Les Diablerets 2015.08.27

Lecture I

Completions of Groups and Groupoids

§ Unipotent and Prounipotent Groups

Fix a field F of characteristic zero.

Main examples for us: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_\ell$

ℓ-adic #s
ℓ prime.

An algebraic group $/F$ U is unipotent if it is isomorphic to a closed subgroup of

$$\begin{pmatrix} 1 & & & * \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \subseteq GL_N(F)$$

Equivalently: U is unipotent if

$U \subseteq GL(V)$ $\quad$ V f.dim vect sp

and U stabilizes a flag "filtration" if you prefer

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_r = V$$

and acts trivially on each graded quotient V_j/V_{j-1}.

The Lie algebra $\underline{u}$ of U is nilpotent. It is a subalgebra of the Lie algebra of nilpotent matrices

$$\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \subseteq \mathfrak{gl}_N(F)$$

<u>Alt:</u> $\underline{u} \subseteq \mathfrak{gl}(V)$ stabilizes the flag and acts trivially on each V_j/V_{j-1}.

Important fact:

(1) The exponential mapping

$\exp : \underline{u} \to U ; \; x \mapsto e^x$

is a polynomial mapping.

(2) $\log : U \to \underline{u}$

is well defined polynomial map

(3) $\underline{u} \underset{\log}{\overset{\exp}{\rightleftarrows}} U$ isomorphisms of schemes/F.

So the ring of functions on U is

$\mathcal{O}(U) = \mathrm{Sym}(\underline{u}) =$ polynomial function on $\underline{u}$

A <u>prounipotent</u> <u>group</u> is the inverse limit of unipotent groups:

$$U = \varprojlim U_\alpha$$

Its Lie algebra is the "pronilpotent" Lie algebra

$$\underline{u} = \varprojlim \underline{u}_\alpha$$

where $\underline{u}_\alpha = \mathrm{Lie}\, U_\alpha$.

<u>Example</u>:

$V =$ finite dimensional vector space/F

$\{X_1, \dots, X_n\}$ basis of V.

$T(V) =$ tensor algebra on V

$= F\langle X_1, \dots, X_n \rangle$

$=$ free associative (non-com) algebra on $X_1, \dots, X_n$

Augmentation:

$\varepsilon : F\langle X_1, \dots, X_n \rangle \to F$

$\varepsilon(X_j) = 0$.

Augmentation ideal:

$I = \ker \varepsilon$.

$T(V)$ is a Hopf algebra (co-comm):

coproduct:

$$\Delta : T(V) \longrightarrow T(V) \otimes T(V)$$

$$\Delta x_j = x_j \otimes 1 + 1 \otimes x_j$$

Primitive elements:

$$P\,T(V) = \left\{ u \in T(V) : \Delta u = u \otimes 1 + 1 \otimes u \right\}$$

$$= \mathbb{L}(V)$$ ← free Lie algebra on V

$$= \mathbb{L}(x_1, \dots, x_n)$$

Completion:

$$T(V)^\wedge := \varprojlim_r T(V)/I^r$$

$$\cong F\langle\langle x_1, \dots, x_n \rangle\rangle$$

↑ formal powerseries in the non-commuting $x_1, \dots, x_n$

Δ is continuous, so induces

$$\Delta : T(V)^\wedge \longrightarrow T(V)^\wedge \hat{\otimes} T(V)^\wedge$$

$(T(V)^\wedge, \Delta)$ = complete cocommutative Hopf algebra

$$P\,T(V)^\wedge = \mathbb{L}(V)^\wedge$$

↖ completion of the free Lie alg

Group-like elements

$$G\,T(V) := \left\{ g \in T(V)^\wedge : \varepsilon(g) = 1 \ \& \ \Delta g = g \otimes g \right\}$$

This is a prounipotent group /F

<u>proof</u>: Set

$G_n := G_n T(V)$ = image of $G\,T(V)$ in $T(V)/I^{n+1}$

Then G_n acts faithfully on $T(V)/I^{n+1}$ by left multiplication. It stabilizes the filtration

$$T(V)/I^{n+1} \supseteq I/I^{n+1} \supseteq \cdots \supseteq I^n/I^{n+1} \supseteq 0$$

and acts trivially on its graded quotients. It is algebraic as its elements satisfy the equations:

(1) $\varepsilon(g) = 1$

(2) $\bar{\Delta} g = \overline{g \otimes g}$

where

$$\bar{\Delta} : T(V)/I^{n+1} \to \bigoplus_{\substack{j+k \\ = n+1}} T(V)/I^j \otimes T(V)/I^k$$

and $\overline{g \otimes g}$ denotes image in RHS. $\square$

<u>Lemma</u> : If $u \in \hat{I} \subseteq T(V)^\wedge$, Then u is primitive $\Leftrightarrow$ e^u is group-like.

