Les Diablerets 2016.08.27

Lecture III

Motivic Structures on Completions of Path Torsors of Curves

Completions of Path torsors

Let X be a topological space. Just as we can take the unipotent completion of

$$ {}_x\Pi_x(X) = \pi_1(X, x) $$

we can take the unipotent completion of ${}_x\Pi_y(X)$:

Def: A local system (i.e. a locally constant sheaf) of F-vector spaces $\mathbb{V}$ over X is unipotent if

(1) $\operatorname{rank}_F \mathbb{V} < \infty$ ← fiber dimension

(2) $\mathbb{V}$ has a filtration

$$ 0 = \mathbb{V}_0 \subseteq \mathbb{V}_1 \subseteq \dots \subseteq \mathbb{V}_N = \mathbb{V} $$

by local systems where each $\mathbb{V}_j / \mathbb{V}_{j-1}$ is a trivial local system.

These form a category $\mathcal{L}_X^{un}$ which is tannakian. To a point $x \in X$ we can associate the fiber functor

$$ \omega_x : \mathcal{L}_X^{un} \longrightarrow \mathrm{Vec}_F $$

which takes a local system $\mathbb{V}$ to its fiber V_x over x.

a.k.a. the stalk of $\mathbb{V}$ at x.

The unipotent completion of ${}_x\Pi_y(X)$

is the affine F-scheme

$$ {}_x\Pi_y^{un}(X) = \mathrm{Isom}^{\otimes}(\omega_x, \omega_y) $$

natural isoms from ω_x to ω_y which preserve $\otimes$.

<u>NB:</u> $\quad {}_x\Pi_x^{un}(X) = \pi_1^{un}(X,x).$

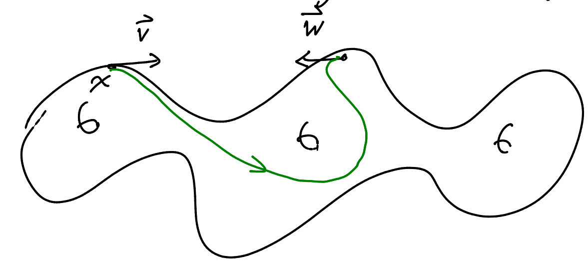
<u>Tangential base points</u>

Algebraic / arithmetic geometry does not like boundary components. Instead (following Deligne) we use "tangential base points".

Suppose that C is a complex curve, $x, y \in C$ and

$$ \vec{v} \in T_x C, \quad \vec{w} \in T_y C $$

are non-zero tangent vectors.



Set $\quad C' = C - \{x,y\}$.

$$ {}_{\vec{v}}\Pi_{\vec{w}}(C') = \{ \text{paths } \gamma \text{ in } C \text{ leaving } x \text{ with velocity vector } \vec{v} \text{ and arriving at } y \text{ with tangent vector } -\vec{w} \text{ s.t. } \gamma(t) \in C' \ \ 0<t<1 \} $$

mod homotopies preserving the conditions.

<u>Alternative (better) description</u>

(i) The real oriented blowup of C at $p \in C$, denoted $\widehat{Bl}_p C$ is the surface with boundary obtained by replacing p by the circle

$$\left(T_p C - \{0\}\right) / \mathbb{R}_+$$

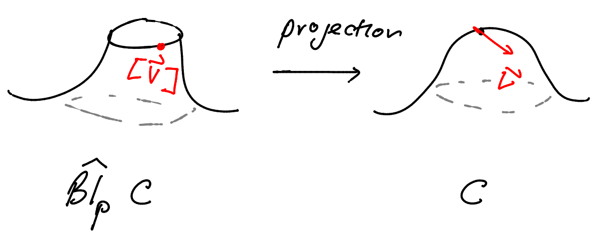
of oriented rays in $T_p C$.

Local computation: $\mathbb{D}$ = unit disk

$$\widehat{\mathbb{D}} = \widehat{Bl}_0 \mathbb{D} = [0,1) \times S^1.$$

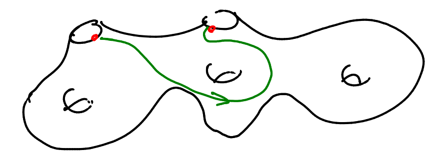
The projection $\widehat{\mathbb{D}} \to \mathbb{D}$ takes $(r,\theta) \in \widehat{\mathbb{D}}$ to $re^{i\theta}$.

