Les Diablerets

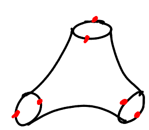
2016.08.27

Lecture IV

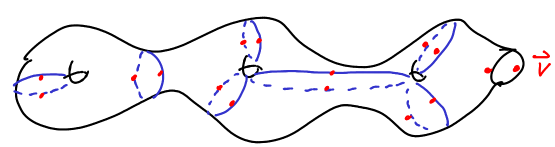
The Genus 1 Story

Last lecture:

S = topological surface of genus g with 1 boundary component obtained by glueing together "tripods"



$= \widehat{Bl}_{\{0,1,\infty\}} \mathbb{P}^1$



an "indexed pants decomposition"

Then $\mathfrak{p} := \mathrm{Lie}\, \pi_1^{un}(S, \vec{v})$ is an object of MTM which is determined by a homomorphism

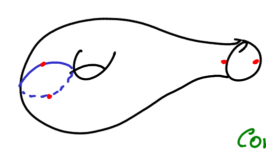
the "arithmetic monodromy"

$$\varphi_P : \underline{k} = \mathbb{L}(\sigma_3, \sigma_5, \ldots) \to \mathrm{Der}^\theta \mathbb{L}(H)^\wedge$$

where $H = H_1(S)$. It is important to remember that φ_P depends upon the indexed pants decomposition P of S.

The goal of this lecture is to describe what we know about this homomorphism when $g = 1$ and related results.

Remark: In genus 1 the pants decomposition is unique. There are only 2 possible indexings.



The indexings differ by a half-twist. Correspond to $\pm \partial/\partial q$

The work in genus 1 is joint with

(1) Makoto Matsumoto: MEM paper

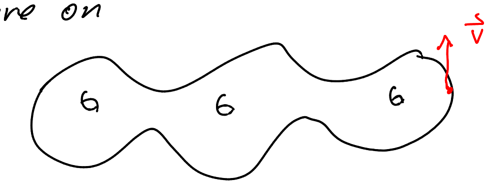
(2) Francis Brown: current/continuing

Before explaining the $g=1$ case, we'll make a few more general comments.

Geometric Monodromy

Recall that $\mathfrak{g}_{g,\vec{1}}$ is the Lie algebra of the $Sp(H)$ completion of the mapping class group $\Gamma_{g,\vec{1}}$ of a surface of type $(g,\vec{1})$, where $g \geq 1$.

THEOREM: For each choice of a complex structure on



There is a canonical MHS on $\mathfrak{g}_{g,\vec{1}}$. The action of $\Gamma_{g,\vec{1}}$ on S induces a homom

$$\mathfrak{g}_{g,1} \longrightarrow \mathrm{Der}\, \mathrm{Lie}\, \pi_1^{un}(S,\vec{v})$$

which is a morphism of MHS. We call it the "geometric monodromy".

Remark: When S is an "Ihara curve" there two weight filtrations $W_\bullet$ and $M_\bullet$ on $\mathfrak{g}_{g,1}$ and $\mathrm{Lie}\, \pi_1^{un}(S,\vec{v})$. I will explain these in the elliptic case below.

already seen this for $p(C,\vec{v})$, C Ihara curve

"relative weight filtration"

Theorem: If S is an Ihara curve, then

$$\mathrm{im}\, \overset{\text{arithmetic}}{\mathfrak{k}} \cap \mathrm{im}\, \overset{\text{geometric}}{\mathfrak{g}_{g,\vec{1}}} \text{ in } \mathrm{Der}\, \mathfrak{p}.$$

(The proof combines the constructions of the previous lecture with Brown's injectivity theorem.)

This statement implies the unipotent Oda conjecture, which was proved in the 90's by various combinations of Ihara, Takao, Nakamura and Matsumoto.

