LECTURES ON THE HODGE-DE RHAM THEORY OF THE FUNDAMENTAL GROUP OF $\mathbb{P}^1 - \{0, 1, \infty\}$

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Contents

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These notes are an introduction to ideas concerning the Hodge theory of the unipotent fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$ and its relation to multiple zeta numbers. They cover more material than the lectures. Most of the major ideas in these notes come from the work of Chen, Deligne, Goncharov, and Racinet.

1. Iterated Integrals and Chen's π_1 de Rham Theorem

The goal of this section is to state Chen's analogue for the fundamental group of de Rham's classical theorem and to prove it in some special cases.

1.1. **The Classical de Rham Theorem.** Let F denote either $\mathbb{R}$ or $\mathbb{C}$. Denote the complex of smooth, F-valued differential k-forms on a smooth manifold M by $E_F^k(M)$. These fit together to form a complex

$$0 \longrightarrow E_F^0(M) \xrightarrow{d} E_F^1(M) \xrightarrow{d} E_F^2(M) \xrightarrow{d} \cdots$$

The map d from k-forms to $(k+1)$-forms is the exterior derivative. This complex is called the *de Rham complex of* M. We shall denote it by $E_F^\bullet(M)$. The cohomology of this complex is called the *de Rham cohomology* of M, and will be denoted by $H_{\mathrm{DR}}^\bullet(M; F)$.

Denote the standard k-simplex by Δ^k. A smooth singular chain is a linear combination of smooth singular k-simplices $\sigma : \Delta^k \to M$. The singular homology and cohomology of M can be computed using *smooth* singular chains in place of the usual continuous ones.[1] More precisely, denote the complex of smooth singular chains on M with values in the abelian group A by $S_\bullet(M; A)$. Define the smooth singular cochains on M with values in A to be its dual:

$$S^\bullet(M; A) := \mathrm{Hom}_{\mathbb{Z}}(S_\bullet(M; \mathbb{Z}), A).$$

Then there are natural isomorphisms

$$H_k(M; A) \cong H_k(S_\bullet(M; A)) \text{ and } H^k(M; A) \cong H^k(S^\bullet(M; A))$$

Integration induces a natural mapping

$$\int : E_F^\bullet(M) \to S^\bullet(M; F)$$

Stokes' Theorem implies that it is a chain mapping. Although both $E_F^\bullet(M)$ and $S^\bullet(M; F)$ are F-algebras, this mapping is *not* an algebra homomorphism.

The Universal Coefficient Theorem implies that the natural mapping

$$H^k(M; F) \to \mathrm{Hom}_{\mathbb{Z}}(H_k(M; \mathbb{Z}), F)$$

[1] There are various ways to prove this, such as the method of acyclic models.

is an isomorphism. We are now ready to state the classical de Rham Theorem.

Theorem 1 (de Rham, 1929)**.** *The integration mapping induces a natural F-algebra isomorphism*

$$H^\bullet_{\mathrm{DR}}(M;F) \to H^\bullet(M;F).$$

Using his iterated integrals, Chen generalized this theorem to homotopy groups.[2] We will concentrate on his de Rham theorem for the fundamental group. Along the way, we will see that certain number theoretically interesting expressions can be expressed in terms of iterated integrals. This is not an accident.

1.2. **Path Spaces and the Fundamental Groupoid.** As in the previous section, M will denote a smooth manifold. The *path space* of M is

$$PM := \{\gamma : [0,1] \to M : \gamma \text{ is piecewise smooth}\}$$

This can be endowed with the compact-open topology. The end point mapping

$$PM \to M \times M, \quad \gamma \mapsto \big(\gamma(0), \gamma(1)\big)$$

is continuous. Denote the inverse image of (a,b) by $P_{a,b}M$. The set $\pi_0(P_{a,b}M)$ of its connected components is simply the set of homotopy classes of piecewise smooth paths from a to b in M. It is denoted by $\pi(M;a,b)$.

Multiplication of paths defines mappings

$$\pi(M;a,b) \times \pi(M;b,c) \to \pi(M;a,c).$$

The category whose objects are the points of M and where the set of morphisms from a to b is $\pi(M;a,b)$ is called the *fundamental groupoid* of M. Here we encounter a point where conventions differ. I am composing paths in the "natural order", as do most topologists.[3] The fundamental groupoid can be defined for any topological space using continuous mappings. For smooth manifolds, all definitions agree. This is the content of the following exercise.

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[2]A good reference is Chen's Bulletin article [2]. About the same time, Dennis Sullivan [20] developed a parallel theory of *minimal models*, which allows the computation of the "real homotopy groups" of a manifold from its de Rham complex. The theories of Chen and Sullivan are equivalent and compute the same invariants. Both theories work well for simply connected manifolds and reasonably well for the fundamental group of non-simply connected manifolds.

[3]Many algebraic geometers (including Deligne and Goncharov) compose paths in the functional order. This does not create any serious problems, but you should be aware of this when reading the literature.

Exercise 1. Prove that the following mappings are all bijections:

$$\left\{\begin{matrix}\text{piecewise smooth}\\ \text{paths from } a \text{ to } b\end{matrix}\right\}/\text{smooth homotopies}$$
$$\to \left\{\begin{matrix}\text{piecewise smooth}\\ \text{paths from } a \text{ to } b\end{matrix}\right\}/\text{homotopy}$$
$$\to \left\{\begin{matrix}\text{continuous paths}\\ \text{from } a \text{ to } b\end{matrix}\right\}/\text{homotopy}$$

This statement is the analogue for the fundamental groupoid of the fact that, for smooth manifolds, singular homology can be computed using smooth singular chains.

1.3. **Iterated Integrals.** Suppose that M is a smooth manifold and that $\alpha, \beta \in P_{a,a}M$. Then for any 1-form (closed or not) on M,

$$\int_{\alpha\beta} w = \int_\alpha w + \int_\beta w = \int_{\beta\alpha} w. \tag{1}$$

This means that ordinary line integrals are *intrinsically* abelian — they cannot detect the order in which we compose α and β. Because of this, ordinary line integrals cannot detect elements of the commutator subgroup of $\pi_1(M, a)$. This raises the question of how one can use differential forms to detect elements of $\pi_1(M, a)$ that are not visible in $H_1(M; \mathbb{R})$.

Chen gave a non-abelian generalization of the standard line integral. These are called *iterated line integrals.*

Definition 2. Suppose that $w_1, \dots, w_r$ are smooth 1-forms on M with values in an associative $\mathbb{R}$ algebra A.[4] (That is, $w_j \in E^1_{\mathbb{R}}(M) \otimes A$.) Suppose that $\gamma \in PM$. Define

$$\int_\gamma w_1 w_2 \dots w_r \in A$$

to be the *time ordered* integral

$$\int_{0 \le t_1 \le t_2 \le \dots \le t_n \le 1} f_1(t_1) \dots f_r(t_r) dt_1 \dots dt_r,$$

where $\gamma^* w_j = f_j(t)dt$. The iterated integral is to be viewed as a function

$$\int w_1 w_2 \dots w_r : PM \to A.$$

A general iterated integral is an $\mathbb{R}$-linear combination of the constant function and basic iterated integrals $\int w_1 \dots w_r$.

[4]Typically, A is $\mathbb{C}$, $\mathbb{M}_n(\mathbb{R})$ or $\mathbb{M}_n(\mathbb{C})$, where $\mathbb{M}_n(R)$ denotes the R-algebra of $n \times n$ matrices over a ring R.

The time ordered nature of iterated integrals is the key to their non-abelian properties.

Definition 3. Let S be a set. A function $\mathcal{F} : PM \to S$ is a *homotopy functional* the value of $\mathcal{F}$ on a path γ depends only in its homotopy class in $P_{a,b}M$. More precisely, for each pair of points a, b in M, there is a function $f_{a,b} : \pi(M; a, b) \to S$ such that the diagram

$$\begin{tikzcd} P_{a,b}M \arrow[r, "\mathcal{F}"] \arrow[d] & S \\ \pi(M;a,b) \arrow[ur, "f_{a,b}"'] & \end{tikzcd}$$

commutes.

A homotopy functional $\mathcal{F} : PM \to S$ induces a function $\phi_{\mathcal{F}} : \pi_1(M, a) \to S$ by taking the homotopy class of a loop γ to $\mathcal{F}(\gamma)$. More generally, it induces functions $\phi_{\mathcal{F}} : \pi(M; a, b) \to S$.

The basic problem, then, is to find all (or enough) iterated integrals that are homotopy functionals.

Exercise 2. Show that if w is a 1-form on a connected manifold M, then

$$\int w : PM \to \mathbb{R}$$

is a homotopy functional if and only if w is closed.

Not all iterated integrals of closed forms are homotopy functionals.

Exercise 3. Suppose that M is the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$. Compute

$$\int_{\alpha\beta} dx\,dy \quad \text{and} \quad \int_{\beta\alpha} dx\,dy,$$

where (x, y) are the coordinates on $\mathbb{R}^2$ and $\alpha(t) = (t, 0),\ \beta(t) = (0, t)$. Deduce that $\int dx\,dy$ is not a homotopy functional.

Exercise 4. Show that if $w_1, \dots, w_r$ are closed A-valued 1-forms on M with the property that $w_j \wedge w_{j+1} = 0$ for $j = 1, \dots, r-1$, then

$$\int w_1 w_2 \dots w_r : PM \to A$$

is a homotopy functional. In particular, if M is a Riemann surface and each w_j is holomorphic, then $\int w_1 \dots w_r$ is a homotopy functional.

1.4. **Basic Properties of Iterated Integrals.** The most basic property of iterated integrals is naturality, which is easily proved using the definition.