Each non-zero $\vec{v} \in T_p C$ determines a point $[\vec{v}] \in \widehat{Bl}_p C$.



Equivalent definition:

$$_{\vec{v}}\Pi_{\vec{w}}(C') := {}_{[\vec{v}]}\Pi_{[\vec{w}]} \widehat{Bl}_{\{x,y\}} C$$

$\widehat{Bl}_{\{x,y\}} C =$ 

<u>Central</u> <u>Example</u>:

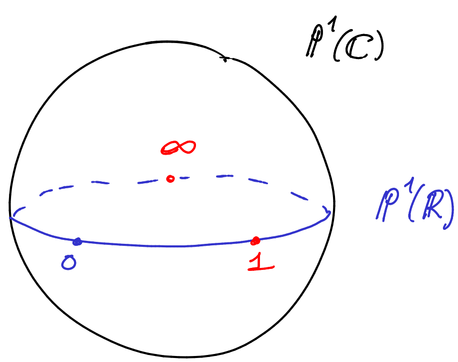
$C = \mathbb{P}^1$

- special points $0, 1, \infty$
- special tangent vectors

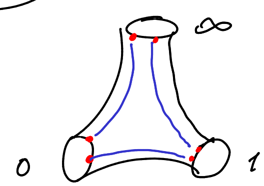
$\pm \partial/\partial x \in T_0 \mathbb{P}^1$, $\pm \partial/\partial x \in T_1 \mathbb{P}^1$

(where x is the coordinate on $\mathbb{P}^1$ that takes the value 0 at 0, 1 at 1 and ∞ at ∞)

and $\pm \partial/\partial u \in T_\infty \mathbb{P}^1$, where $u = 1/x$.



$\widehat{Bl}_{\{0,1,\infty\}}\, \mathbb{P}^1 =$



Hodge Theory:

Suppose that:

(1) C is a compact Riemann surface,

(2) $C' = C - S$, where $S \subseteq \bar{C}$ is finite.

(3) $x, y \in C'$, $p, q \in S$,

$\vec{v} \in T_p C$, $\vec{w} \in T_q C$

both non zero

Caution: The MHS depends on the length of the $\vec{v}$

Theorem: If

$a, b, c \in \{x, y, \vec{v}, \vec{w}\}$

Then

(1) $\mathcal{O}({}_a\Pi_b^{un} C')$ is an algebra in ind-MHS

(2) The coproducts

$$\mathcal{O}({}_a\Pi_b^{un} C') \to \mathcal{O}({}_a\Pi_c^{un} C') \otimes \mathcal{O}({}_c\Pi_b^{un} C')$$

dual to path multiplication

$${}_a\Pi_c^{un} \times {}_c\Pi_b^{un} \longrightarrow {}_a\Pi_c^{un}$$

are morphisms of MHS. $\square$

Ex: ${}_{\vec{v}_0}\Pi_{\vec{v}_1}^{un}(\mathbb{P}^1 - \{0,1,\infty\})$ contains associators

Glueing

Suppose that we have

(1) a finite set $C_1, \ldots, C_N$ of compact Riemann surfaces

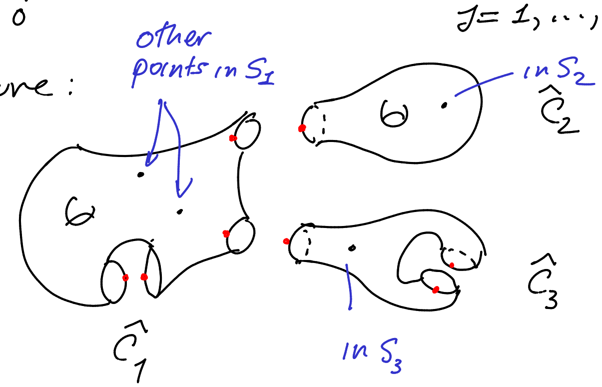
(2) finite subsets S_j of C_j

(3) non-zero tangent vectors

$\underset{\neq 0}{\vec{v}_p} \in T_p C_j$ for some $p \in S_j$.

