Les Diablerets 2015.08.27

Lecture II

Mixed Hodge Theory (and motivic Galois groups) for Group Theorists

Hodge Structures

A $\mathbb{Q}$-Hodge structure (HS) V of weight $m \in \mathbb{Z}$ consists of a finite dimensional vector space $V_{\mathbb{Q}}$ together with a decreasing filtration

$$V_{\mathbb{C}} = F^m V \supseteq F^{m+1} V \supseteq \dots \supseteq F^s V = 0$$

(called the Hodge filtration), where $V_{\mathbb{C}} := V_{\mathbb{Q}} \otimes \mathbb{C}$. These are required to satisfy

so $F^p V \cap \overline{F^{m-p+1}} V = 0$

$$V_{\mathbb{C}} = \bigoplus_{p+q=m} F^p V \cap \overline{F^q V}.$$

The subspace $F^p V \cap \overline{F^q V}$ is typically denoted $V^{p,q}$ (where $p+q=m$). With this notation

$$V_{\mathbb{C}} = \bigoplus_{p+q=m} V^{p,q}, \quad \overline{V^{p,q}} = V^{q,p}$$

$$F^p V = \bigoplus_{s \geq p} V^{s,m-s}$$

Morphisms: A morphism

$$\varphi : U \to V$$

of Hodge structures

$$U = (U_{\mathbb{Q}}, F^\bullet), \quad V = (V_{\mathbb{Q}}, F^\bullet)$$

is a $\mathbb{Q}$ linear map

$$\varphi_{\mathbb{Q}} : U_{\mathbb{Q}} \to V_{\mathbb{Q}}$$

whose complexification

$$\varphi_{\mathbb{C}} : U_{\mathbb{C}} \to V_{\mathbb{C}}$$

preserves the Hodge filtrations:

$$\varphi_{\mathbb{C}}(F^p U) \subseteq F^p V. \qquad (*)$$

This condition implies that

$$\varphi_{\mathbb{C}}(\overline{F^q U}) \subseteq \overline{F^q V}$$

as $\varphi_{\mathbb{C}}$ is defined over $\mathbb{R}$. So $(*)$ is equivalent to the condition

$$\varphi_{\mathbb{C}}(U^{p,q}) \subseteq V^{p,q}.$$

<u>Remark</u>: This implies that if U and V have different weights, then

$$\mathrm{Hom}_{MHS}(U, V) = 0 .$$

<u>Products</u> <u>and</u> <u>duals</u>:

Suppose that U and V are Hodge structures of weights m and n, respectively. Then

(1) $U \otimes V$ is a HS of weight $m+n$:

- $(U \otimes V)_{\mathbb{Q}} = U_{\mathbb{Q}} \otimes V_{\mathbb{Q}}$
- $(U \otimes V)^{p,q} = \bigoplus U^{s,t} \otimes V^{p-s,q-t}$

(2) U^* is a HS of weight $-m$:

- $(U^*)_{\mathbb{Q}} = \mathrm{Hom}_{\mathbb{Q}}(U_{\mathbb{Q}}, \mathbb{Q})$
- $(U^*)^{p,q} = \mathrm{Hom}_{\mathbb{C}}(U^{-p,-q}, \mathbb{C})$

(3) Define

$$\mathrm{Hom}_{\mathbb{Q}}(U, V) = U^* \otimes V.$$

This is a Hodge structure of weight $n-m$.

Canonical Example:

X = smooth projective variety

eg: X = compact Riemann surface

$$V_{\mathbb{Q}} = H^m(X;\mathbb{Q})$$

$$H^{p,q}(X) = \frac{\text{closed } m\text{-forms of type } (p,q)}{\text{exact } (p,q) \text{ forms}}$$

where $p+q=m$.

Hodge Theorem: a Hodge structure of weight m

$$H^m(X,\mathbb{C}) = \bigoplus_{p+q=m} H^{p,q}(X)$$

This implies that $\mathrm{rk}\, H^{2j+1}(X)$ is even. So the Hopf surface $(\mathbb{C}^2-\{0\})/\mathbb{Z}$ is not algebraic.

eg: $X = \mathbb{C}/\Lambda$ ← a lattice

$$H^1(X;\mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$$
$$= \mathbb{C}\, dz \oplus \mathbb{C}\, d\bar{z}$$

$$H_1(X;\mathbb{C}) = H^{-1,0}(X) \oplus H^{0,-1}(X)$$

weight -1.