Proposition 4. *Suppose that $f : M \to N$ is a smooth mapping between smooth manifolds. If $w_1, w_2, \dots, w_r \in E^1(N)$ and $\alpha \in PM$, then*

$$\int_{f\circ\alpha} w_1 w_2 \dots w_r = \int_\alpha f^* w_1 f^* w_2 \dots f^* w_r. \quad \square$$

The next three properties of iterated integrals are of a combinatorial nature and reflect the combinatorics of simplices. They are formulas for how to evaluate an iterated integral on the product of two paths, how to pointwise multiply two iterated integrals (as functions on PM) and how to evaluate an iterated integral on the inverse of a path.

Our model for the standard r-simplex is the *time ordered r-simplex*:

$$\Delta^r = \{(t_1, t_2, \dots, t_r) \in \mathbb{R}^r : 0 \le t_1 \le t_2 \le \cdots \le t_r \le 1\}.$$

The definition of a basic iterated line integral may be restated as:

$$\int_\gamma w_1 \cdots w_r = \int_{\Delta^r} (p_1^*\gamma^* w_1) \wedge (p_2^*\gamma^* w_2) \wedge \cdots \wedge (p_r^*\gamma^* w_r) \tag{2}$$

where $p_j : \mathbb{R}^r \to \mathbb{R}$ denotes projection onto the j^{th} coordinate.

The two relevant combinatorial properties of simplices are established in the next two exercises.

Exercise 5. In this exercise, $t_0 = 0$ and $t_{r+1} = 1$. Show that

$$\Delta^r = \bigcup_{j=0}^{r} \big\{(t_1, t_2, \dots, t_r) : 0 \le t_1 \le \cdots \le t_j \le 1/2 \le t_{j+1} \le \cdots \le t_r\big\}$$

and that there is a natural identification of $\Delta^j \times \Delta^{r-j}$ with

$$\big\{(t_1, t_2, \dots, t_r) : 0 \le t_1 \le \cdots \le t_j \le 1/2 \le t_{j+1} \le \cdots \le t_r\big\}.$$

Suppose that r and s are two non-negative integers. A permutation σ of $\{1, 2, \dots, r+s\}$ is a *shuffle of type* (r, s) if

$$\sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(r) \text{ and}$$
$$\sigma^{-1}(r+1) < \sigma^{-1}(r+2) < \cdots < \sigma^{-1}(r+s).$$

To make sense out of this definition, it helps to note that $\sigma^{-1}(k)$ is the position of k in the ordered list

$$\sigma(1), \sigma(2), \sigma(3), \dots, \sigma(r+s).$$

Thus σ is a shuffle of type (r,s) if the numbers $1,2,3,\dots,r$ occur in order, and so do the numbers $r+1, r+2, \dots, r+s$. For example, the 6 shuffles of $\{1,2,3,4\}$ of type $(2,2)$ are

$$1234,\ 1324,\ 1342,\ 3124,\ 3142,\ 3412.$$

Denote the set of shuffles of type (r,s) by $\mathrm{Sh}(r,s)$. There are $\binom{r+s}{r}$ of these.

Exercise 6. View $\Delta^r \times \Delta^s$ as a subset of $\mathbb{R}^r \times \mathbb{R}^s = \mathbb{R}^{r+s}$. Show that

$$\Delta^r \times \Delta^s = \bigcup_{\sigma \in \mathrm{Sh}(r,s)} \{(t_1, t_2, \dots, t_{r+s}) : 0 \le t_{\sigma(1)} \le t_{\sigma(2)} \le \cdots \le t_{\sigma(r+s)} \le 1\}.$$

Proposition 5. *Suppose that $w_1, w_2, \cdots$ are smooth 1-forms on the manifold M. Then:*

Coproduct: *If $\alpha, \beta \in PM$ are composable (i.e., $\alpha(1) = \beta(0)$), then*

$$\int_{\alpha\beta} w_1 w_2 \dots w_r = \sum_{j=0}^{r} \int_\alpha w_1 \dots w_j \int_\beta w_{j+1} \cdots w_r.$$

Here we introduce and use the convention that $\int_\gamma \phi_1 \cdots \phi_k = 1$ when $k = 0$.

Product: *If $\alpha \in PM$, then*

$$\int_\alpha w_1 \dots w_r \int_\alpha w_{r+1} \cdots w_{r+s} = \sum_{\sigma \in \mathrm{Sh}(r,s)} w_{\sigma(1)} w_{\sigma(2)} \cdots w_{\sigma(r+s)}.$$

Antipode: *If $\alpha \in PM$, then*

$$\int_{\alpha^{-1}} w_1 w_2 \cdots w_r = (-1)^r \int_\alpha w_r w_{r-1} \cdots w_1.$$

These statements follow directly from the alternative definition (2) of iterated line integrals and the results of Exercises 5 and 6.

Example 6. If α and β are composable paths, then

$$\int_{\alpha\beta} w_1 w_2 w_3 = \int_\alpha w_1 w_2 w_3 + \int_\alpha w_1 w_2 \int_\beta w_3 + \int_\alpha w_1 \int_\beta w_2 w_3 + \int_\beta w_1 w_2 w_3,$$

$$\int_\alpha w_1 \int_\alpha w_2 w_3 = \int_\alpha w_1 w_2 w_3 + \int_\alpha w_2 w_1 w_3 + \int_\alpha w_2 w_3 w_1,$$

and

$$\int_{\alpha^{-1}} w_1 w_2 w_3 = -\int_\alpha w_3 w_2 w_1.$$

Exercise 7 (Commutator formula). Suppose that $\alpha, \beta \in P_{x,x}M$ and that $w_1, w_2 \in E^1(M)$. Show that

$$\int_{\alpha\beta\alpha^{-1}\beta^{-1}} w_1 w_2 = \begin{vmatrix} \int_\alpha w_1 & \int_\alpha w_2 \\ \int_\beta w_1 & \int_\beta w_2 \end{vmatrix}.$$

It is now clear that iterated line integrals can detect elements of $\pi_1(M, x)$ not visible in $H_1(M; \mathbb{R})$. For example, suppose $U = \mathbb{P}^1 - \{0, 1, \infty\}$ and

$$w_0 = \frac{dz}{z} \text{ and } w_1 = \frac{dz}{1-z} \in H^0(\Omega^1_U).$$

If σ_0 and σ_1 are generators of $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, 1/2)$ satisfying

$$\int_{\sigma_j} w_k = (-1)^k 2\pi i \delta_{jk},$$

then

$$\int_{\sigma_0\sigma_1\sigma_0^{-1}\sigma_1^{-1}} w_0 w_1 = \begin{vmatrix} \int_{\sigma_0} w_0 & \int_{\sigma_0} w_1 \\ \int_{\sigma_1} w_0 & \int_{\sigma_1} w_1 \end{vmatrix} = 4\pi^2.$$

Exercise 8 (Change of base point formula). Suppose that $\alpha \in P_{x,x}M$, $\gamma \in P_{y,x}M$ and that $w_1, w_2 \in E^1(M)$. Show that

$$\int_{\gamma\alpha\gamma^{-1}} w_1 w_2 = \int_\alpha w_1 w_2 + \begin{vmatrix} \int_\gamma w_1 & \int_\gamma w_2 \\ \int_\alpha w_1 & \int_\alpha w_2 \end{vmatrix}.$$

Iterated line integrals do not depend on the parameterization of paths. For two paths $\alpha, \beta \in P_{x,y}M$, write $\alpha \sim \beta$ if there exists $\phi \in P_{0,1}[0, 1]$ such that $\beta = \alpha \circ \phi$. This relation generates an equivalence relation on PM that we shall also denote by $\sim$.

The following property is easily proved using elementary calculus.

Proposition 7. *Iterated integrals* $\int w_1 w_2 \dots w_r : PM \to A$ *factor through the quotient mapping* $PM \to PM/\sim$. *That is, if* $\alpha, \beta \in PM$ *and* $\alpha \sim \beta$, *then*

$$\int_\alpha w_1 \dots w_r = \int_\beta w_1 \dots w_r. \quad \square$$

The set $(P_{x,x}M)/\sim$ has a well defined associative product

$$\left[(P_{x,x}M)/\sim\right] \times \left[(P_{x,x}M)/\sim\right] \to (P_{x,x}M)/\sim.$$

The identity is the constant path at x. We shall denote by 1_x. Set

$$P(M, x) = \coprod_{(P_{x,x}M)/\sim} \mathbb{Z}$$

This is an associative algebra whose elements are formal finite linear combinations

$$c = \sum_{\gamma} n_\gamma \gamma$$

Iterated integrals with values in A define functions

$$\int w_1 \dots w_r : P(M, x) \to A, \quad c \mapsto \langle \int w_1 \dots w_r, c \rangle.$$

Another fundamental property of iterated integrals is *nilpotence*.

Proposition 8 (Nilpotence). *If $r, s \geq 1$, $w_1, \dots, w_r \in E^1(M)$ and $\alpha_1, \dots, \alpha_s \in P(M, x)$, then*

$$\langle \int w_1, \dots, w_r, (\alpha_1 - 1_x)(\alpha_2 - 1_x) \cdots (\alpha_s - 1_x) \rangle = \begin{cases} \prod_{j=1}^r \int_{\alpha_j} w_j & r = s \\ 0 & s > r. \end{cases}$$

This generalizes the property (1) of standard line integrals, which is the case $r = 1$:

$$\begin{aligned} \langle \int w, (\alpha - 1_x)(\beta - 1_x) \rangle &= \langle \int w, \alpha\beta - \alpha - \beta + 1_x \rangle \\ &= \int_{\alpha\beta} w - \int_\alpha w - \int_\beta w + \int_{1_x} w \\ &= 0. \end{aligned}$$

The proof of this proposition contains some important and useful techniques due to Chen.