<u>Pf</u>: Use

(a) $\Delta e^u = e^{\Delta u}$ (Δ continuous)

(b) $e^{u \otimes 1 + 1 \otimes u} = e^u \otimes e^u$. $\square$

<u>Cor</u>: The Lie algebra of $GT(V)^\wedge$ is $PT(V)^\wedge = \mathbb{L}(V)^\wedge$. $\square$

<u>Remark</u>: To write down the ring of functions on $GT(V)^\wedge$, identify it with $F\langle\langle X_1, \ldots, X_r \rangle\rangle$. Each $g \in GT(V)^\wedge$ is a powerseries

$$g = \sum_{r=0}^{\infty} \sum_{|I| = r} a_I(g) X_{i_1} \cdots X_{i_r}$$

Then $g \mapsto a_I(g) \in \mathcal{O}(GT(V)^\wedge)$.

The PBW Theorem implies that

$$\mathcal{O}(\mathbb{L}T(V)^\wedge) = \varinjlim \mathrm{Sym}(\mathfrak{g}_n) \qquad \mathrm{Lie}(G_n) \subseteq T(V)/I^{n+1}$$

product = shuffle prod

$$= \text{"continuous polynomials" on } \mathbb{P}T(V)^\wedge.$$

§ Unipotent Completion

Suppose that Γ is a discrete group. The unipotent completion of Γ (over F) is a prounipotent group $\Gamma^{un}_{/F}$ over F and a homomorphism

$$\theta : \Gamma \to \Gamma^{un}_{/F}(F)$$

F rational points of $\Gamma^{un}_{/F}$

These are characterized by the universal mapping property:

Homomorphisms $\Gamma \xrightarrow{\rho} \mathcal{U}(F)$ of Γ into the F points of a unipotent F-group $\mathcal{U}$ correspond to homomorphisms

$$\Gamma^{un}_{/F} \xrightarrow{\hat{\rho}} \mathcal{U}$$

of affine F-groups where the diagram

$$\begin{tikzcd} \Gamma \arrow[r, "\theta"] \arrow[dr, "\rho"'] & \Gamma^{un}_{/F}(F) \arrow[d, "\hat{\rho}"] \\ & \mathcal{U}(F) \end{tikzcd}$$

commutes.

<u>Remark</u>: We are not assuming that Γ is finitely generated.

<u>Group algebras</u>: Γ discrete group, $F\Gamma$ = group algebra/F. This is a cocommutative Hopf algebra with coprod

$$\Delta\gamma = \gamma \otimes \gamma \qquad \gamma \in \Gamma.$$

Standard augmentation $\varepsilon : F\Gamma \to F$ ~~takes~~ takes each $\gamma \in \Gamma$ to 1. Augmentation ideal $I = \ker \varepsilon$. The I-adic completion

$$F\Gamma^\wedge := \varprojlim F\Gamma / I^n$$

is a complete Hopf algebra.

<u>Example</u>: Let

$$\Gamma = \langle x_1, \dots, x_n \rangle$$

the free group generated by $\{x_1, \dots, x_n\}$.

Define a homomorphism

$$\theta : \Gamma \longrightarrow \mathbb{Q}\langle\langle X_1, \dots, X_n \rangle\rangle$$

by $\theta(x_j) = \exp X_j$. This induces an algebra homomorphism

$$\hat{\theta} : \mathbb{Q}\Gamma \to \mathbb{Q}\langle\langle X_1, \dots, X_n \rangle\rangle$$

which is continuous as it is augmentation preserving. So it induces an algebra homomorphism

$$\hat{\theta} : \mathbb{Q}\Gamma^\wedge \longrightarrow \mathbb{Q}\langle\langle X_1, \dots, X_n \rangle\rangle$$

standard coprod $\Delta X_j = X_j \otimes 1 + 1 \otimes X_j$

Since e^{X_j} is grouplike, $\hat{\theta}$ takes grouplikes to grouplikes. Consequently it preserves the coproduct.

