mathrm{Mod}(S)$ maps to a finite index subgroup of $\mathrm{Mod}(S_G)$, with kernel the center $Z(G)$ of G. Thus $\mathrm{Mod}(S)^G$ and $\mathrm{Mod}(S_G)$ have in common a subgroup of finite index, so that the natural representation of $\mathrm{Mod}(S)^G$ on $H_1(S, \mathbb{Q})$ can be thought of as a virtual linear representation of $\mathrm{Mod}(S_G)$. This representation takes its values in the centralizer of G in the symplectic group $\mathrm{Sp}(H_1(S, \mathbb{Q}))$ and respects the isotypical decomposition of $H_1(S, \mathbb{Q})$ into irreducible $\mathbb{Q}G$-modules. We shall see that the above minimality property implies that when S_G has genus ≥ 3:

(i) The subspace of $H_1(S, \mathbb{Q})$ spanned by the finite $\mathrm{Mod}(S_G)$-orbits is a sum of $\mathbb{Q}G$-isotypical summands on which $\mathrm{Mod}(S_G)$ acts through a finite group. This subspace comes with a natural polarized Hodge structure preserved by this action.

(ii) The representation of $\mathrm{Mod}(S_G)$ on any other $\mathbb{Q}G$-isotypical summand is also isotypical for the $\mathrm{Mod}(S_G)$-action. It gives rise to a simple variation of polarized Hodge structure over the moduli space of G-Riemann surfaces of the given topological type.

The polarized Hodge structure appearing in (i) is independent of any complex structure on S that is compatible with the given orientation (so it is "topological"), but we hasten to add that according to conjecture of Putman and Wieland, this space should be trivial. In fact they show that its vanishing is almost equivalent to a conjecture of N.V. Ivanov, which states that the first Betti number of every cofinite subgroup of a mapping class group of genus ≥ 3 is always trivial.

The image of the representation in (ii) lies in an almost-simple $\mathbb{Q}$-group and there is evidence that under perhaps mild additional assumptions, this image is arithmetic.

This represents joint work with Marco Boggi.

References

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