Proof. Denote the free associative $\mathbb{R}$-algebra generated by indeterminates $X_1, \dots, X_r$ by

$$\mathbb{R}\langle X_1, \dots, X_r \rangle$$

and its completion with respect to the ideal $I := (X_1, \dots, X_r)$ by

$$\mathbb{R}\langle\langle X_1, \dots, X_r \rangle\rangle$$

Elements of this ring are formal power series in the non-commuting indeterminates $X_1, \dots, X_r$.

Consider the function $T : PM \to \mathbb{R}\langle\langle X_1, \dots, X_r \rangle\rangle$ that takes γ to

$$1 + \sum_j \int_\gamma w_j X_j + \sum_{j,k} \int_\gamma w_j w_k X_j X_k + \sum_{j,k,l} \int_\gamma w_j w_k w_l X_j X_k X_l + \cdots .$$

The coproduct property of iterated integrals implies that if $\alpha, \beta \in PM$ are composable paths, then

$$T(\alpha\beta) = T(\alpha)T(\beta).$$

By linearity, this extends to an algebra homomorphism

$$T : P(M, x) \to \mathbb{R}\langle\langle X_1, \dots, X_r\rangle\rangle.$$

Since $T(\alpha) - 1$ is in the maximal ideal I,

$$T\big((\alpha_1 - 1_x)(\alpha_2 - 1_x)\cdots(\alpha_s - 1_x)\big) \in I^s.$$

The result follows by examining the coefficient of $X_1X_2 \dots X_r$. $\square$

1.5. **The Group Algebra and its Dual.** Suppose that π is a discrete group and R a commutative ring with 1. Denote the group algebra of π over R by $R\pi$. This is the set of all *finite* linear combinations

$$\sum_{g\in\pi} r_g g$$

where $r_g \in R$. The *augmentation* is the homomorphism $\epsilon : R\pi \to R$ defined by

$$\epsilon : \sum_{g\in\pi} r_g g \mapsto \sum_{g\in\pi} r_g.$$

The kernel of ϵ is called the *augmentation ideal* and denoted J_R. (We will denote it by J when R is clear from context.) The powers of J_R

$$R\pi = J_R^0 \supseteq J_R \supseteq J_R^2 \supseteq J_R^3 \supseteq \cdots \tag{3}$$

define a topology — called the J*-adic topology* — on $R\pi$. Note that this topology is frequently not separated — that is, the intersection of the powers of J_R is not always trivial. The J-adic completion of $R\pi$ is

$$R\pi\hat{} := \varprojlim_m R\pi/J^m.$$

It is a topological R-algebra.

Exercise 9. Show that the function

$$\pi^{\mathrm{ab}} \to J_R/J_R^2, \quad g \mapsto (g-1) + J_R^2$$

is a homomorphism and induces an isomorphism

$$\pi^{\mathrm{ab}} \otimes_{\mathbb{Z}} R \cong J_R/J_R^2.$$

Hint: first prove the case where $R = \mathbb{Z}$.

Note that $\pi^{\mathrm{ab}} \otimes_{\mathbb{Z}} R = H_1(\pi; R)$, which is isomorphic to $H_1(X; R)$ when π is the fundamental group of a path connected space X.

Exercise 10. Show that the graded algebra

$$\bigoplus_{m=0}^{\infty} J_R^m/J_R^{m+1}$$

is generated by J_R/J_R^2. Deduce that a section of the projection $J_R{}^\frown \to J_R/J_R^2$ induces an algebra homomorphism

$$T(J_R/J_R^2) \to R\pi^\frown$$

with dense image, where

$$T(V) := R \oplus \bigoplus_{m>0} V^{\otimes m}$$

denotes the free associative R-algebra generated by the R-module V. Deduce that if $H_1(\pi; R)$ is a free R-module, then $R\pi^\frown$ is the quotient of the completed tensor algebra

$$T(H_1(\pi; R))^\frown$$

generated by $H_1(\pi; R)$ where the projection $T(H_1(\pi; R))^\frown \to R\pi^\frown$ induces the identity

$$H_1(\pi; R) \cong I/I^2 \to J/J^2 \cong H_1(\pi; R).$$

For a discrete R-module M, define

$$\mathrm{Hom}_R^{\mathrm{cts}}(R\pi, M) := \varinjlim_m \mathrm{Hom}_R(R\pi/J^m, M).$$

Exercise 11. Show that the continuous dual

$$\mathrm{Hom}_R^{\mathrm{cts}}(R\pi, R) = \mathrm{Hom}_R^{\mathrm{cts}}(\mathbb{Z}\pi, R)$$

is a commutative R-algebra whose product is pointwise multiplication of functions. Show that $g \mapsto g^{-1}$ induces a homomorphism

$$i : \mathrm{Hom}_R^{\mathrm{cts}}(R\pi, R) \to \mathrm{Hom}_R^{\mathrm{cts}}(R\pi, R)$$

and that multiplication

$$R\pi \otimes R\pi \to R\pi$$

is continuous and induces a coproduct

$$\Delta : \mathrm{Hom}_R^{\mathrm{cts}}(R\pi, R) \to \mathrm{Hom}_R^{\mathrm{cts}}(R\pi, R) \otimes \mathrm{Hom}_R^{\mathrm{cts}}(R\pi, R),$$

Together with the augmentation $\epsilon : \mathrm{Hom}_R^{\mathrm{cts}}(R\pi, R) \to R$ induced by evaluation at the identity, these give $\mathrm{Hom}_R^{\mathrm{cts}}(\mathbb{Z}\pi, R)$ the structure of an augmented commutative Hopf algebra.[5]

[5]An augmented bialgebra is an R-algebra $A \to R$ with a homomorphism, $\Delta : A \to A \otimes A$, called the *comultiplication*. A commutative Hopf algebra is an augmented bialgebra together with a homomorphism $i : A \to A$, called the *antipode*, which is compatible with the augmentation, multiplication and comultiplication. Rather than write down the axioms, I will simply say that the standard example is the coordinate ring of an affine algebraic group G — the coproduct is induced by multiplication $G \times G \to G$, the augmentation by evaluation at the identity, and the antipode by the inverse mapping $g \mapsto g^{-1}$.

Exercise 12. Show that J_R is the free R-module generated by the set

$$\{g-1 : g \in \pi,\ g \neq 1\}.$$

Deduce that every element of J_R^m is an R-linear combination of 'monomials'

$$(g_1-1)(g_2-1)\cdots(g_k-1)$$

of 'degree' $k \geq m$.

Dual to the filtration (3) is the filtration

$$R = B_0 \operatorname{Hom}_R^{\mathrm{cts}}(R\pi, R) \subseteq B_1 \operatorname{Hom}_R^{\mathrm{cts}}(R\pi, R) \subseteq B_2 \operatorname{Hom}_R^{\mathrm{cts}}(R\pi, R) \subseteq \cdots$$

of $\operatorname{Hom}_R^{\mathrm{cts}}(R\pi, R)$, where

$$B_m \operatorname{Hom}_R^{\mathrm{cts}}(R\pi, R) = \operatorname{Hom}_R^{\mathrm{cts}}(R\pi/J^{m+1}, R).$$

With these filtrations, $\operatorname{Hom}_R^{\mathrm{cts}}(R\pi, R)$ is a filtered Hopf algebra. That is the multiplication, comultiplication and antipode induce mappings

$$B_n \otimes B_m \to B_{m+n},\ B_n \to \sum_{j+k=n} B_j \otimes B_k \text{ and } B_m \to B_m.$$

1.6. **Chen's de Rham Theorem for the Fundamental Group.** Suppose that M is a connected manifold, that $x, y, z \in M$ and that $F = \mathbb{R}$ or $\mathbb{C}$. Denote the set of iterated integrals $PM \to F$ restricted to $P_{x,y}M$ by $\mathrm{Ch}(P_{x,y}M; F)$. The shuffle product formula implies that this is an F-algebra. The coproduct formula implies that the mapping

$$\mathrm{Ch}(P_{x,z}M; F) \to \mathrm{Ch}(P_{x,y}M; F) \otimes_F \mathrm{Ch}(P_{y,z}M : F) \tag{4}$$

given by

$$\int w_1 w_2 \ldots w_r \mapsto \sum_{j=0}^{r} \int w_1 \ldots w_j \otimes \int w_{j+1} \ldots w_r$$

is well defined and is dual to path multiplication

$$P_{x,y}M \times P_{y,z}M \to P_{x,z}M.$$

When $x = y$, this is augmented by evaluation at the constant loop 1_x. With this augmentation, product and coproduct, $\mathrm{Ch}(P_{x,x}M; F)$ is a commutative Hopf algebra.

Iterated integrals are naturally filtered by *length*. Denote the linear span of the $\int w_1 \dots w_r$ where $r \leq n$ by $L_n \operatorname{Ch}(P_{x,y}M; F)$. With these filtrations, $\operatorname{Ch}(P_{x,y}M; F)$ is a filtered Hopf algebra.[6]

Denote the subspace consisting of those iterated integrals that are homotopy functionals by $H^0(\operatorname{Ch}(P_{x,y}M; F))$. It is clearly a subring of $\operatorname{Ch}(P_{x,y}M; F)$ as the product of two homotopy functionals is a homotopy functional. The length filtration restricts to a length filtration $L_\bullet$ of $H^0(\operatorname{Ch}(P_{x,y}M; F))$.