<u>Genus 1</u>: The group $\Gamma_{1,\vec{1}}$ is a central extension

$$0 \to \mathbb{Z} \longrightarrow \Gamma_{1,\vec{1}} \longrightarrow SL_2(\mathbb{Z}) \to 1$$

of $\Gamma_{1,1} = SL_2(\mathbb{Z})$. It is isomorphic to B_3, the group of 3-string braids, and also to the fundamenl group of the complement of the trefoil knot in S^3.

Recall that $\mathfrak{u}_{g,n+\vec{r}}$ is the Lie algebra of the prounipotent radical of $\mathcal{G}_{g,n+\vec{r}}$. One can show that there is a central extension (as Lie algebras with MHS)

$$0 \to \mathbb{Q}(1) \to \mathfrak{u}_{1,\vec{1}} \to \mathfrak{u}_{1,1} \to 0.$$

We will compute $\mathfrak{u}_{1,1}$. First:

(1) Recall that the irreducible representations of $SL(H) \cong SL_2$ are the symmetric powers $S^m H$ of $H := H_1(E;\mathbb{Q})$.

(2) Recall that each homology (resp. cohomology) group of $\mathfrak{u}_{1,1}$ is a pro- (resp. ind-) representation of $SL(H)$.

General results about relative completion imply that

$$\mathrm{Hom}\left(H_1(\mathfrak{u}_{1,1}), S^m H\right) \cong H^1(SL_2(\mathbb{Z}), S^m H).$$

So

$$H_1(\mathfrak{u}_{1,1}) = \prod_{m \geq 0} H^1(SL_2(\mathbb{Z}), S^m H)^* \otimes S^m H.$$

SL_2 acts here

We now recall some facts about $H^\bullet(SL_2(\mathbb{Z}), S^m H)$:

① "center-kills" implies that these groups vanish when m is odd;

② Since $SL_2(\mathbb{Z})$ is "virtually free", these groups vanish when $j \geq 2$;

③ The standard presentation of $SL_2(\mathbb{Z})$ implies that $H^1(SL_2(\mathbb{Z}); \mathbb{Q}) = 0$.

For the rest, we need:

Eichler–Shimura & Manin–Drinfeld (+ Zucker)

(1) If $n>0$, then there is an exact sequence

$$0 \to H^1_{cusp}(SL_2(\mathbb{Z}), S^{2n}H) \to H^1(SL_2(\mathbb{Z}), S^{2n}H) \to \frac{S^{2n}H}{N\cdot S^{2n}H} \to 0$$

kernel of the restriction map $H^1(SL_2(\mathbb{Z}), S^{2n}H) \to H^1\left(\begin{pmatrix}1 & \mathbb{Z}\\ 0 & 1\end{pmatrix}, S^{2n}H\right)$

$N = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = \log\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$

(2) $H^1_{cusp}(SL_2(\mathbb{Z}), S^{2n}H)$ has a pure Hodge structure of weight $2n+1$ and type $(2n+1, 0)$, $(0, 2n+1)$.

(3) $S^{2n}H/N\cdot S^{2n}H \cong \mathbb{Q}(-2n-1)$ Hodge structure of type $(2n+1, 2n+1)$

(4) $H^1(SL_2(\mathbb{Z}), S^{2n}H) = H^1(\mathcal{M}_{1,1}, S^{2n}H)$ has a natural MHS and the sequence above is a sequence of MHS.

(5) The sequence splits as a MHS. so $H^1(SL_2(\mathbb{Z}), S^{2n}H)$ is in MHS^{ss}

(6) $F^{2n+1} H^1_{cusp}(SL_2(\mathbb{Z}), S^{2n}H)$

= holomorphic cusp forms of $SL_2(\mathbb{Z})$ of weight $2n+2$.

dang! That makes at least 3 different notions of weight:
- Hodge
- representation
- modular

(7) $S^{2n}H/N S^{2n}H = \mathbb{Q}\, G_{2n+2}$ $n \geq 1$

The Eisenstein series of wt $2n+2$.

on modular forms

Digression: Set $\mathfrak{H} = \{\tau \in \mathbb{C} : \operatorname{Im} \tau > 0\}$.