Semi-simple mixed Hodge structures

These are direct sums of Hodge structures of possibly different weights: Suppose $V_1, \ldots, V_r$ are HSs of weights $m_1, \ldots, m_r$, respectively. Their direct sum

$$V = V_1 \oplus \cdots \oplus V_r$$

is a (split) mixed Hodge structure with

$$V_{\mathbb{Q}} = V_{1,\mathbb{Q}} \oplus \cdots \oplus V_{r,\mathbb{Q}}$$
$$F^p V = F^p V_1 \oplus \cdots \oplus F^p V_r.$$

There is an additional filtration

$$0 = W_a V_{\mathbb{Q}} \subseteq \cdots \subseteq W_b V_{\mathbb{Q}} = V_{\mathbb{Q}}$$

called the *weight filtration*. It

is defined by

$$W_m V_{\mathbb{Q}} = \bigoplus_{m_j \le m} V_{j,\mathbb{Q}}$$

Note that $\quad$ the m^{th} weight graded quotient

$$Gr_m^W V \cong \bigoplus_{\{j : m_j = m\}} V_j$$

is a "pure" Hodge structure of weight m.

<u>Polarizations</u>. Often forgotten, but important.

A <u>polarization</u> of a Hodge structure V of weight m. is a $(-1)^m$ symmetric bilinear form $\quad$ necessarily non-degen

$$\langle \, , \rangle : V_{\mathbb{Q}} \otimes V_{\mathbb{Q}} \to \mathbb{Q}$$

satisfying:

(1) if $u \in V^{p,q}$ and $v \in V^{s,t}$, then

$$\langle u, v \rangle = 0 \quad \text{unless} \quad \begin{array}{l} s = m-p \\ t = m-q, \end{array}$$

(2) The pairing $\quad$ $p+q=m$

$$S : V^{p,q} \otimes V^{p,q} \longrightarrow \mathbb{C}$$

$$S(u,v) = i^{p-q} \langle u, \bar{v} \rangle$$

is positive definite hermitian.

$\Rightarrow \langle \, , \rangle$ non degen.

<u>Example</u>: $X =$ compact Riemann surface

$V = H^1(X)$ and

$$\langle \, , \rangle : H^1(X) \otimes H^1(X) \longrightarrow \mathbb{Q}$$

$$u \otimes v \longmapsto \int_X u \cup v$$

$$S : H^{1,0} \otimes H^{1,0} \to \mathbb{C}$$

$$S(\omega_1, \omega_2) = i \int_X \omega_1 \wedge \bar{\omega}_2$$

which is positive definite by the Riemann bilinear relations.

The significance of the polarization is the easily proved fact that if A is a Hodge substructure of a <u>polarized</u> HS V, then

(1) the restriction of the polarization to A is non degenerate, so

(2) $V = A \oplus A^\perp$

(3) $A^\perp$ is a sub Hodge structure of V.

Hodge structures that arise in geometry typically have a natural polarization.

<u>Tannakian Point of View</u>

Let

MHS^{ss} = category whose objects are $\oplus$ of <u>polarizable</u> HS's.

This is tannakian. The fiber functor is

$$\omega : V = (V_{\mathbb{Q}}, F^\bullet) \rightsquigarrow V_{\mathbb{Q}}$$

Every object of MHS^{ss} is completely reducible.

<u>PROP</u>: ① $\pi_1(MHS^{ss}, \omega)$ is a reductive group.

② The functor

$$MHS^{ss} \to \text{Graded Vector spaces}$$
$$V \longmapsto V_{\mathbb{Q}} = \oplus Gr^W_m V_{\mathbb{Q}}$$

induces a central cocharacter

$$\chi : \mathbb{G}_m \to \pi_1(MHS^{ss}, \omega).$$ ∎

$t \in \mathbb{G}_m(\mathbb{Q})$ acts on $Gr^W_m V$ by mult by t^m.