Exercise 13. Show that the coproduct (4) and antipode restrict to a coproduct

$$H^0(\operatorname{Ch}(P_{x,z}M; F)) \to H^0(\operatorname{Ch}(P_{x,y}M; F)) \otimes_F H^0(\operatorname{Ch}(P_{y,z}PM : F))$$

and antipode

$$H^0(\operatorname{Ch}(P_{x,z}M; F)) \to H^0(\operatorname{Ch}(P_{x,y}M; F)).$$

Deduce that $H^0(\operatorname{Ch}(P_{x,x}M; F))$ is a filtered commutative Hopf algebra.

Integration induces a mapping

$$\int : H^0(\operatorname{Ch}(P_{x,y}M; F)) \to \operatorname{Hom}_F^{\mathrm{cts}}(\mathbb{Z}\pi_1(M, x), F). \tag{5}$$

This is injective, as the set of path components of $P_{x,x}M$ is $\pi_1(M, x)$ and as $H^0(\operatorname{Ch}(P_{x,y}M; F))$ is, by definition, a subset of functions on PM.

Exercise 14. Show that $\int$ is a Hopf algebra homomorphism that maps L_m into B_m.

One version of Chen's de Rham Theorem for the fundamental groups is:

Theorem 9 (Chen)**.** *The homomorphism (5) is surjective, and therefore an isomorphism of Hopf algebras. Moreover, it is an isomorphism of filtered Hopf algebras. That is, for each $m \geq 0$, integration induces an isomorphism*

$$L_m H^0(\operatorname{Ch}(P_{x,y}M; F)) \cong \operatorname{Hom}_F^{\mathrm{cts}}(\mathbb{Z}\pi_1(M, x)/J^{m+1}, F).$$

[6] It may appear that iterated integrals are graded by length. However, because of identities such as

$$\int w_1 \dots w_{j-1}(df) w_j \dots w_r$$
$$= \int w_1 \dots w_{j-1}(f w_j) w_{j+1} \dots w_r - \int w_1 \dots w_{j-1}(f w_{j-1}) w_j \dots w_r,$$

iterated integrals are only filtered by length.

We will prove a stronger version of this in the case where M is a Zariski open subset of $\mathbb{P}^1(\mathbb{C})$.

1.7. **Proof of Chen's Theorem when $M = \mathbb{P}^1(\mathbb{C}) - S$.** Suppose that S is a finite subset of $\mathbb{P}^1(\mathbb{C})$. If S is empty, then $\mathbb{P}^1(\mathbb{C})$ is simply connected, and there is nothing to prove. So we suppose that S is non-empty. Since $\operatorname{Aut}\mathbb{P}^1$ acts transitively, we may assume that $\infty \in S$:

$$S = \{a_1, \dots, a_N, \infty\}.$$

Set $U = \mathbb{P}^1(\mathbb{C}) - S$.

The holomorphic 1-forms on U with logarithmic poles on S

$$H^0(\Omega^1_{\mathbb{P}^1}(\log S))$$

has basis

$$w_j := \frac{dz}{z - a_j},\ j = 1, \dots, N.$$

Denote the set of iterated integrals built up from elements of

$$H^0(\Omega^1_{\mathbb{P}^1}(\log S))$$

by $\mathrm{Ch}(H^0(\Omega^1_{\mathbb{P}^1}(\log S)))$. By Exercise 4, these are all homotopy functionals.

Theorem 10. *For each $x \in U$, the composite*

$$\mathrm{Ch}(H^0(\Omega^1_{\mathbb{P}^1}(\log S))) \hookrightarrow H^0(\mathrm{Ch}(P_{x,x}U; \mathbb{C})) \hookrightarrow \mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Z}}(\mathbb{Z}\pi_1(U, x), \mathbb{C})$$

is a Hopf algebra isomorphism.

Set

$$A = \mathbb{C}\langle\langle X_1, \dots, X_N\rangle\rangle.$$

Define the augmentation $\epsilon : A \to \mathbb{C}$ by taking a power series to its constant term. The augmentation ideal $\ker \epsilon$ is the maximal ideal $I = (X_1, \dots, X_N)$. Consider the formal power series

$$T = 1 + \sum_j \int w_j X_j + \sum_{j,k} \int w_j w_k X_j X_k + \cdots$$
$$\in\ \mathrm{Ch}(H^0(\Omega^1_{\mathbb{P}^1}(\log S)))\langle\langle X_1, \dots, X_N\rangle\rangle,$$

where the coefficient of the monomial $X_{i_1} X_{i_2} \dots X_{i_r}$ is $\int w_{i_1} w_{i_2} \dots w_{i_r}$. We shall view this as an A-valued iterated integral. Since each coefficient of T is a homotopy functional, evaluating each coefficient on a path defines a mapping

$$\pi_1(U, x) \to A, \quad \gamma \mapsto \langle T, \gamma\rangle.$$

The coproduct property of iterated integrals implies that this is a homomorphism. It thus induces a homomorphism $\mathbb{C}\pi_1(U, x) \to A$.

The nilpotence property (Prop. 8) of iterated integrals implies that $\Theta(J^m) \subseteq I^m$, which implies that Θ is continuous. It therefore induces a homomorphism

$$\widehat{\Theta} : \mathbb{C}\pi_1(U,x)^\frown \to A$$

Proposition 11. *The mapping $\widehat{\Theta}$ is an isomorphism.*

Proof. By Exercise 10, $\mathbb{C}\pi_1(U,x)^\frown$ is the quotient of

$$T(H_1(\mathbb{P}^1(\mathbb{C}) - S))^\frown.$$

One thus has a commutative diagram

$$\begin{tikzcd} T(H_1(U;\mathbb{C}))^\frown \arrow[d, "\Phi"'] \arrow[dr, "\widehat{\Theta}\circ\Phi"] & \\ \mathbb{C}\pi_1(U,x)^\frown \arrow[r, "\widehat{\Theta}"'] & A \end{tikzcd}$$

It is easy to check that $\widehat{\Theta} \circ \Phi$ induces an isomorphism on I/I^2 and is therefore an isomorphism. This implies that the coefficients of T span

$$\mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Z}}(\mathbb{Z}\pi_1(U,x), \mathbb{C}),$$

which completes the proof. $\square$

1.8. **The de Rham Theorem for the Fundamental Groupoid.** Chen's de Rham theorem generalizes to the fundamental groupoid. Suppose that $x, y \in M$, where M is a smooth manifold. The group $H_0(P_{x,y}M; R)$ is the free R-module generated by $\pi(M; x, y)$. When $x = y$, this is just the group algebra $R\pi_1(M,x)$.

Multiplication of paths gives $H_0(P_{x,y}M; R)$ the structure of a left $\pi_1(M,x)$-module and a right $\pi_1(M,y)$-module. Both of these modules are free of rank 1. Denote the augmentation ideal of $R\pi_1(M,z)$ by J_z.

Exercise 15. Show that for all $n \geq 1$,

$$J_x^n H_0(P_{x,y}M; R) = H_0(P_{x,y}M; R) J_y^n.$$

Denote their common value by $J^n H_0(P_{x,y}M; R)$ or $J_{x,y}$.

The filtration

$$H_0(P_{x,y}M; R) \supseteq J H_0(P_{x,y}M; R) \supseteq J^2 H_0(P_{x,y}M; R) \supseteq \cdots$$

defines a topology (the $J-adic$ topology) on $H_0(P_{x,y}M; R)$. The J-adic completion of $H_0(P_{x,y}M; R)$ is

$$H_0(P_{x,y}M; R)^\frown := \varprojlim_n H_0(P_{x,y}M; R)/J^n_{x,y}.$$

When F is $\mathbb{R}$ or $\mathbb{C}$, integration induces a mapping

$$H^0(\mathrm{Ch}(P_{x,y}M;F)) \to \mathrm{Hom}_{\mathbb{Z}}^{\mathrm{cts}}(H_0(P_{x,y}M),F). \tag{6}$$

Chen's de Rham theorem implies that this is an isomorphism of F-algebras. Moreover, the coproduct

$$H^0(\mathrm{Ch}(P_{x,y}M;F)) \to H^0(\mathrm{Ch}(P_{x,z}M;F)) \otimes H^0(\mathrm{Ch}(P_{z,y}M;F))$$

is dual to the product $H_0(P_{x,z}M) \otimes H_0(P_{z,y}M) \to H_0(P_{x,y}M)$.

Exercise 16. Prove that the choice of $\gamma \in \pi(M;x,y)$ gives an isomorphism

$$R\pi_1(M,x)/J_x^n \cong H_0(P_{x,y}M;R)/J_{x,y}^n.$$

Deduce that (6) is an isomorphism which restricts to isomorphisms

$$L_n H^0(\mathrm{Ch}(P_{x,y}M;F)) \to \mathrm{Hom}_{\mathbb{Z}}(H_0(P_{x,y}M)/J_{x,y}^{n+1},F).$$

Remark 12. When $U = \mathbb{P}^1(\mathbb{C}) - S$, as in the previous section, then T induces an isomorphism

$$\widehat{\Theta}_{x,y} : H_0(P_{x,y}U;\mathbb{C}) \to A$$

which is defined by taking $\gamma \in \pi(U;x,y)$ to $T(\gamma)$. This mapping is compatible with path multiplication. That $\widehat{\Theta}$ is an isomorphism follows directly from Proposition 11 and Exercise 16.

1.9. **Postscript.** You can learn more about iterated integrals in Chen's Bulletin paper [2] and my expository papers [11, 13]. The first two of these [2, 11] contain a more conceptual, though less direct, approach to properties of iterated integrals; [11] contains an elementary proof of Chen's de Rham Theorem for the fundamental group.