A *modular form* of weight m of $SL_2(\mathbb{Z})$ is a holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ such that

necessarily even

(1) $$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^m f(\tau).$$

all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

So $f(\tau+1) = f(\tau) \ldots$

$$\Rightarrow \quad f = \sum_{-\infty}^{\infty} a_n q^n \qquad q = e^{2\pi i}$$

(2) f is holomorphic at $q=0$.

(i.e. $a_n = 0$, $n < 0$)

A *cusp form* of weight m is a modular form of weight m that vanishes at $q=0$.

The first non-trivial cusp form is the Ramanujan τ-function

$$\Delta(q) = q \prod_{n \geq 1} (1-q^n)^{24}$$

The *normalized* Eisenstein series of weight $2k$ ($k \geq 2$)

$$G_{2k}(\tau) = \frac{1}{2} \frac{(2k-1)!}{(2\pi i)^{2k}} \sum_{\substack{\lambda \in \mathbb{Z} \oplus \mathbb{Z}\tau \\ \lambda \neq 0}} \frac{1}{\lambda^{2k}}$$

$\sigma_m(n) = \sum_{d \mid n} d^m.$ $\qquad q = e^{2\pi i \tau}$

$$= -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n .$$

The ring of modular forms is the graded polynomial ring $\mathbb{Q}[G_4, G_6]$. Cusp forms comprise the ideal (Δ). Their Poincaré series are

$$\frac{1}{(1-t^4)(1-t^6)} \quad \text{and} \quad \frac{t^{12}}{(1-t^4)(1-t^6)} \quad \text{respectively.}$$

There is a natural basis of the cusp forms of weight $2k$:

i.e.: coefficient of q is 1

$$\mathcal{B}_{2k} = \{\text{normalized Hecke eigenforms } f \text{ of weight } 2k\}$$

Basis of modular forms of weight $2k$ is $\mathcal{B}_{2k} \cup \{G_{2k}\}$.

Each $f \in \mathfrak{B}_{2k+2}$ determines a (real) 2-dimensional Hodge substructure V_f of $H^1(SL_2(\mathbb{Z}), S^{2k}H)$. :

$$V_f = V_f^{2k+1,0} \oplus V_f^{0,2k+1}$$

Write the dual Hodge structure

$$\check{V}_f = \mathrm{Hom}(V_f, \mathbb{R})$$ or better, $\mathbb{Q}_f$ the field gen by the Fourier coeffs of f.

as

$$\check{V}_f = \mathbb{R}e_f' \oplus \mathbb{R}e_f''$$

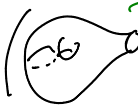
where

e_f' has type $(-2k+1, 0)$

e_f'' " " $(0, -(2k+1))$

The Hodge structure associated to the Eisenstein series G_{2k} is $\mathbb{Q}(-2k-1)$, which is of type $(2k+1, 2k+1)$. The dual Hodge structure is $\mathbb{Q}(2k+1)$. Denote the corresponding $\mathbb{Q}$-DR generator by e_{2n}.

Comments on notation : First order Tate curve

① $H = H_1\Big($  $\Big)$ has a natural basis :

Hodge wt -2 $\rightarrow$ A = class of the vanishing cycle — it spans $\mathbb{Q}(1) \subseteq H$

Hodge wt 0 $\rightarrow$ T = class of loop that generates $\mathbb{Q}(0) \subseteq H$

So $\mathfrak{sl}(H)$ has a natural torus : it gives T weight 1 and A weight -1. Set

This is the monodromy $\log \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\rightarrow$ $e_0 = A\, \partial/\partial T$ ← $\mathfrak{sl}(H)$ - weight -2

$\check{e}_0 = T\, \partial/\partial A$ ← $\mathfrak{sl}(H)$ weight 2

$$\boxed{\mathfrak{sl}(H) = \mathbb{Q}\check{e}_0 \oplus \text{Cartan} \oplus \mathbb{Q}e_0}$$

② The Lie algebra $\underline{u}_{1,1}$ contains a countable number of copies of

$S^{2n}H$, even many of the same W-weight

To specify them, we denote by

$S^m(v)$ the $\mathfrak{sl}(H)$-module with

a highest weight vector v of $\mathfrak{sl}(H)$

weight m.

$$S^m(v) = \mathbb{Q}v \oplus \mathbb{Q}\, e_0 \cdot v \oplus \cdots \oplus \mathbb{Q}\, e_0^m \cdot v$$

where $\quad e_0^j \cdot v = \mathrm{ad}_{e_0}^j(v)$.