<u>Mixed Hodge Structures</u>

A $\mathbb{Q}$-mixed Hodge structure V is a triple $(V_{\mathbb{Q}}, W_\bullet, F^\bullet)$, where

$V_{\mathbb{Q}}$ = finite dimensional $\mathbb{Q}$-vect sp

$W_\bullet$ = "weight filtration"

= an increasing filt

$$0 = W_a V_{\mathbb{Q}} \subseteq W_{a+1} V_{\mathbb{Q}} \subseteq \cdots \subseteq W_b V_{\mathbb{Q}} = V_{\mathbb{Q}}$$

$F^\bullet$ = "Hodge filtration"

= decreasing filtration of $V_{\mathbb{C}}$

$$V_{\mathbb{C}} = F^m V \supseteq F^{m+1} V \supseteq \cdots \supseteq F^k V = 0$$

satisfying

induced Hodge filtration

$(*)$ each $Gr_r^W V := \left(W_r V_{\mathbb{Q}} / W_{r-1},\ F^\bullet \right)$

$$F^\bullet \overset{\text{''}}{=} F^p V_r / F^p V_{r-1}$$

is a HS of weight m.

<u>Example</u>: any semi-simple MHS.

<u>Theorem</u> (Deligne) The cohomology ring of every complex algebraic variety has a (graded polarizable) MHS that is functorial w.r.t morphisms of varieties. $\square$

<u>Theorem</u> (Deligne) The category of $\mathbb{Q}$-MHS is a neutral tannakian category with fiber functor

$$(V_{\mathbb{Q}}, W_\bullet, F^\bullet) \mapsto V_{\mathbb{Q}}$$

Let

$$MHS = \text{category of "graded polarizable" MHS}$$

ie: each $Gr_m^W V$ is a polarizable HS of wt m.

This is also tannakian. The inclusion

$$MHS^{ss} \hookrightarrow MHS$$

is fully faithful. So

$$\pi_1(MHS, \omega) \to \pi_1(MHS^{ss}, \omega)$$

is surjective.

recall that this is reductive.

Theorem (essentially Deligne). The kernel U of

This is the reductive quotient

$$\pi_1(MHS,\omega) \to \pi_1(MHS^{ss},\omega)$$

is prounipotent. The abelianization $H_1(U)$ of U is a $\pi_1(MHS^{ss},\omega)$ module. It has strictly negative weights w.r.t

$$\mathbb{G}_m \xrightarrow{\chi} \pi_1(MHS^{ss},\omega) \to \operatorname{Aut} H_1(U)$$

What this means is that, for example,

$$\operatorname{Ext}^1_{MHS}(A,B) = 0$$

unless $wt(A) > wt(B)$. Here A and B are pure Hodge structures.

Natural Splittings.

Suppose that G is an affine F-group that is an extension

$$1 \to U \to G \to R \to 1$$

of a reductive group R by a (pro) unipotent group U.

Suppose that

(1) $\chi : \mathbb{G}_m \to R$

is a (non-trivial) central cocharacter.

(2) $H_1(U) = \prod_{n<0} H_1(U)_{(n)}$

where $H_1(U)_{(n)}$ = subspace where $t \in \mathbb{G}_m(F)$ acts by mult by t^n.

This is a "negatively weighted extension" of R by U.

Eg: $G = \pi_1(MHS,\omega)$, $R = \pi_1(MHS^{ss},\omega)$ as above.

$$\begin{array}{c} R = \pi_1(MHS^{ss},\omega) \\ \uparrow \chi \\ \mathbb{G}_m \end{array}$$

Prop: For each irreducible R-module V, there is an integer $n(V)$ s.t.

$$\chi(t)(v) = t^{n(V)} v \qquad \text{all } v \in V.$$

proof: Since χ is central

$$\chi : \mathbb{G}_m \longrightarrow \underbrace{\mathrm{Aut}_G V \cong \mathbb{G}_m}_{\varphi}$$

Schur's Lemma.

So $\varphi(t) = t^n$ for all t. $\square$

Def: $n(V)$ is the weight of V.

Prop: There is a lift $\tilde{\chi} : \mathbb{G}_m \longrightarrow G$ of $\chi : \mathbb{G}_m \to R$:

$$\begin{tikzcd} G \arrow[r] & R \\ & \mathbb{G}_m \arrow[ul, "\tilde{\chi}"] \arrow[u, "\chi"'] \end{tikzcd}$$

Any two such lifts are conjugate by an element of U.

proof: Use Levi's Theorem. $\square$

Rk: In general, $\tilde{\chi}$ will not be central.