2. Iterated Integrals and Multiple Zeta Numbers

In this section we introduce multiple zeta numbers, develop some of their basic properties, and show how they occur as iterated integrals. Most of the material in this section is due to Zagier [24], Goncharov [10] and Racinet [19].

2.1. Iterated integrals and Multiple Zeta Numbers.

Multiple zeta numbers generalize the classical values of the Riemann zeta function at integers larger than 1. They were first considered by Euler. They have recently resurfaced in the works of Zagier [24] and Goncharov [10].

Definition 13. For positive integers $n_1, \dots, n_r$, where $n_r > 1$, define

$$\zeta(n_1, \dots, n_r) = \sum_{0<k_1<\dots<k_r} \frac{1}{k_1^{n_1} k_2^{n_2} \dots k_r^{n_r}}.$$

The integer r is called the *depth* of the multiple zeta number, and $n_1 + \dots + n_r$ its *weight.*

Exercise 17. Show that

$$\zeta(n_1)\zeta(n_2) = \zeta(n_1, n_2) + \zeta(n_1 + n_2) + \zeta(n_2, n_1)$$

and that

$$\begin{aligned}\zeta(n_1, n_2)\zeta(n_3) &= \zeta(n_1, n_2, n_3)\\ &+ \zeta(n_1, n_2 + n_3) + \zeta(n_1, n_3, n_2) + \zeta(n_1 + n_3, n_2) + \zeta(n_3, n_1, n_2).\end{aligned}$$

Note that how, in the second relation, the z_3 "percolates" left. It may occupy the same position as n_1 or n_2, but n_1 and n_2 cannot occupy the same position.

2.2. Percolation Relations.

The combinatorics of the domain of summation of multiple zeta numbers is similar to the combinatorics of time ordered simplices. Because of this, multiple zeta values satisfy shuffle-like relations that we shall call *percolation relations.*

Definition 14. Suppose that r, s are positive integers. A *percolant of type* (r, s) and depth d (a positive integer) is a function

$$p : \{1, 2, \dots, r + s\} \to \{1, 2, \dots, r + s\}$$

that surjects onto $\{1, 2, \dots, d\}$ and whose restrictions to $\{1, 2, \dots, r\}$ and $\{1, 2, \dots, s\}$ are strictly order preserving (and thus injective). Denote the depth d of p by $|p|$; it satisfies

$$\max(r, s) \le |p| \le r + s.$$

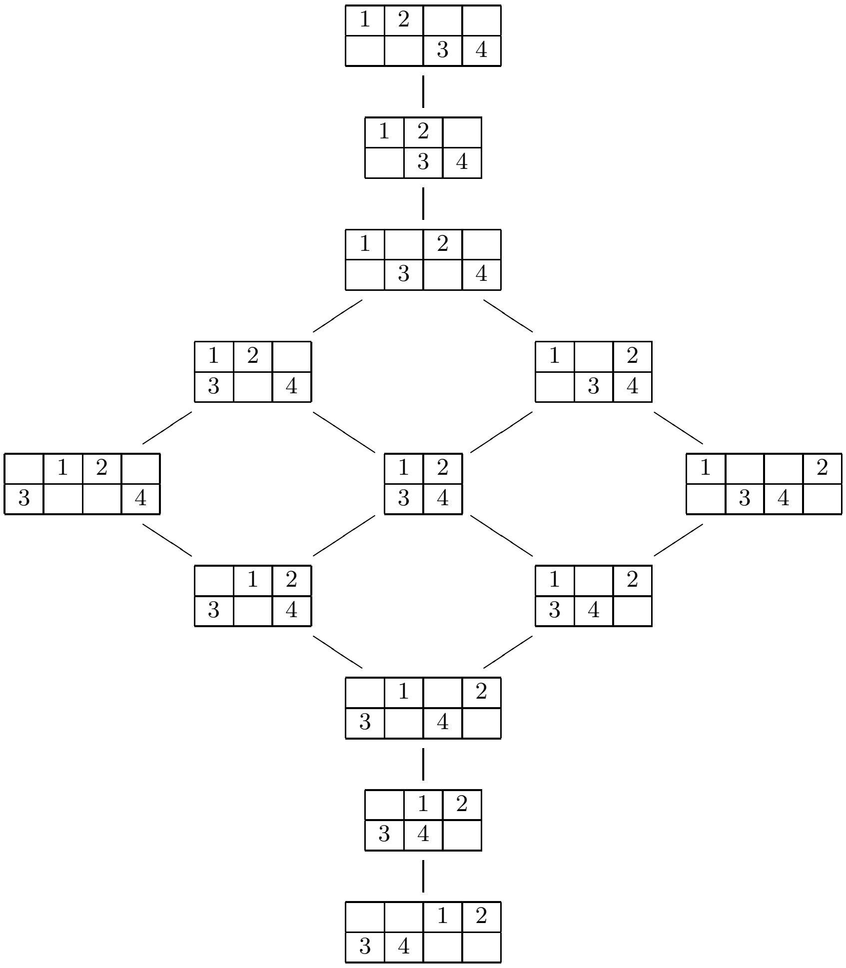
Denote the set of percolants of type (r, s) by $\operatorname{Perc}(r, s)$.

The percolants of depth $r+s$ are precisely the shuffles of type (r,s). The number of percolants of type (r,s) is

$$\sum_{d=\max(r,s)}^{r+s} \binom{d}{d-r \quad d-s} ;$$

the term corresponding to d is the number of percolants of depth d and type (r,s).

Example 15. The partially ordered set of the 13 percolants of type $(2,2)$ is:



Here the number of columns of the box corresponding to a percolant p is its depth. The set of numbers appearing in the j^{th} column of the diagram corresponding to p is $p^{-1}(j)$. The edges in the diagram are "elementary moves."

Exercise 18. Show that if $(k_1, \dots, k_r) \in \mathbb{Z}^r$ and $(k_{r+1}, \dots, k_{r+s}) \in \mathbb{Z}^s$ satisfy

$$0 < k_1 < k_2 < \cdots < k_r \text{ and } 0 < k_{r+1} < k_{r+2} < \cdots < k_{r+s},$$

then there is a unique percolant p of type (r, s) such that

$$k_{p(1)} \leq k_{p(2)} \leq \cdots \leq k_{p(r+s)},$$

where $k_{p(j_1)} = k_{p(j_2)}$ if and only if $p(j_1) = p(j_2)$.

Exercise 19. Show that

$$\zeta(n_1, \dots, n_r)\zeta(n_{r+1}, \dots, n_{r+s}) = \sum_{p \in \mathrm{Perc}(r,s)} \zeta(p^* \boldsymbol{z})$$

where the j^{th} coordinate of $p^*\boldsymbol{z} \in \mathbb{C}^{|p|}$ is $\sum_{k \in p^{-1}(j)} z_k$.

Example 16. The formula for $\zeta(n_1, n_2)\zeta(n_3, n_4)$ has 13 terms, one for each of the percolants listed in Example 15:

$$\begin{aligned}
\zeta(n_1, n_2)\zeta(n_3, n_4) = \zeta(n_1, n_2, n_3, n_4) &+ \zeta(n_1, n_3, n_2, n_4) + \zeta(n_3, n_1, n_2, n_4) \\
+ \zeta(n_1, n_3, n_4, n_2) &+ \zeta(n_3, n_1, n_4, n_2) + \zeta(n_3, n_4, n_1, n_2) \\
&+ \zeta(n_1, n_2 + n_3, n_4) + \zeta(n_1 + n_3, n_2, n_4) \\
&+ \zeta(n_1, n_3, n_2 + n_4) + \zeta(n_3, n_1, n_2 + n_4) \\
&+ \zeta(n_1 + n_3, n_4, n_2) + \zeta(n_3, n_1 + n_4, n_2) \\
&+ \zeta(n_1 + n_3, n_2 + n_4).
\end{aligned}$$

Since the product of two multiple zeta numbers is a $\mathbb{Z}$-linear combination of multiple zeta numbers, the $\mathbb{Q}$-linear span of the multiple zeta numbers (including 1, the mixed zeta number of weight 0) in $\mathbb{R}$ is a subalgebra MZN. Since $\zeta(2) = \pi^2/6$,

$$\mathrm{MZN}_{\mathbb{C}} := \mathrm{MZN} \oplus i\pi\mathrm{MZN}$$

is a $\mathbb{Q}$-subalgebra of $\mathbb{C}$ that contains $(2\pi i)^n$ for all $n \geq 1$. We also have the graded $\mathbb{Q}$-algebra

$$\mathrm{MZN}_{\bullet} = \bigoplus_{m \geq 0},$$

where MZN_m is the $\mathbb{Q}$-linear span of the mixed zeta numbers of weight m. This has an increasing filtration by depth.

2.3. **Multiple Zeta Numbers as Periods of Iterated Integrals.** Multiple zeta numbers can also be expressed as periods of iterated integrals. The shuffle product and antipode formulas for iterated integrals then give further relations between multiple zeta numbers.