$j = 1, \dots, N$

Picture:



$$\hat{C}_j = \widehat{Bl}_{\{p : \vec{v}_p \neq 0\}} C_j$$

Now suppose have a pairing of some (not necessarily all) boundary components of

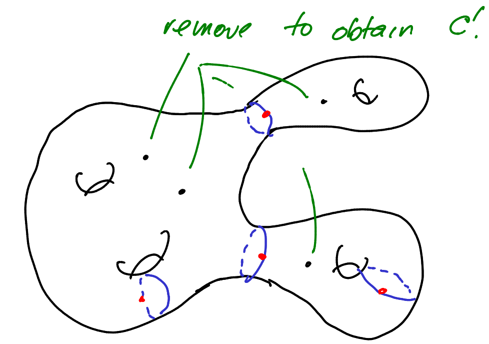
$$\hat{C}_1 \cup \dots \cup \hat{C}_N$$

Glue so that the corresponding tangent vectors line up to obtain a surface C.

↑ red dots

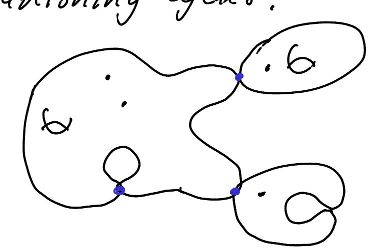
Eg:

$C =$



Rk: This can be regarded as a topological model of the nodal curve C_0 obtained by collapsing all of the blue "vanishing cycles".

C_0



Theorem If

$a, b, c \in C' - \{\text{vanishing cycles}\}$

possibly tangent vectors

then

(1) $\mathcal{O}({}_a\Pi_b^{un}\, C)$ has a natural ind-MHS "compatible with the MHSs on the $\mathcal{O}({}_s\Pi_t^{un}\, C_j')$"; product is a morphism

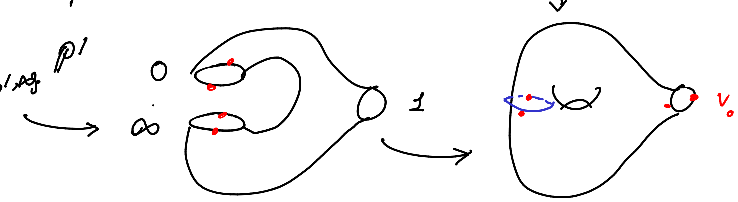
(2) The coproduct

$$\mathcal{O}({}_a\Pi_b^{un}\, C') \to \mathcal{O}({}_a\Pi_c^{un}\, C') \otimes \mathcal{O}({}_c\Pi_b^{un}\, C')$$

is a morphism MHS.

Example: "Tate curve" E'_{Tate}

$\hat{\mathbb{P}}^1 = \widehat{Bl}_{\{0,1,\infty\}}\, \mathbb{P}^1$



So $\mathcal{O}(\pi_1^{un}(E'_{Tate}, \vec{v}))$ has a natural ind-MHS.

Mixed Tate Motives (quick and dirty)

Conjecturally, the category of mixed motives (whatever they are) over (say) $\mathbb{Z}$ is tannakian with a forgetful functor to MHS (conj to be fully faithful). We are a long way from realizing that dream. But there is one case where it has been realized; that of "mixed Tate motives."

Voevodsky + Levine
|
Levine
|
Deligne-Goncharov.

Let $\quad MTM$ = tannakian category of mixed Tate motives unramified $/\mathbb{Z}$.