Exercise: Show that $\hat{\theta}$ is an isomorphism of complete Hopf algebras. (Hint: Show that both sides have the same universal mapping property.)

<u>Conclusion</u>:

$$\operatorname{Lie} \Gamma^{un}_{/\mathbb{Q}} \cong \mathbb{L}(X_1, \dots, X_n)^\wedge$$

and

$$\theta : \Gamma \to \Gamma^{un}(\mathbb{Q}) = G\, \mathbb{Q}\langle\langle X_1, \dots, X_n \rangle\rangle$$
$$x_j \longmapsto e^{X_j}.$$

Exercise:

Suppose

$$\Gamma = \langle x_1, \ldots, x_n : r_\alpha : \alpha \in A \rangle$$

relations

not necessarily finite.

Then have

$$\theta : \Gamma \longrightarrow \mathbb{Q}\langle\langle X_1, \ldots, X_n \rangle\rangle / (\log r_\alpha : \alpha \in A)$$

closed ideal

Show the RHS is a complete ~~Hopf~~ algebra and that θ induces a complete Hopf algebra isomorphism

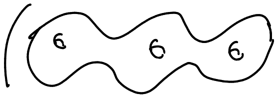
$$\mathbb{Q}\Gamma^\wedge \longrightarrow \mathbb{Q}\langle\langle X_1, \ldots, X_n \rangle\rangle / (\log \theta(r_\alpha) : \alpha \in A)$$

Deduce:

$$\operatorname{Lie} \Gamma^{un}_{/\mathbb{Q}} = \mathbb{L}(X_1, \ldots, X_n)^\wedge / (\log \theta(r_\alpha) : \alpha \in A)$$

Example:

genus g

$\Gamma = \pi_1\Big($  $, x_0\Big)$

$$= \langle a_1, \ldots, b_g : \prod_{j=1}^{g} a_j b_j a_j^{-1} b_j^{-1} \rangle$$

$\operatorname{Lie} \Gamma^{un}_{/\mathbb{Q}}$

$\sum_j [A_j, B_j] + \text{higher order}$

$$\cong \mathbb{L}(A_1, \ldots A_g, B_1, \ldots, B_g)^\wedge / \Big(\log \prod_j \theta(a_j b_j a_j^{-1} b_j^{-1})\Big)$$

Remark: At the end of Lecture II, we will use Hodge theory to show that

$$\operatorname{Lie} \Gamma^{un}_{/\mathbb{Q}} \cong \mathbb{L}(A_1, \ldots A_g, B_1, \ldots, B_g)^\wedge / \Big(\sum_{j=1}^{g} [A_j, B_j]\Big)$$

Remark: Matrix entries of unipotent representations are functions on $\Gamma^{un}_{/F}$:

$\rho : \Gamma \to GL(V)$ unipotent rep.

$F \xrightarrow{v} V \xrightarrow{\varphi} F$ $\quad v \in V$, $\varphi \in \check{V}$

a matrix entry

denoted $[V; \varphi, v]$

$$[V; \varphi, v] : \gamma \mapsto \varphi(\rho(\gamma)(v))$$

This is an element of $\mathcal{O}(\Gamma^{un}_{/F})$.

Tannakian Approach

Let G be any group (discrete, profinite, algebraic, ...) and F a field of char 0 appropriate to the setting:

- any F for discrete G,
- $\mathbb{Q}_\ell$ when G is profinite,
- an extension of k if G is algebraic $/k$.

Let $\mathrm{Rep}_F(G)$ denote the corresponding category of finite dimensional reps $/F$:

- arbitrary when G is discrete,
- continuous when G is profinite,
- rational when G is algebraic.

Essential features of $\mathrm{Rep}_F(G)$:

(1) it is an abelian, F linear tensor category

(2) it has a unit object 1 (viz, the 1-dimensional trivial rep)

(3) The $\otimes$ product is commutative and associative; that satisfy natural conds such as

$$V \otimes W \xrightarrow{\approx} W \otimes V \to V \otimes W$$

is the identity.