Exercise 20 (cf. [24]). Let $U = \mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}$ and

$$w_0 = \frac{dz}{z} \text{ and } w_1 = \frac{dz}{1-z} \in H^0(\Omega^1_U).$$

Suppose that $i_1, \ldots, i_r \in \{0, 1\}$. We say that the iterated integral

$$\int_0^1 w_{i_1} \ldots w_{i_r}$$

converges to L if

$$\lim_{\substack{\epsilon \to 0^+ \\ \delta \to 0^+}} \int_{[\epsilon, 1-\delta]} w_{i_1} \ldots w_{i_r}$$

exists and equals L.[7] Show that

$$\int_0^1 w_{i_1} \ldots w_{i_r}$$

converges if and only if $i_1 = 1$ and $i_r = 0$. Show that if $n_r > 1$, then

$$\int_{[0,x]} w_1 \overbrace{w_0 \ldots w_0}^{n_1 - 1} w_1 \overbrace{w_0 \ldots w_0}^{n_2 - 1} w_1 \ldots w_1 \overbrace{w_0 \ldots w_0}^{n_r - 1} = \sum_{0 < k_1 < \cdots < k_r} \frac{x^{k_r}}{k_1^{n_1} k_2^{n_2} \ldots k_r^{n_r}}.$$

Deduce that

$$\zeta(n_1, \ldots, n_r) = \int_0^1 w_1 \overbrace{w_0 \ldots w_0}^{n_1 - 1} w_1 \overbrace{w_0 \ldots w_0}^{n_2 - 1} w_1 \ldots w_1 \overbrace{w_0 \ldots w_0}^{n_r - 1}.$$

The antipode and naturality formulas satisfied by iterated integrals (Prop. 5) give relations between multiple zeta numbers. The automorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ defined by $f(z) = 1 - z$ has the property that

$$f_*[0, 1] = [0, 1]^{-1} = [1, 0] \text{ and } f^* w_a = -w_{a+1} \text{ for } a \in \mathbb{Z}/2Z.$$

[7]The path $t \mapsto (1 - t)a + tb$ in $\mathbb{C}$ from a to b will be denoted by $[a, b]$.

Consequently, if $w = n_1 + \cdots + n_r$, then

$$
\begin{aligned}
&\zeta(n_1, \ldots, n_r) \\
&= \int_0^1 w_1 \overbrace{w_0 \ldots w_0}^{n_1-1} w_1 \overbrace{w_0 \ldots w_0}^{n_2-1} w_1 \ldots w_1 \overbrace{w_0 \ldots w_0}^{n_r-1} \\
&= (-1)^w \int_{[0,1]} f^* w_0 \overbrace{f^* w_1 \ldots f^* w_1}^{n_1-1} f^* w_0 \ldots f^* w_0 \overbrace{f^* w_1 \ldots f^* w_1}^{n_r-1} \\
&= (-1)^w \int_{f_*[0,1]} w_0 \overbrace{w_1 \ldots w_1}^{n_1-1} w_0 \ldots w_0 \overbrace{w_1 \ldots w_1}^{n_r-1} \\
&= \int_{[0,1]} \overbrace{w_1 \ldots w_1}^{n_r-1} w_0 \ldots w_0 \overbrace{w_1 \ldots w_1}^{n_1-1}.
\end{aligned}
$$

This is a multiple zeta number. For example

$$\zeta(3) = \int_0^1 w_1 w_0 w_0 = \int_0^1 w_1 w_1 w_0 = \zeta(1,2).$$

More generally,

(7)

$$\zeta(m_1+1, m_2+1, \ldots, m_r+1) = \zeta(\overbrace{1,\ldots,1,2}^{m_r}, \ldots, \overbrace{1,\ldots,1,2}^{m_2}, \overbrace{1,\ldots,1,2}^{m_1}).$$

Writing multiple zeta numbers as iterated integrals and using the shuffle product formula also gives formulas for their products. Surprisingly, these are different from those given by the percolation formula. This leads to interesting (and mysterious) relations between multiple zeta numbers. For example, the shuffle product formula gives

$$
\begin{aligned}
\zeta(2)^2 &= \int_0^1 w_1 w_0 \int_0^1 w_1 w_0 \\
&= 2\int_0^1 w_1 w_0 w_1 w_0 + 4\int_0^1 w_1 w_1 w_0 w_0 \\
&= 2\zeta(2,2) + 4\zeta(1,3).
\end{aligned}
$$

On the other hand, the percolant formula gives

$$\zeta(2)^2 = 2\zeta(2,2) + \zeta(4).$$

Combining these with (7), we see that

$$\zeta(4) = 4\zeta(1,3) = 4\zeta(1,1,2).$$

These are examples of the mysterious *double shuffle relations*, which are studied in detail by Racinet in [19].

2.4. **Polylogarithms.** A multivalued function on a topological space is simply a function on some (unramified) covering of the space. When M is connected, a homotopy functional $\mathcal{F} : PM \to A$ and a choice of a base point $x_o \in M$ give rise to an A-multivalued function $\phi_{\mathcal{F}}$ on M: if $x \in M$ and γ is a path in M from x_o to x, then the value of $\phi_{\mathcal{F}}$ at x is $\mathcal{F}(\gamma)$. This depends only on the homotopy class of γ. You should think of $\phi_{\mathcal{F}}$ as the result of "analytically continuing" the germ of the function $\phi_{\mathcal{F}}|_V$ defined in a simply connected neighbourhood V of x_o along γ.

Example 17. Set $U = \mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}$ and let w_0 and w_1 denote the rational differentials on U defined in Exercise 20. By Exercise 4, $\int w_1 w_0$ is a homotopy functional on PU. Thus

$$x \mapsto \int_0^x w_1 w_0$$

is a multi-valued holomorphic function on U.[8] In fact, it is Euler's dilogarithm, whose principal branch in the unit disk is defined by

$$\ln_2(x) = \sum_{n \geq 1} \frac{x^n}{n^2}.$$

More generally, the k-logarithm

$$\ln_k(x) := \sum_{n \geq 1} \frac{x^n}{n^k} \qquad |x| < 1$$

can be expressed as the length k iterated integral

$$\int_0^x w_1 \overbrace{w_0 \cdots w_0}^{k-1}$$

From this integral expression, it is clear that $\ln_k$ can be analytically continued to a multi-valued function on $\mathbb{C} - \{0, 1\}$.

Note that $\zeta(k)$, the value of the Riemann zeta function at an integer $k > 1$, is the value $\ln_k(1)$ of the principal branch of $\ln_k(x)$ at $x = 1$. More information about iterated integrals and polylogarithms can be found, for example, in [12].

[8]Here we need to be careful about the path of integration. The path from 0 to x should not pass through 0 or 1 once it has left 0.

2.5. **Multiple Polylogarithms.** Multiple polylogarithms are to multiple zeta values as polylogarithms are to zeta values. Before defining multiple polylogarithms, we need a criterion for an iterated line integral of length ≤ 2 to be a homotopy functional.[9]

Exercise 21. Suppose that $\xi, \phi_1, \dots, \phi_r$ are closed scalar-valued 1-forms on M and that a_{jk} are scalars. Show that

$$\sum_{j,k} a_{jk} \int \phi_j \phi_k + \int \xi + \text{ a constant}$$

is a homotopy functional if and only if

$$d\xi + \sum_{j,k} a_{jk}\, \phi_j \wedge \phi_k = 0.$$

Hint: pullback to the universal covering of M.

The multiple polylogarithm $L_{m_1,\dots,m_n}$ is defined by analytically continuing the holomorphic function defined on the unit polydisk in $\mathbb{C}^n$ by

$$L_{m_1,\dots,m_n}(x_1,\dots,x_n) := \sum_{0<k_1<\dots<k_n} \frac{x_1^{k_1} x_2^{k_2} \dots x_n^{k_n}}{k_1^{m_1} k_2^{m_2} \dots k_n^{m_n}} \qquad |x_j| < 1.$$

Note that the value at $(1,1,\dots,1)$ of this branch is $\zeta(n_1,\dots,n_m)$.

Like polylogarithms, multiple polylogarithms can be expressed as iterated integrals. For example,

$$L_{1,1}(x,y) = \int_{(0,0)}^{(x,y)} \left(\frac{dy}{1-y}\frac{dx}{1-x} + \frac{d(xy)}{1-xy}\left(\frac{dy}{1-y} - \frac{dx}{1-x} - \frac{dx}{x}\right)\right).$$

This expression defines a well defined multi-valued function on

$$\mathbb{C}^2 - \{(x,y) : xy(1-x)(1-y)(1-xy) = 0\}$$

as the relation

$$\frac{dy}{1-y} \wedge \frac{dx}{1-x} + \frac{d(xy)}{1-xy} \wedge \left(\frac{dy}{1-y} - \frac{dx}{1-x} - \frac{dx}{x}\right) = 0$$

holds in the rational 2-forms on $\mathbb{C}^2$. Similar formulas for all multiple polylogarithms can be found in Zhao's paper [25].

2.6. **Postscript.** You can learn more about multiple zeta numbers and multiple polylogarithms in the papers of Zagier [24], Goncharov [9, 10], Racinet [19] and Zhao [25], and also in [12].

[9]There is a more general formula valid for all iterated line integrals. See [2] for details.

3. Mixed Hodge-Tate Structures and Their Periods

The remaining two sections are devoted to explaining how multiple zeta numbers occur periods of the mixed Hodge structure on the fundamental group of $\mathbb{P}^1(\mathbb{C})-\{0,1,\infty\}$ with respect to a suitable asymptotic base point. In this section, we introduce mixed Hodge-Tate structures, develop their basic properties, and construct the mixed Hodge-Tate structure on the de Rham fundamental group of $\mathbb{P}^1(\mathbb{C})-\{0,1,\infty\}$.

Throughout this section, Λ will denote $\mathbb{Z}$ or $\mathbb{Q}$.

3.1. **Preamble.** Despite the new terminology we are about to introduce, the idea of periods is natural and comprehensible. The basic idea, which you should keep in mind while reading this section, is simple — one has two finite dimensional rational vector spaces V and V^{DR} together with an isomorphism

$$\Phi : V_{\mathbb{C}} \xrightarrow{\simeq} V^{\mathrm{DR}}_{\mathbb{C}}$$

of their complexifications. The vector space V is to be thought of as arising from topology and V^{DR} as its de Rham analogue, consisting of algebraic differential forms defined over $\mathbb{Q}$. The isomorphism Φ typically arises from a de Rham type theorem and is defined by integration.