Know (Levine, Deligne–Goncharov) that its fundamental group is an extension

DR splitting

$$1 \to K \to \pi_1(MTM, \omega^{DR}) \to \mathbb{G}_m \to 1$$

where K is unipotent. This "de Rham" fundamental group is canonically split. So the DR realization of every MTM is canonically graded. In particular, the Lie algebra $\underline{k}$ of K is graded.

$$\underline{k} \cong \mathbb{L}(\sigma_3, \sigma_5, \sigma_7, \sigma_9, \dots)^{\wedge}$$

where the $\mathbb{G}_m$ acts on σ_{2n+1} by

$$t : \sigma_{2n+1} \longmapsto t^{-(2n+1)} \sigma_{2n+1}$$

<u>Rk:</u> So this extension is negatively weighted.

The σ_{2n+1} are <u>not</u> canonical.

Eg: you can change σ_{11} to

$$\sigma_{11} + c\,[\sigma_3[\sigma_3, \sigma_5]] \qquad c \in \mathbb{Q}.$$

The simple objects of MTM are the objects $\mathbb{Q}(n)$, $n \in \mathbb{Z}$, where

$$\mathbb{Q}(n)^{DR} = \mathbb{Q}$$

on which $\pi_1(MTM, DR)$ acts via:

Tate weight is $-n$

$$\pi_1(MTM, DR) \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{Q}^\times = Aut(\mathbb{Q})$$
$$t \longmapsto t^{-n}$$

The forgetful functor $MTM \to MHS$ takes $\mathbb{Q}(n) \in MTM$ to the 1-dim Hodge structure (also denoted $\mathbb{Q}(n)$) with

$$V_{\mathbb{C}} = V^{-n,-n}.$$

This has Hodge weight $-2n$.

- Deligne and Goncharov proved that if $\underline{v}, \underline{w}$ are any 2 of the 6 canonical tangent vectors of $\mathbb{P}^1$, Then

$$\mathcal{O}\left({}_{\vec{v}}\Pi^{un}_{\vec{w}}(\mathbb{P}^1 - \{0, 1, \infty\})\right)$$

is in MTM.

- Brown proved that $\pi_1(MTM)$ acts faithfully on each of these. That is, the action

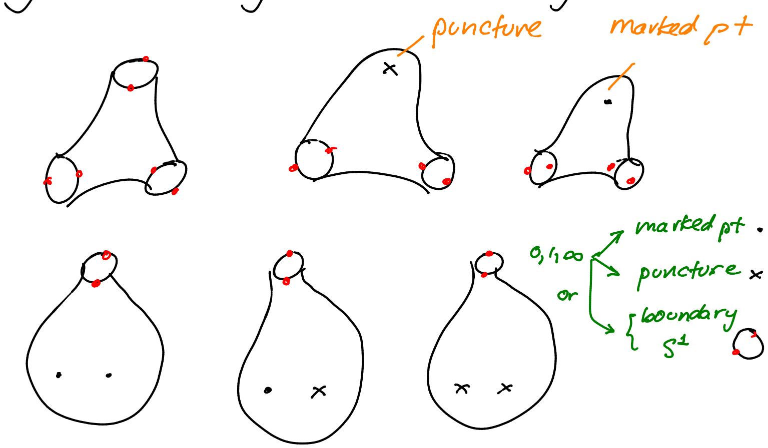
$$\pi_1(MHS, \omega^B) \longrightarrow \operatorname{Aut} \mathcal{O}\left({}_{\underline{v}}\Pi^{un}_{\underline{w}}\, \mathbb{P}^1 - \{0,1,\infty\}\right)$$

factors through $\pi_1(MTM, \omega^B)$:

$$\begin{tikzcd}
\pi_1(MHS, \omega^B) \arrow[r] \arrow[dr] & \operatorname{Aut} \mathcal{O}\left({}_{\underline{v}}\Pi^{un}_{\underline{w}}\, \mathbb{P}^1 - \{0,1,\infty\}\right) \\
& \pi_1(MTM, \omega^B) \arrow[u, hook]
\end{tikzcd}$$

Inspired by a paper of Ihara & Nakamura.