———— ∘ ————

We can now write down $\underline{u}_{1,1}$: it is isomorphic to the completion of the free Lie algebra generated by the $\mathfrak{sl}(H)$-module

$$\bigoplus_{n\geq 1}\Big[\bigoplus_{f\in B_{2n+2}} \underbrace{\big(S^{2n}(e_f') \oplus S^{2n}(e_f'')\big)}_{W\text{-wt} = -1} \oplus \underbrace{S^{2n}(e_{2n+2})}_{W\text{-wt} = -2n-2}\Big]$$

This is a semi-simple MHS.

So $\quad \mathfrak{g}_{1,1} \cong \mathfrak{sl}(H) \ltimes \underline{u}_{1,1}$.

For $(1,\vec{1})$ we have

$$\underline{u}_{1,\vec{1}} = \underline{u}_{1,1} \oplus \mathbb{Q}\,\underline{e}_2$$

corresponds to G_2

central in $\underline{u}_{1,\vec{1}}$

generates trivial $\mathfrak{sl}(H)$-module

weight -2, type $(-1,-1)$

$$\mathfrak{g}_{1,\vec{1}} = \mathfrak{sl}(H) \ltimes \underline{u}_{1,\vec{1}}$$

———— ∘ ————

<u>What about the relations?</u>

The monodromy homomorphism

$$\mathfrak{g}_{1,\vec{1}} \longrightarrow \mathrm{Der}\, \mathfrak{p}$$

is a morphism of MHS. An important observation is that the Hodge structure on every $\mathfrak{sl}(H)$-invariant summand of $\mathrm{Gr}_\bullet^W \mathfrak{p}$, and so also $\mathrm{Gr}_\bullet^W \mathrm{Der}\, \mathfrak{p}$, is of the form

$$S^m H(r) := S^m H \otimes \mathbb{Q}(r).$$

Since $H = \mathbb{Q}(0) \oplus \mathbb{Q}(1)$, it follows that $\mathfrak{p}$ and $\operatorname{Der}\mathfrak{p}$ are mixed Tate.

<u>Rk:</u> We already knew this as both are in MTM.

The exactness of $\operatorname{Gr}^W_\bullet$ on MHS implies that

$$\operatorname{Gr}^W_\bullet \ker\{ \underline{u}_{1,\vec{1}} \to \operatorname{Der}\mathfrak{p} \}$$

$$= \ker\{ \operatorname{Gr}^W_\bullet \underline{u}_{1,\vec{1}} \to \operatorname{Der} \operatorname{Gr}^W_\bullet \mathfrak{p} \}$$

$$= \ker\{ \operatorname{Gr}^W_\bullet \underline{u}_{1,\vec{1}} \to \operatorname{Der} \mathbb{L}(H) \}$$

But

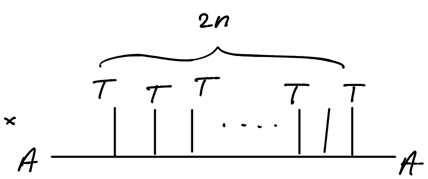
$$\operatorname{Gr}^W_{-1} \underline{u}_{1,\vec{1}} \cong \bigoplus_{n \geq 1} S^{2n} H \otimes \check{V}_f$$

not Tate!

So all "cuspidal generators" $\varepsilon'_f, \varepsilon''_f$ must go to 0 in $\operatorname{Der}\mathbb{L}(H)$.

