Approximate groups, incidence geometry and model theory

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Additive Combinatorics studies finite sets and arithmetic progressions in additive groups. One of its fundamental theorem, going back to G. Freiman in the 60's, is the Green-Ruzsa-Freiman theorem, which provides structural information on large finite subsets A of an additive group whose sumset $A+A$ not much larger than A. Such subsets must be "commensurable" with the sum of a finite additive subgroup and boundedly many arithmetic progressions.

Another landmark of Additive Combinatorics is the sum-product phenomenon, discovered by Erdős and Szemerédi in the 80's : there is an absolute numerical constant $\epsilon_0 > 0$ such that for every finite subset of real numbers A either the sumset $A+A$ or the product AA has size at least $|A|^{1+\epsilon_0}$.

The last decade or so witnessed a renewed interest for these questions and related problems, in part due to the numerous applications the emerging techniques of additive combinatorics have found in harmonic analysis, asymptotic group theory and analytic number theory (see e.g. the books and surveys [6], [4], [1]).

In particular a lot of effort has been spent trying to extend the above results to non-commutative groups (the study of approximate groups) and even to situations where no group is involved.

One example is the structure theorem of Breuillard-Green-Tao [3], which extends Freiman's theorem to non-commutative groups, showing that the lack of growth under products of a finite subset of an arbitrary group is entirely explained by a finite nilpotent structure related to the subset.

Another is the Elekes-Szabó theorem, which studies finite sets of numbers with many tuples satisfying a given algebraic constraint. The sum-product phenomenon can be reformulated in this way by counting the quadruples (x, y, z, w) of numbers from A satisfying the constraints $z = x+y$ and $w = xy$.

While most of the work in Additive Combinatorics is usually done via elementary methods, the increased complexity of the problems at hand together with the quest for a greater generality have led people to borrow tools and concepts from Model Theory. Some years ago Hrushovski pioneered a conceptual model-theoretical framework [5] in which to recast a large part of Additive Combinatorics. This has brought at the same time new tools and new insights, which have turned out to be very successful ; they were key in particular to the above mentioned structure theorem for approximate groups [7].

In this talk I will begin with an introduction to the above context, then focus on a recent joint work with Martin Bays [2] in which we use non-standard analysis and the above-mentioned model-theoretical framework, along with some recent advances in incidence geometry (Szemerédi-Trotter type bounds), to revisit and extend the Elekes-Szabó theorem. We classify all possible algebraic constraints in any dimension and arity, showing that they must always be related, via finite-to-finite algebraic correspondences to commutative algebraic groups of a certain form.

References

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