Construction: Every G-module V can be decomposed into a sum of weight spaces

$$V = \bigoplus_{n \in \mathbb{Z}} V_n$$

In general, this grading depends on $\tilde{\chi}$.

under the $\mathbb{G}_m$-action

$$\mathbb{G}_m \xrightarrow{\tilde{\chi}} G \to \mathrm{Aut}(V).$$

Define

$$W_m V = \bigoplus_{n \le m} V_n.$$

Then

$$\cdots \subseteq W_{m-1} V \subseteq W_m V \subseteq \cdots$$

"weight filtration"

__PROP:__ The filtration $W_\bullet$ of V does not depend on the choice of the lift $\tilde{x}$.

__proof__ (sketch): (Details can be found in Hain-Matsumoto: "Negatively weighted ...")

Note that $\mathbb{G}_m$ acts on $\underline{u}$ via

$$\mathbb{G}_m \xrightarrow{\tilde{x}} G \longrightarrow \operatorname{Aut} \underline{u}.$$

Since $H_1(U)$ has only negative weights, all weights in $\underline{u}$ are also negative.

If $u \in \underline{u}_{(-n)}$ is a vector of weight where $n > 0$ and $v \in V_m$, then

$$t \cdot (u(v)) = (tu)(tv) = t^{m-n} v.$$

So

$$u : V_m \longrightarrow V_{m-n}$$

Thus $\underline{u}$ acts trivially on $Gr_\bullet^W V$.

Similarly, U " " " "

So $u \in U$ preserves $W_\bullet V$. ie: $uW_rV = W_rV$

If $u \tilde{x} u^{-1}$ is another lift of x, Then its eigenspaces are $u \cdot V_n$. So its weight filtration is

$$u W_\bullet V = W_\bullet V. \qquad \square$$

__COR:__ Every G-module has a natural weight filtration that is preserved by G-module maps. The subgroup U acts trivially on each $Gr_m^W V$, so that it is an $R = G/U$ module.

but generally not canonical!

Each choice of a lift $\tilde{x}$ of x to G gives a natural splitting of the weight filtration of every G-module:

$$V = \bigoplus V_n \cong \bigoplus Gr_m^W V$$

This isomorphism is preserved by all morphisms in $\mathrm{Rep}(G)$.

$$\begin{array}{ccc} V_1 & \xrightarrow{\approx} & \bigoplus_m Gr^W_m V_1 \\ \varphi \downarrow & & \downarrow Gr\,\varphi \\ V_2 & \longrightarrow & \bigoplus_m Gr^W_m V_2 \end{array}$$

These splittings are compatible with $\otimes$ and duals.

Graded reps

<u>COR</u>: $Gr^W_\bullet : \mathrm{Rep}(G) \to Gr\,\mathrm{Rep}(R)$ is exact. $\square$

<u>Back to Hodge Theory</u>

The discussion above implies that the weight filtration of every (graded polarizable) MHS has a natural splitting that is compatible with $\otimes$ and duals.

<u>Theorem</u> (Morgan, Hain) If X is a complex algebraic variety and $x \in X$, then

$$\mathfrak{p}(X,x) := \mathrm{Lie}\ \pi_1^{un}(X,x)$$

has a natural MHS s.t.

(1) the bracket $\mathfrak{p} \otimes \mathfrak{p} \to \mathfrak{p}$ is a morphism

(2) $\mathfrak{p}(X,x) \to \mathfrak{p}(X,x)^{ab} = H_1(X)$ is a morphism. $\square$

<u>Exercise</u>:

Use exactness of $Gr^W_\bullet$ to show that

eg X = cpact complex curve once punctured curve.

(a) if $H_1(X)$ is pure of weight -1, then

$$W_{-n}\,\mathfrak{p} = LCS^n\,\mathfrak{p}$$

(b) if $H_1(X)$ is pure of weight -2

Then

$$W_{-2n+1}\mathfrak{p} \cong W_{-2n}\mathfrak{p} = LCS^n \mathfrak{p}.$$

(c) In both cases

$$\mathrm{Gr}^W_\bullet \mathfrak{p} = \mathbb{L}(H)$$

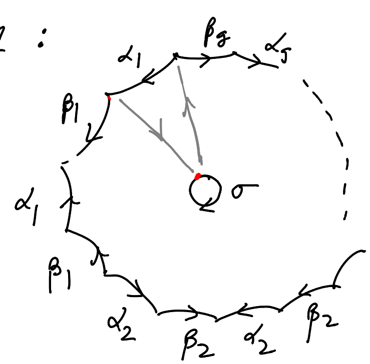
canonical!