The elliptic KZB equation implies that the "Eisenstein generator" ε_{2n} goes to a non-zero rational multiple of the derivation

$$\varepsilon_{2n} := \frac{1}{2} \times$$



<u>Rk:</u> This is the unique element of $\operatorname{Der}^0 \mathbb{L}(A,T)$ satisfying

$$\varepsilon_{2n}(T) = -T^{2n+1} \cdot A$$

<u>Question:</u> Why cannot the image of $\operatorname{Gr}^W_\bullet \underline{u}_{1,\vec{1}} \to \operatorname{Der}\mathbb{L}(H)$ be $\mathbb{L}\left(\bigoplus_n S^{2n}(\varepsilon_{2n+2})\right)$?

<u>Answer:</u> Periods!

Brown's computation of periods of twice iterated integrals of Eisenstein series implies that there are many non-trivial extensions of MHS in $\underline{u}_{1,1}$. These force relations between the ε_{2n}. In fact, every $f \in \mathcal{B}_{2n+2}$ gives a countable set of relations between the ε_{2n}.

The purely quadratic relations between the ε_{2n} were found by Aaron Pollack. He also found relations of every "degree" between $\{\varepsilon_{2n} : n \geq 0\}$ mod depth 3. These relations lift to $\operatorname{Der} \mathbb{L}(H)$ (Hain–Matsumoto, Baumard–Schneps $d=3$).

<u>Explanation</u> <u>of</u> <u>the</u> <u>first</u> <u>Pollack</u> <u>relation</u>

$$\Delta = q \prod_{n \geq 1} (1-q^n)^{24} = \text{Ramanujan } \tau\text{-function}$$

weight 11

$$V_\Delta = V^{11,0} \oplus V^{0,11} = H^1_{cusp}(SL_2(\mathbb{Z}), S^{10}H)$$

	S^0	S^2	S^4	S^6	S^8	S^{10}
-1						$S^{10} \otimes \check{V}_\Delta$
-2	e_2					
-4		$S^2(e_4)$				
-6			$S^4(e_6)$			
-8		$\Lambda^2 S^2(e_4)$	0	$S^6(e_8)$		
-10		*	*	$S^6[e_4, e_6]$	$S^8(e_{10})$	
-12		*	*	*	*	$S^{10}(e_{12})$
-14		*	*	*	*	$\left\{ \begin{array}{l} S^{10}[e_4, e_6] \\ \oplus\ S^{10}[e_6, e_8] \end{array} \right.$
-16		*	*	*	*	*

central

first possibly non-trivial extn S^{10} column.

this extension vanishes by Manin–Drinfeld

Extension

$$0 \to \begin{array}{c} S^{10}([e_4, e_{10}]) \\ \oplus \\ S^{10}([e_6, e_8]) \end{array} \to E \to S^{10} \otimes \check{V}_\Delta \to 0$$

W weight $\quad -14 \qquad\qquad -1$

Take $\mathfrak{e}_0$ coinvariants we get an

ie: take highest weight vectors

extension $\mathbb{Q}(7)^2$ as weight -14

$$0 \to \mathbb{Q}[e_4, e_{10}] \oplus \mathbb{Q}[e_6, e_8] \to E' \to \check{V}_\Delta(-5) \to 0$$

Rearrange to write this as a sum of 2 extensions of the form

$$0 \to V_\Delta(12) \to M \to \mathbb{Q}(0) \to 0$$

Brown's computation of the periods of twice iterated integrals of Eisenstein series

$$\int [G_4 | G_{10}] \quad + \quad \int [G_6 | G_8]$$

imply that these two extensions are proportional:

$$14\,[e_4, e_{10}] - 9\,[e_6, e_8] = 0$$

in the "maximal Tate quotient" of $\mathfrak{u}_{1,\vec{1}}$.