<u>Ihara</u> <u>Curves</u>:
These are the possibly pointed, punctured curves obtained by assembling the "building blocks"



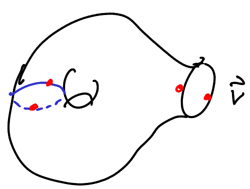
<u>Theorem</u>. Let C be an Ihara curve. If a, b are distinguished points or tangent vectors, then

$$\mathcal{O}({}_a\Pi_b\, C) \in MTM$$

The products and coproducts are

morphisms in MTM. $\square$

Example: First order Tate curve E'_{Tate}



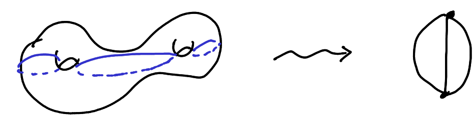
$$\mathcal{O}\left(\pi_1^{un}(E'_{Tate}, \vec{v})\right) \in \text{ind-MTM}$$

$$\text{Lie } \pi_1^{un}(E'_{Tate}, \vec{v}) \in \text{pro-MTM}$$

Remark: Note that Ihara curves correspond to a pants decomposition of the corresponding topological surface, These, in turn, correspond to smoothings of "maximally degenerate stable nodal curves.

The homology of an Ihara curve

The dual graph Γ of an Ihara curve has one vertex for each component and one edge for each vanishing cycle:



There is a map (in the homotopy category) from an Ihara curve X to its dual graph Γ. It induces a map

$$\alpha: H_1(X) \longrightarrow H_1(\Gamma) \leftarrow \text{isom to } \mathbb{Q}(0)^g$$

There is also a map

$$\beta: \bigoplus_{\substack{\text{components} \\ C}} H_1(C) \to H_1(X).$$

isom to $\mathbb{Q}(1)^2$

Exercise: Show $\ker \alpha = \operatorname{im} \beta$.

Prop

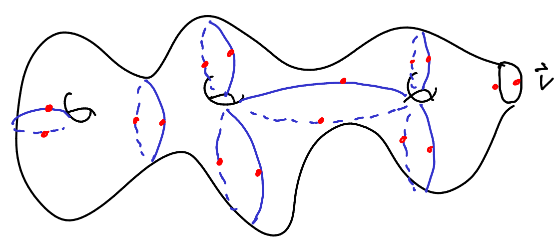
Prop$^{\underline{n}}$ As an MTM, $H_1(X;\mathbb{Q})$ is a split extension

$$0 \to \underset{\substack{\| \wr \\ \mathbb{Q}(1)^{g+\vec{r}+n-1}}}{\operatorname{im}\beta} \to H_1(X;\mathbb{Q}) \longrightarrow \underset{\substack{\| \wr \\ \mathbb{Q}(0)^g}}{H_1(\Gamma;\mathbb{Q})} \to 0$$

$\square$

Derivation Algebras

Suppose that C is an Ihara curve with one boundary component. (Eg: E'_{Tate})



Let $\vec{v}$ be one of the two distinguished boundary points. (These are tangential base points.) Let

$$\mathfrak{p} = \mathrm{Lie}\, \pi_1^{un}(C, \vec{v})$$

More accurately, this is the $\mathbb{Q}$–DR realization

This is an object of MTM. This has a canonical splitting of its weight filtration. (via: $\mathbb{G}_m \to \pi_1(\mathrm{MTM}, \mathrm{DR})$)

This gives a canonical isomorphism

$$\mathfrak{p} \cong \mathbb{L}(H)^{\wedge}$$

where $H = H_1(C) \cong \mathbb{Q}(0)^g \oplus \mathbb{Q}(1)^g$

↑ limit MHS.

The Lie algebra $\mathbb{L}(H)$ has a natural $Sp(H)$ action. There is a unique trivial rep in

$$Gr^W_{-2}\, \mathbb{L}(H) \cong \Lambda^2 H.$$

It is a copy of $\mathbb{Q}(1)$ and is spanned by

$$\theta := \sum_{j=1}^{g} [a_j, b_j] \qquad a_j, b_j \text{ symplectic basis of } H.$$

As pointed out before it is spanned by log (boundary loop).