The periods of such a structure are the entries of the matrix of Φ with respect to $\mathbb{Q}$-bases of V and V^{DR}. Typically, they are given by integrals of $\mathbb{Q}$-rational differential forms over topological cycles. They measure the degree to which the two rational structures differ. Abstractly, one can describe a matrix entry of ϕ as a number of the form

$$\langle \phi, \Phi(v)\rangle \in \mathbb{C}$$

where $v \in V$ and $\phi : V^{\mathrm{DR}} \to \mathbb{Q}$.

In our case, we will take

$$V = \mathrm{Hom}(\mathbb{Q}\pi_1(\mathbb{P}^1(\mathbb{C}) - \{0,1,\infty\}, x)/J^{n+1}, \mathbb{Q})$$

and

$$V^{\mathrm{DR}} = \begin{Bmatrix} \mathbb{Q}\text{-linear combinations of iterated} \\ \text{integrals of } w_0 \text{ and } w_1 \text{ of length} \leq n \end{Bmatrix}$$

where $w_0 = dz/z$ and $w_1 = dz/(1-z)$. The isomorphism of their complexifications is given by Chen's de Rham Theorem. The periods of V will be $\mathbb{Q}$-linear combinations of complex numbers of the form

$$\int_\gamma w_{j_1} w_{j_2} \dots w_{j_r} \in \mathbb{C}$$

where $\gamma \in \pi_1(\mathbb{P}^1(\mathbb{C}) - \{0,1,\infty\}, x)$, $r \leq n$ and $j_k \in \{0,1\}$.

In practice, V and V^{DR} will be endowed with additional structure — "weight filtrations" of V and V^{DR}, and a "Hodge filtration" of V^{DR}.

The periods will be computed with respect to bases adapted to these filtrations. When x is taken to be a suitable "asymptotic base point," the periods will be multiple zeta numbers.

3.2. **Mixed Hodge-Tate Structures.** [10] A Λ-mixed Hodge-Tate structure V consists of a finitely generated Λ-module V_Λ and two filtrations. The first is an increasing filtration

$$\cdots \subseteq W_{2m-2}V_\mathbb{Q} \subseteq W_{2m}V_\mathbb{Q} \subseteq W_{2m+2}V_\mathbb{Q} \subseteq \cdots$$

of $V_\mathbb{Q} := \mathbb{V}_\Lambda \otimes_\Lambda \mathbb{Q}$. It is called the *weight filtration* of V and is denoted $W_\bullet V_\mathbb{Q}$. It induces a filtration of $V_\mathbb{C}$ by extension of scalars. The second is a decreasing filtration

$$\cdots \supseteq F^{p-1}V_\mathbb{C} \supseteq F^pV_\mathbb{C} \supseteq F^{p+1}V_\mathbb{C} \supseteq \cdots$$

of $V_\mathbb{C} := V_\Lambda \otimes_\Lambda \mathbb{C}$. It is called the *Hodge filtration* of V. The two filtrations are required to satisfy the condition

$$V_\mathbb{C} = \bigoplus_{m\in\mathbb{Z}} F^m V_\mathbb{C} \cap W_{2m}V_\mathbb{C}. \tag{8}$$

The most basic examples are the Hodge-Tate structures $\Lambda(n)$, where $n \in \mathbb{Z}$. These are the Hodge theoretic analogues of the Galois modules $\Lambda_\ell(n)$. In the Hodge case, the underlying Λ-module is the subgroup $(2\pi i)^n\Lambda$ of $\mathbb{C}$. Its complexification, $\Lambda(n)_\mathbb{C}$, is identified with $\mathbb{C}$ via the inclusion $(2\pi i)^n\Lambda \hookrightarrow \mathbb{C}$. The Hodge and weight filtrations are defined by

$$W_{-2n-2}\mathbb{Q}(n) = 0, \quad W_{-2n}\mathbb{Q}(n) = \mathbb{Q}(n)$$

and

$$F^{-n}\Lambda(n)_\mathbb{C} = \Lambda(n)_\mathbb{C} \quad F^{-n+1}\Lambda(n)_\mathbb{C} = 0.$$

Once we have defined tensor products and duals of mixed Hodge-Tate structures, it will be clear that $\Lambda(n) = \Lambda(1)^{\otimes n}$ and that $\Lambda(-n)$ is the dual of $\Lambda(n)$. Thus all $\Lambda(n)$ are obtained from $\Lambda(1)$ by tensor powers and duals.

Example 18. The Hodge-Tate structure $\mathbb{Z}(1)$ occurs naturally as

$$\pi_1(\mathbb{G}_m(\mathbb{C}), \mathrm{id}) = H_1(\mathbb{C}^*; \mathbb{Z}).$$

Its complexification is the dual of

$$H^1(\mathbb{C}^*; \mathbb{C}) = \mathbb{C}\,\frac{dz}{z},$$

[10] The terminology "mixed Hodge-Tate structure" is not standard. In standard terminology, a Λ-mixed Hodge-Tate structure is a Λ-mixed Hodge structure all of whose weight graded quotients are direct sums of the Hodge-Tate structures $\Lambda(n)$. Here I have taken a more direct approach.

where z is the standard coordinate on $\mathbb{G}_m$. The integral lattice is generated by the class of the positively oriented unit circle σ in $\mathbb{C}^*$. Denote the generator of $H_1(\mathbb{C}^*;\mathbb{C})$ dual to dz/z by Z:

$$\langle \frac{dz}{z}, Z\rangle = 1.$$

Then $H_1(\mathbb{C}^*;\mathbb{C}) = \mathbb{C}Z$ and the inclusion $H_1(\mathbb{C}^*;\mathbb{Z}) \hookrightarrow H_1(\mathbb{C}^*;\mathbb{C})$ takes σ to

$$\langle \frac{dz}{z}, \sigma\rangle Z = \int_\sigma \frac{dz}{z}\, Z = 2\pi i Z.$$

Thus, we can identify $H_1(\mathbb{C}^*,\mathbb{Z})$ with the subgroup $2\pi i\mathbb{Z}\ Z$ of $\mathbb{C}Z = H_1(\mathbb{C}^*;\mathbb{C})$. The weight filtration is defined by

$$0 = W_{-4}H_1(\mathbb{C}^*;\mathbb{Q}) \subseteq W_{-2}H_1(\mathbb{C}^*;\mathbb{Q}) = H_1(\mathbb{C}^*;\mathbb{Q})$$

and the Hodge filtration by

$$H_1(\mathbb{C}^*;\mathbb{C}) = F^{-1}H_1(\mathbb{C}^*;\mathbb{C}) \supseteq F^0H_1(\mathbb{C}^*;\mathbb{C}) = 0.$$

These definitions should make more sense in a moment. But, at least heuristically, F^p consists of classes represented by differential forms, each of whose terms is a product of $\geq p$ differentials of the form df/f. Since Z is the dual of dz/z, it lies in $F^{-1}H_1(\mathbb{C}^*)$.

A more general example is where $U = \mathbb{P}^1(\mathbb{C}) - S$, a Zariski open subset of the projective line. In this case[11]

$$H_1(U) \cong \mathbb{Z}(1)^{|S|-1} \text{ and } H^1(U) \cong \mathbb{Z}(-1)^{|S|-1}.$$

This follows as

$$H^1(U;\mathbb{C}) = \{d\log f : f \text{ is an invertible regular function on } U\}.$$

3.3. **The Mixed Hodge-Tate Structure on the Dual of the Fundamental Group.** Suppose that U is a Zariski open subset of $\mathbb{P}^1(\mathbb{C})$. Given Theorem 10, the construction of a natural mixed Hodge structure on the dual of the truncated group ring $\mathbb{Z}\pi_1(U,x)/J^{n+1}$ is straightforward. We use the notation of Section 1.7 — $U = \mathbb{P}^1(\mathbb{C}) - S$, where S is a non-empty finite subset of $\mathbb{P}^1(\mathbb{C})$ that contains ∞:

$$S = \{a_1, \dots, a_N, \infty\}.$$

The group $H^1(U;\mathbb{C})$ is isomorphic to

$$H^0(\Omega^1_{\mathbb{P}^1}(\log S))$$

and has basis

$$w_j := \frac{dz}{z - a_j},\ j = 1, \dots, N.$$

[11]Direct sums of mixed Hodge-Tate structures are defined in the obvious way.

We shall construct a natural mixed Hodge-Tate structure on

$$V_{\mathbb{Z}} = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\pi_1(U,x)/J^{n+1}, \mathbb{Z}).$$

We identify its complexification

$$V_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\pi_1(U,x)/J^{n+1}, \mathbb{C}),$$

with $L_n \operatorname{Ch}(H^0(\Omega^1_{\mathbb{P}^1}(\log S)))$ via Theorem 10. Define the weight filtration by

$$W_{2m}V_{\mathbb{Q}} = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\pi_1(U,x)/J^{m+1}, \mathbb{Q}).$$

Note that, by Chen's de Rham Theorem, the complexified weight filtration is simply the length filtration:

$$W_{2m}V_{\mathbb{C}} = L_m \operatorname{Ch}(H^0(\Omega^1_{\mathbb{P}^1}(\log S))).$$

The p^{th} term of the Hodge filtration is defined to be the linear span of the iterated integrals in $L_n \operatorname{Ch}(H^0(\Omega^1_{\mathbb{P}^1}(\log S)))$ of length $\geq p$.[12] The group

$$F^m V_{\mathbb{C}} \cap W_{2m} V_{\mathbb{C}}$$

consists of those elements of $L_n \operatorname{Ch}(H^0(\Omega^1_{\mathbb{P}^1}(\log S)))$ of length exactly m. These filtrations define a mixed Hodge-Tate structure on V.