Conclusion: First Pollack relation:

$e_{2n} \longmapsto \frac{(2n-2)!}{2} \cdot \varepsilon_{2n}$ under $\tilde{\pi}_{1,\vec{1}} \to \mathrm{Der}\, \mathbb{L}(H)$

$$[\varepsilon_4, \varepsilon_{10}] \pm 3^{\pm} [\varepsilon_6, \varepsilon_8] = 0$$

The general situation is more complicated and best expressed in terms of Deligne–Beilinson cohomology.

Eisenstein quotients

The image of $\mathfrak{g}_{1,\vec{1}} \to \mathrm{Der}\, \mathfrak{p}$ is a $\pi_1(\mathrm{MTM})$ module, and therefore a MTM. Let

$$\mathfrak{g}^{\mathrm{eis}}_{1,\vec{1}} = \text{maximal quotient of } \mathfrak{g}_{1,1} \text{ in MHS that is an MTM:}$$

kills all non Tate MHS in $\mathfrak{g}_{1,\vec{1}}$

injective

$$\mathfrak{g}_{1,\vec{1}} \twoheadrightarrow \mathfrak{g}^{\mathrm{eis}}_{1,\vec{1}} \longrightarrow \mathrm{Der}\, \mathfrak{p}.$$

Lots of relations

Let $\underline{u}^{eis}_{1,\vec{1}}$ be the image of $\mathfrak{g}_{1,\vec{1}}$ in $\mathfrak{g}^{eis}_{1,\vec{1}}$. Note:

$$u^{eis}_{1,\vec{1}} = \mathbb{L}\Big(\bigoplus_{n\geq 0} S^{2n}(e_{2n+2})\Big)^{\wedge} \Big/ \begin{matrix}\text{relations}\\ \text{stable by}\\ SL_2\end{matrix}$$

Let

$$\hat{\mathfrak{g}}_{1,\vec{1}} = \operatorname{Lie} \pi_1(MTM) \ltimes \mathfrak{g}^{eis}_{1,\vec{1}}$$

$$\hat{u}_{1,\vec{1}} = \underline{k} \ltimes \underline{u}^{eis}_{1,\vec{1}}$$

<u>THM</u> (Hain-Matsumoto) This has a <u>canonical</u> bigrading that splits $F^{\bullet}$, $M_{\bullet}$ and $W_{\bullet}$

<u>Presentation</u> (of associated bigraded) of $\hat{\underline{u}}_{1,\vec{1}}$

$$0 \to \underline{u}^{eis}_{1,\vec{1}} \to \hat{\underline{u}}^{eis}_{1,1} \xrightarrow{\curvearrowleft\ \text{action}} \underline{k} \to 0$$

$$z_{2m-1} \longmapsto \sigma_{2m-1}$$

$$\sigma_{2m-1} \equiv z_{2m-1} \bmod \underline{u}^{eis}_{1,\vec{1}}.$$

each fixed by $sl(H)$

$$\hat{\underline{u}}_{1,\vec{1}} = \mathbb{L}\big(z_3, z_5, z_7, \ldots;\ \varepsilon_0^j \cdot e_{2n} : n\geq 1,\ 0\leq j\leq 2n-2\big)^{\wedge} \Big/ \begin{matrix}\text{- geometric relations}\\ \text{- arithmetic relns}\end{matrix}$$

relations between the $\varepsilon_0^j \cdot e_{2n}$ — come from cusp forms

come from Eisenstein series

describe how the z_{2m-1} act on the e_{2n}.

$e_{2n}, \ldots, \varepsilon_0^{2n-2}\cdot e_{2n}$ span $S^{2n-2}(e_{2n})$

<u>What we know:</u>

1) The quadratic heads of lifts of all Pollack relations

2) The quadratic heads of all arithmetic relations

3) There relations are all linearly independent in the set of $sl(H)$ highest weight vectors in $H_2(\hat{\underline{u}}_{1,\vec{1}})$.

What we don't know

(*) whether there are any more relations. That is, do the known relations span the set of highest weight vectors in $H_2(\hat{\mathfrak{u}}_{1,\vec{1}})$.

THERE IS A HUGE / VERY RICH STRUCTURE IN THE ELLIPTIC CASE.