(4) There are natural isomorphisms

$$1 \otimes V \cong V \cong V \otimes 1$$

(5) There are dual objects

$$\check{M} = \mathrm{Hom}_F(M, 1)$$

that satisfy natural compatibilities with the other structure.

(6) There is a faithful tensor functor

take a representation to its underlying vector space

$$\omega : \mathrm{Rep}_F(G) \to \mathrm{Vec}_F$$

that preserves duals, etc.

Axiomatize this to get the definition of a "neutral tannakian category over F." A <u>fiber functor</u>

Tannakian

$$\omega : \mathcal{T} \to \mathrm{Vec}_F \ \mathcal{T}$$

is any faithful $\otimes$ functor.

<u>Basic Theorem</u>: If $\mathcal{T}$ is a neutral tannakian category over F, and $\omega : \mathcal{T} \to \mathrm{Vec}_F$ is a fiber functor, then $\mathcal{T}$ is equivalent to $\mathrm{Rep}(G)$ where G is affine F group

$$G := \pi_1(\mathcal{T}, \omega)$$

$$= \mathrm{Aut}^{\otimes} \omega$$

The group of natural isomorphisms $\omega \to \omega$ that preserve $\otimes$.

whose ring of functions is

$$\mathcal{O}(G) = \text{matrix entries of } G$$

$[V; \varphi, v]$ $v \in \omega(V)$, $\varphi \in \omega(\check{V})$) dual vect sp

Its value on $\eta \in \mathrm{Aut}^{\otimes}\omega$ is

$$F \xrightarrow{v} \omega(V) \xrightarrow{\eta_V} \omega(V) \xrightarrow{\varphi} F.$$

Examples

① $\mathcal{T} = \mathrm{Vec}_F \qquad \omega = id$

$\pi_1(\mathcal{T}, \omega) =$ trivial group

② Γ = discrete group

$\mathcal{T} =$ cat of finite dimensional unipotent reps of Γ

$\omega : \mathcal{T} \to \mathrm{Vec}_F$ is the obvious forgetful functor

$\pi_1(\mathcal{T}, \omega) = \Gamma^{un}_{/F}$

More exotic examples:

(a) $\mathcal{T} =$ category of $\mathbb{Q}$-MHS

$F = \mathbb{Q}$

(b) $\mathcal{T} =$ category of mixed Tate motives unramified $/\mathbb{Z}$

$F = \mathbb{Q}$.

More about them later.

Problems with Unipotent Completion

While unipotent completion is useful, it has its drawbacks. Two significant ones are:

(1) $\Gamma^{un}_{/F}$ = trivial group

$\Leftrightarrow \quad H_1(\Gamma; \mathbb{Q}) = 0$

(2) $\Gamma^{un}_{/F} \cong \mathbb{G}_{a/F}$

$\Leftrightarrow \quad \dim_{\mathbb{Q}} H^1(\Gamma; \mathbb{Q}) = 1$

Examples of interest:

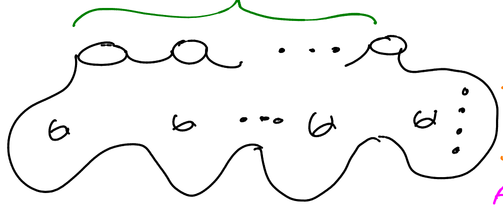
(1) knot groups:

$$H_1(S^3 - \text{knot}) \cong \mathbb{Z}$$

so $\quad \pi_1^{un}(S^3 - \text{knot}, *) \cong \mathbb{G}_a$

No information!

(2) Mapping class groups



$$\Gamma_{g, n+\vec{r}} := \pi_0 \operatorname{Diff}^+(S, \partial S \cup P)$$

group of connected components.

fix pointwise

eg: $\quad \Gamma_{0, n+\vec{1}} =$ pure braid group on n strings

FACT: $\quad$ If $g \geq 1$

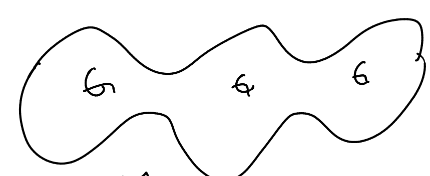
$$H_1(\Gamma_{g, n+\vec{r}}; \mathbb{Q}) = 0$$

all n, r satisfying $\quad 2g-2+n+r > 0$.