(d) Deduce that there is a natural (but not canonical) isomorphism

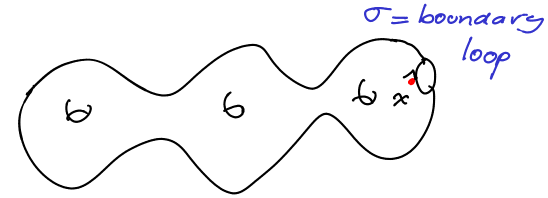
$$\mathfrak{p} \cong \prod_{n>0} \mathrm{Gr}^W_{-n}\mathfrak{p} = \mathbb{L}(H)^\wedge.$$

$\square$

<u>Application</u>:

$S =$ 

$/\sim$

$S =$ 

$$\alpha_1, \dots, \beta_g \in \pi_1(S, x)$$

$$\pi_1(S, x) = \langle \alpha_1, \dots, \beta_g \rangle$$

$$\sigma^{-1} = \prod_{j=1}^{g} \alpha_j \beta_j \alpha_j^{-1} \beta_j^{-1}$$

Let $A_j = [\alpha_j] \in H_1(S)$

$\quad B_j = [\beta_j] \in H_1(S)$

By above,

$$\mathfrak{p} \cong \mathbb{L}(A_1, \dots, B_g)^\wedge$$

Q: Can we choose this isomorphism such that

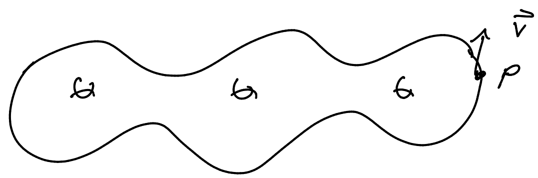
$$-\log\sigma = \sum_{j=1}^{g} [A_j, B_j]$$

<u>Rk</u>: Always true that

$$-\log\sigma = \sum_{j=1}^{g} [A_j, B_j] + \text{higher order}.$$

A: Yes!

<u>proof</u>: Replace S by smooth curve

$C =$ 

Set $C' = C - \{p\}$.

$$\mathfrak{p} = \operatorname{Lie} \pi_1(C', \vec{v})$$

This has a natural MHS compatible with its bracket:

Have inclusion

$$(\Delta, 0, \partial/\partial t) \longrightarrow (C, p, \vec{v})$$

induces

$$\pi_1(\Delta', \partial/\partial t) \to \pi_1(C', \vec{v})$$
$$\gamma \longmapsto \sigma^{-1}$$

pos generator

and

$$\operatorname{Lie} \pi_1^{un}(\Delta', \partial/\partial t) \longrightarrow \mathfrak{p}$$

<u>FACT</u>: $\operatorname{Lie} \pi_1^{un}(\Delta^*, \partial/\partial t) \cong \mathbb{Q}$ has a HS (gen by $\log \gamma$)

of weight -2. | It is the 1-dimensional Hodge structure $\mathbb{Q}(1)$ of type $(-1,-1)$

Fix a natural splitting of $W_\bullet$ in MHS. This fixes an isomorphism

$$\mathfrak{p} \cong \mathbb{L}(H)^\wedge$$

$$-\log \sigma = \text{image of } \log \gamma \in Gr^W_{-2} \mathbb{L}(H)$$ by naturality of splitting

so $$-\log \sigma = \sum_{j=1}^{g} [A_j, B_j].$$

<u>Corollary</u>: If C is the closed surface (C, p), then

$$\operatorname{Lie} \pi_1^{un}(C, p) \cong \frac{\mathbb{L}(A_1, \dots A_g, B_1, \dots, B_g)^\wedge}{\left(\sum_{j=1}^{g} [A_j, B_j]\right)}$$

<u>proof</u>: Use the fact that

$$\operatorname{Lie} \pi_1^{un}(C, p) = \operatorname{Lie} \pi_1^{un}(C', \vec{v}) / (\log \sigma).$$