REGULARITY OF INTERFACES IN PHASE TRANSITIONS VIA OBSTACLE PROBLEMS

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The classical Stefan problem aims to describe the temperature distribution in a homogeneous medium undergoing a phase change, for example ice melting to water: this is accomplished by coupling the heat equation in the water with a transmission condition, the so-called Stefan condition, on the evolving boundary between its two phases. Note that this evolving boundary is an unknown (hyper-)surface; hence, Stefan problems are examples of free boundary problems.

In its most classical formulation, the Stefan problem can be reduced to study the so-called parabolic obstacle problem. For simplicity, we focus here on the static case.

The obstacle problem is a classic motivating example in the mathematical study of variational inequalities and free boundary problems. The problem is to find the equilibrium position of an elastic membrane whose boundary is held fixed, and which is constrained to lie above a given obstacle (say a plane) under the action of gravity.

If the graph of $u : \Omega \to \mathbb{R}$ represents the elastic membrane, then u is known to solve the PDE

$$\Delta u = \chi_{\{u>0\}}, \qquad u \geq 0.$$

While the regularity of solutions is well understood, the main issue is to understand the regularity of the *free boundary* $\partial\{u > 0\}$.

As shown by Caffarelli in its seminal papers [1, 2], points of the free boundary $\partial\{u > 0\}$ are divided into two classes: regular points and singular points. A free boundary point $x_\circ$ is either regular or singular depending on the type of behavior of u near that point. More precisely:

(0.1)

$$x_\circ \text{ is called } \textit{regular} \text{ point} \quad \Leftrightarrow \quad r^{-2}u(x_\circ + rx) \xrightarrow{r\downarrow 0} \frac{1}{2}\max\{e\cdot x, 0\}^2$$

for some $e = e_{x_\circ} \in \mathbb{S}^{n-1}$, and

(0.2)

$$x_\circ \text{ is called } \textit{singular} \text{ point} \quad \Leftrightarrow \quad r^{-2}u(x_\circ + rx) \xrightarrow{r\downarrow 0} p_{*,x_\circ}(x) := \frac{1}{2}x\cdot Ax$$

for some symmetric nonnegative definite matrix $A = A_{x_\circ}$ satisfying $\mathrm{tr}(A) = 1$.

It is well know that the free boundary is an analytic hypersurface near regular points. On the other hand, near singular points the *contact set* $\{u = 0\}$ can be pretty wild and form cusps. Moreover, singular points may exhibit Cantor-like structures with infinitely many connected components.

Despite these "negative" results showing that singular points could be rather bad, it is still possible to show some nice structure. More precisely, singular points are naturally stratified according to the dimension of the linear space

$$L_{x_\circ} := \{p_{*,x_\circ} = 0\} = \ker(A_{x_\circ}).$$

For $m \in \{0, 1, 2, \ldots, n-1\}$ we define the m-th stratum as

$$\Sigma_m := \big\{x_\circ \ : \ \text{ singular point with } \dim(L_{x_\circ}) = m\big\}.$$

As shown by Caffarelli in [2], each stratum Σ_m is locally contained in a m-dimensional manifold of class C^1. This result has been first improved in dimension $n = 2$ by Weiss [4], who has been able to prove that Σ_1 is locally contained in a $C^{1,\alpha}$ curve, for some universal exponent $\alpha > 0$. Along the same lines, in a recent paper Colombo, Spolaor, and Velichkov [CSV17] have obtained a logarithmic epiperimetric inequality at singular points in any dimension $n \geq 3$, thus improving the C^1 regularity to a more quantitative $C^{1,\log^\epsilon}$ one.

In the recent paper [3], we prove that, up to the presence of some "anomalous" points of higher codimension, the singular points can be covered by $C^{1,1}$ manifolds. (This result is essentially optimal.) More precisely, for $n = 2, 3$ singular points are contained in a union of $C^{1,1}$ manifolds. In higher dimension $n \geq 4$ we show that the same result holds up to the presence of some "anomalous" points of higher codimension. In addition, we prove that the higher dimensional stratum is always contained in a $C^{1,\alpha_\circ}$ manifold, thus extending to every dimension the result in [4].

For a overview of the classical results, we refer the reader to [2].

References

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