So if $g \geq 1$, $\quad \Gamma^{un}_{g, n+\vec{r}} = 1$.

The Symplectic Representation

Let $\overline{S}$ be the closed surface obtained by capping of the boundary components:

$\hat{S} =$ 

Let $H_R = H_1(\hat{S}; R)$. This has a non degenerate symplectic form — the intersection pairing

$$\langle \, , \rangle : H_1(\hat{S}) \otimes H_1(\hat{S}) \longrightarrow \mathbb{Z}.$$

The action of $\Gamma_{g,n+\vec{r}}$ on S induces an action on $\hat{S}$ and thus a homom

$$\Gamma_{g,n+\vec{r}} \longrightarrow \mathrm{Aut}(H_R, \langle\, , \rangle) \cong Sp_g(R)$$

This is Zariski dense.

<u>Relative Unipotent Completion</u>

Input:

Γ = discrete group

F = field of char 0

R = reductive (eg semi-simple) algebraic group $/F$

$\rho : \Gamma \to R(F)$ a Zariski dense representation

The completion of Γ with respect to ρ consists of

(1) an affine F-group $\mathcal{G}$ which is an extension

$$1 \to \mathcal{U} \to \mathcal{G} \to R \to 1$$

where $\mathcal{U}$ is prounipotent

(2) a homomorphism

$$\rho : \Gamma \to \mathcal{G}(F)$$

(necessarily Zariski dense)

These are required to satisfy:

Given a homomorphism

$$\varphi : \Gamma \to G(F)$$

where G is an extension

$$1 \to U \to G \to R \to 1$$

unipotent

and $\Gamma \longrightarrow G(F) \to R(F)$ is ρ,

there is a unique homomorphism

$$\begin{array}{ccccccccc} 1 & \to & \mathcal{U} & \to & \mathcal{G} & \to & R & \to & 1 \\ & & \downarrow & & \downarrow \hat{\varphi} & & \| & & \\ 1 & \to & U & \to & G & \to & R & \to & 1 \end{array}$$

such that

$$\Gamma \to \mathcal{G}(F) \xrightarrow{\hat{\varphi}} G(F)$$

is φ.

Construction: Let $\mathcal{T}$ be the category of representations V of Γ (over F) that admit a filtration

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_r = V$$

by submodules such that the action of Γ on each V_j/V_{j-1} factors

through an action of R:

$$\Gamma \longrightarrow R(F) \circlearrowright V_j/V_{j-1} \circlearrowleft R .$$

Matrices:

$$\left(\begin{array}{c|c|c} \rho & * & * \\ \hline 0 & \rho & * \\ \hline 0 & 0 & \rho \end{array}\right)$$

This is tannakian.

$$\mathcal{G} = \pi_1(\mathcal{T}, \omega)$$

obvious forgetful functor.

<u>Example</u>: $g > 0$

$\Gamma = \Gamma_{g,n+\vec{r}}$

$F = \mathbb{Q}$

$R = Sp(H) \cong Sp_g$

$\rho : \Gamma \to Sp(H_{\mathbb{Q}})$ the standard rep, Relative completion $\mathcal{G}_{g, r+\vec{n}}$:

$$\begin{tikzcd} 1 \arrow[r] & \mathcal{U}_{g,\vec{r}+n} \arrow[r] & \mathcal{G}_{g,\vec{r}+n} \arrow[r] & Sp_g \arrow[r] \arrow[l, bend right, "\text{Levi}"'] & 1 \end{tikzcd}$$

Lie algebra $\underline{u}_{g,n+\vec{r}}$ (pronilpotent)

$$\mathcal{G}_{g,n+\vec{r}} \cong Sp_g \ltimes \mathcal{U}_{g,n+\vec{r}}$$

$$Sp_g \circlearrowright \underline{u}_{g,n+\vec{r}}$$

Presentation of $\underline{u}_{g,n+\vec{r}}$ known when $g \geq 3$. and when $g=1$ and $n+\vec{r} \leq 2$. Genus 2 still mysterious.