# INTEGRABLE PROBABILITY AND STOCHASTIC PDES

IVAN CORWIN

ABSTRACT. The outcome of coin flipping can be studied by way of binomial coefficients, and their asymptotics reveal the behavior of the broad Gaussian universality class. In this talk we will describe elements of a higher version of this story – coin flipping is replaced by certain stochastic processes including PDEs driven by noise, the binomial coefficients are replaced by symmetric function from representation theory and quantum integrable systems, and the Gaussian is replaced by certain distributions related to random matrix theory and the Kardar-Parisi-Zhang universality class.

This extended abstract provides references associated to the various results and background for the talk that I will deliver.

This talk describes a connection between stochastic partial differential equations—namely the stochastic heat equation (SHE) or Kardar-Parisi-Zhang (KPZ) equation—and certain random point processes coming from random matrix theory—namely the Airy point process. I will discuss the following remarkable identity which connects these two realms: for $u > 0$

$$\mathbb{E}_{\mathrm{SHE}}\left[e^{-z(2t,0)ue^{t/12}}\right] = \mathbb{E}_{\mathrm{Airy}}\left[\prod_{k=1}^{\infty} \frac{1}{1+ue^{t^{1/3}a_k}}\right]. \tag{1}$$

The LHS(1) involves the fundamental solution to the $(1+1)$-dimensional SHE $z(t,x)$ which solves $\partial_t z(t,x) = \frac{1}{2}\partial_x^2 z(t,x) + \xi(t,x)z(t,x)$ with initial data $z(0,x) = \delta(x)$. Here $\xi(t,x)$ is space-time white noise (a Gaussian generalized function with covariance which is delta in space and time) and $\delta(x)$ is the delta function at $x = 0$. The expectation $\mathbb{E}_{\mathrm{SHE}}$ on the LHS is with respect to the noise $\xi$ (of which $z$ is a measurable function). The RHS(1) involves $\{a_1 > a_2 > \cdots\}$ which is a collection of ordered real random variables known as the Airy point process. This point process arises as the limit of the top few eigenvalues for an asymptotically large Hermitian random matrix.

The body of the talk splits into three pieces, each answering one of the following questions:

(1) **What are these objects?** For details regarding the definition of the SHE / KPZ equation, see [14, 16, 26, 27]; for an overview on random matrix theory, including the Airy point process, see the recent books [2, 21].

(2) **Why is this identity true?** The identity in (1) is stated in [8] as an immediate corollary of the formula proved in [1]. It can also be seen as coming from a more general identity [7, 12] which relates stochastic vertex models [17] with Macdonald processes [6]. There is also a half-space SHE variant of this identity given in [4], though only for a particular choice of boundary condition. The full extend and generality for this type of identity is not yet understood. This connection between stochastic vertex models / measures on partitions falls under the study of integrable probability—see the reviews [9–11, 18] as well as some related books [3, 5, 23–25, 28].

(3) **How is this identity useful?** An immediate corollary of the identity is that the long-time fluctuations of the KPZ equation grow like $t^{1/3}$ and are governed by the same fluctuation distribution as the first point $a_1$ in the Airy point process—the GUE Tracy-Widom distribution. This behavior is a hallmark of the "KPZ universality class" and is believed to arise for quite general growth models—see the reviews [14, 16, 22, 26, 27]. A more involved application of the identity is the derivation of precise tail decay bounds [19] as well as the large deviation principle [20] for the KPZ equation.

## References


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I. Corwin, Columbia University, Department of Mathematics, 2990 Broadway, New York, NY 10027, USA

*Email address*: ivan.corwin@gmail.com