Set

$$\{\delta \in \mathrm{Der}\, \mathfrak{p} : \delta \log(\mathrm{bdry}) = 0\}$$

$$\cong$$

$$\mathrm{Der}^{\theta}\, \mathbb{L}(H) = \{\delta \in \mathrm{Der}\, \mathbb{L}(H) : \delta(\theta) = 0\}.$$

Since the log of the boundary loop spans a copy of $\mathbb{Q}(1)$, $\underline{k} = \mathbb{L}(\sigma_3, \sigma_5, \ldots)$ acts trivially on it. So we conclude that there is a canonical homomorphism

$$\underline{k} = \mathbb{L}(\sigma_3, \sigma_5, \sigma_7, \ldots) \to \mathrm{Der}^{\Theta} \mathbb{L}(H) .$$

<u>Relative</u> <u>weight</u> <u>filtrations</u>

The Lie algebra $\mathfrak{p}$ has two weight filtrations:

(1) its weight filtration $W_\bullet$ as a once punctured curve:

$$W_{-n}\mathfrak{p} = LCS^n \mathfrak{p}$$

(2) its "relative weight filtration", which we shall denote by $M_\bullet$. In this case, $M_\bullet$ is its weight filtration as an object of MTM. One can show that there is a canonical (unique!) bigrading of $\mathfrak{p}$ (the DR realization) that splits $M_\bullet$, $W_\bullet$ and the Hodge filtration $F^\bullet$:

Recall that

$$H = H_1(C) = \mathbb{Q}(0)^g \oplus \mathbb{Q}(1)^g \quad \text{in MTM}$$

Choose a basis $T_1, \ldots, T_g$ of $\mathbb{Q}(0)^g$ and a basis $A_1, \ldots, A_g$ of $\mathbb{Q}(1)^g$ s.t.

intersection pairing $\to \langle T_j, A_k \rangle = \delta_{jk}$.

Exactness of $Gr^W_\bullet$, $Gr^M_\bullet$ and $Gr_F^\bullet$ implies

$$Gr_F^0 Gr_0^M Gr_{-1}^W \mathfrak{p} \cong \underbrace{Gr_F^0 Gr_0^M Gr_{-1}^W H}_{= \mathrm{Span}\{T_1, \ldots, T_g\}}$$

and that

$$Gr_F^{-1} Gr_{-2}^M Gr_{-1}^W \mathfrak{p} = \underbrace{Gr_F^{-1} Gr_{-2}^M Gr_{-1}^W H}_{= \mathrm{Span}\{A_1, \ldots, A_g\}}$$

So there are canonical isomorphisms

$$\mathfrak{p} \cong \left(\mathrm{Gr}^{\bullet}_{F} \mathrm{Gr}^{M}_{\bullet} \mathrm{Gr}^{W}_{\bullet} \mathfrak{p}\right)^{\wedge}$$

$$\cong \mathbb{L}(A_1, \ldots, A_g, T_1, \ldots, T_g)^{\wedge}.$$

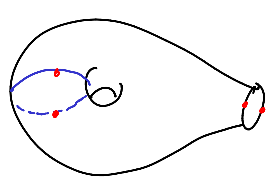
Under this isomorphism

$W_{-n}\mathfrak{p}$ = span monomials of degree $\geq n$

$M_{-2m}\mathfrak{p}$ = span monomials of A degree $\geq m$

$F^{-p}\mathfrak{p}$ = span of monomials of degree $\leq p$ in the A_j.

Example: Tate curve $(g=1)$



$$H = \mathbb{Q}T \oplus \mathbb{Q}A = \mathbb{Q}(0) \oplus \mathbb{Q}(1)$$

$$\mathfrak{p} \cong \mathbb{L}(A, T)^{\wedge}$$

M weight

W weight

	0	-2	-4	-6	-8	-10
-1	T	A				
-2		$T\cdot A$				
-3		$T^2\cdot A$	$A^2\cdot T$			
-4		$T^3\cdot A$	*	$A^3\cdot T$		
-5		$T^4\cdot A$	*	*	$A^4\cdot T$	
-6		$T^6\cdot A$	*	*	*	$A^5\cdot T$

Remarks:

(1) each $\mathrm{Gr}^W_{-m}\mathfrak{p}$ is an $\mathfrak{sl}(H)$ module

(2) if $S^a H \subseteq \mathrm{Gr}^W_{-m}\mathfrak{p}$, the