Note that the $2m^{\text{th}}$ weight graded quotient

$$\operatorname{Gr}^W_{2m} V := W_{2m}V/W_{2m-2}V$$

has complexification the space of iterated integrals

$$\Big\{ \int w_{j_1} w_{j_2} \dots w_{j_m} : w_{j_k} \in H^0(\Omega^1_{\mathbb{P}^1}(\log S)) \Big\} \cong H^1(U;\mathbb{C})^{\otimes m}$$

of length exactly m. The integral lattice of $2m^{\text{th}}$ weight graded quotient is

$$\mathrm{Hom}_{\mathbb{Z}}(J^m_{\mathbb{Z}}/J^{m+1}_{\mathbb{Z}}, \mathbb{Z}).$$

It follows from the nilpotence property (Prop. 8) that

$$\operatorname{Gr}^W_{2m} V \cong H^1(U)^{\otimes m} \cong \mathbb{Z}(-m)^{\oplus N^m}.$$

We will shortly see that this property holds more generally — the weight graded quotients of $\mathbb{Q}$-mixed Hodge-Tate structures are always direct sums of $\mathbb{Q}(n)$s.

[12] Unlike in the general case, the iterated integrals $\operatorname{Ch}(H^0(\Omega^1_{\mathbb{P}^1}(\log S)))$ are graded by length. This can be deduced easily from the fact that $w_1, \dots, w_N$ are linearly independent in $H^1(U;\mathbb{C})$ and from nilpotence property (Prop. 8) of iterated integrals. Thus, in this case, it makes sense to talk about iterated integrals of length m.

Remark 19. The directed system

$$\begin{aligned}(9)\quad 0 \hookrightarrow \mathbb{Z} \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\pi_1(U,x)/J^2,\mathbb{Z}) \hookrightarrow \cdots \\ \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\pi_1(U,x)/J^n,\mathbb{Z}) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\pi_1(U,x)/J^{n+1},\mathbb{Z}) \hookrightarrow \cdots\end{aligned}$$

is compatible with the Hodge and weight filtrations, and is a directed system in the category of $\mathbb{Z}$-mixed Hodge-Tate structures. It is common and convenient to speak of the mixed Hodge structure on the direct limit

$$\mathrm{Hom}_{\mathbb{Z}}^{\mathrm{cts}}(\mathbb{Z}\pi_1(U,x),\mathbb{Z})$$

of this system. Its complexification is $\mathrm{Ch}(H^0(\Omega^1_{\mathbb{P}^1}(\log S)))$. Its product, coproduct and antipode are all morphisms of mixed Hodge-Tate structures.

3.4. **Basic Properties of Mixed Hodge-Tate Structures.** Mixed Hodge-Tate structures form a category with particularly nice properties. Although abstract, the properties are important and provide powerful and useful computational tools.

Definition 20. A morphism $\phi : A \to B$ between mixed Hodge-Tate structures consists of a homomorphism $\phi : A_\Lambda \to B_\Lambda$ such that the induced homomorphisms $A_\mathbb{Q} \to B_\mathbb{Q}$ and $A_\mathbb{C} \to B_\mathbb{C}$ preserve the weight and Hodge filtrations, respectively.

Exercise 22. Suppose that V is the mixed Hodge-Tate structure on $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\pi_1(U,x)/J^{m+1},\mathbb{Z})$ defined above. Show that augmentation $V \to \mathbb{Z}(0)$, and the coproduct $V \to V \otimes V$ are morphisms of mixed Hodge-Tate structures.

A mixed Hodge-Tate structure on a finitely generated Λ-module can be thought of as a grading of its complexification that is compatible with the Hodge and weight filtrations.

Exercise 23. Show that the condition (8) implies that

$$(10)\quad \bigcap_{m\in\mathbb{Z}} W_{2m}V_\mathbb{Q} = 0,\ \bigcup_{m\in\mathbb{Z}} W_{2m}V_\mathbb{Q} = V_\mathbb{Q},\ \bigcap_{m\in\mathbb{Z}} F^mV_\mathbb{C} = 0,\ \bigcup_{m\in\mathbb{Z}} F^mV_\mathbb{C} = V_\mathbb{C}.$$

It is convenient to set $V^{p,p} = F^pV_\mathbb{C} \cap W_{2p}V_\mathbb{C}$ so that condition (8) becomes $V_\mathbb{C} = \oplus_{p\in\mathbb{Z}} V^{p,p}$. Show that

$$(11)\qquad W_{2m}V_\mathbb{C} = \bigoplus_{p\le m} V^{p,p} \text{ and } F^pV_\mathbb{C} = \bigoplus_{q\ge p} V^{q,q}.$$

Note, in particular, that $\Lambda(n)_\mathbb{C} = V^{-n,-n}$. This corresponds to the fact that $\Lambda(n)$ is the one dimensional Hodge structure of type $(-n,-n)$.

The following result should be surprising in view of the fact that the category of filtered vector spaces (and filtration preserving linear maps) is not abelian. (*Exercise*: explain this.)

Exercise 24. Suppose that A and B are mixed Hodge-Tate structures. Show that a homomorphism $\phi : A_\Lambda \to B_\Lambda$ is a morphism of mixed Hodge-Tate structures if and only if the induced homomorphism $\phi_\mathbb{C} : A_\mathbb{C} \to B_\mathbb{C}$ preserves the gradings: $\phi_\mathbb{C}(A^{p,p}) \subseteq B^{p,p}$ for all $p \in \mathbb{Z}$. Deduce that the category of mixed Hodge-Tate structures is abelian.[13]

The previous exercise implies that if V is a mixed Hodge-Tate structure, then so is each $W_{2m}V$ — the underlying Λ-module can be taken to be[14]

$$(W_{2m}V)_\Lambda = i^{-1}(W_{2m}V_\mathbb{Q}),$$

where $i : V_\Lambda \to V_\mathbb{Q}$ is the natural mapping.

By the previous exercise, the category of mixed Hodge-Tate structures is closed under taking subquotients. Thus each graded quotient

$$\operatorname{Gr}^W_m V := W_m V / W_{m-1} V$$

is again a mixed Hodge-Tate structure.

Exercise 25. Show that for each $m \in \mathbb{Z}$, the functor Gr^W_{2m} from the category of $\mathbb{Q}$-mixed Hodge-Tate structures to the category of $\mathbb{Q}$-vector spaces is exact.

Definition 21. We will say that $2m \in 2\mathbb{Z}$ is a *weight* of the mixed Hodge-Tate structure V if $\operatorname{Gr}^W_{2m} V_\mathbb{Q}$ is not zero. A mixed Hodge-Tate structure with only one weight $2m$ is said to be *pure* of weight $2m$.

For example, $\Lambda(m)$ is pure of weight $-2m$.

Exercise 26. Show that if V is a $\mathbb{Q}$-mixed Hodge-Tate structure that is pure of weight $-2m$, then V is isomorphic to a direct sum of copies of $\mathbb{Q}(m)$.

This is often expressed by saying that the category of pure $\mathbb{Q}$-mixed Hodge-Tate structures is *semi-simple*.

[13] This is straightforward when $\Lambda = \mathbb{Q}$. Remember that in an abelian category, a morphism that is monic and epi is an isomorphism. When $\Lambda = \mathbb{Z}$, you should think about the morphism $\times 2 : \mathbb{Z}(1) \to \mathbb{Z}(1)$. Is it an isomorphism? If not, what is its kernel? cokernel? Two important (though generally not well appreciated) points are, in order that the category of $\mathbb{Z}$-mixed Hodge-Tate structures be abelian, one is forced to (1) allow the underlying $\mathbb{Z}$-module $V_\mathbb{Z}$ to have torsion; (2) define the weight filtration on $V_\mathbb{Q}$ and not on $V_\mathbb{Z}$.

[14] There are several ways to induce a weight filtration on V_Λ, but no canonical choice.

3.5. **Duals and Tensor Products.** It is useful to make explicit the standard conventions for dualizing filtrations and for taking the tensor product of two filtered vector spaces.

Suppose that V is an R-module with an increasing filtration $G_\bullet$:

$$\cdots \subseteq G_{m-1}V \subseteq G_m V \subseteq G_{m+1}V \subseteq \cdots$$

by R-submodules.[15]

This filtration induces a decreasing filtration (denoted by $G^\bullet$) on the dual $\mathrm{Hom}_R(V, R)$:

$$G^m \mathrm{Hom}_R(V, R) := \mathrm{Hom}_R(V/G_{m-1}V, R).$$

Exercise 27. Show that when R is a field, there is a natural isomorphism

$$\mathrm{Gr}^m_G \mathrm{Hom}_R(V, R) \cong \mathrm{Hom}_R(\mathrm{Gr}^G_m V, R).$$

Suppose that V' and V'' are R-modules with increasing filtrations, $G'_\bullet$ and $G''_\bullet$. The tensor product filtration $G_\bullet = G'_\bullet \otimes G''_\bullet$ of $V' \otimes_R V''$ is defined by

$$G_m(V' \otimes_R V'') := \sum_{r+s=m} (G'_r V') \otimes_R (G''_s V'').$$

Exercise 28. Show that if R is a field, there is a natural isomorphism

$$\mathrm{Gr}^G_m(V' \otimes_R V'') \cong \bigoplus_{r+s=m} (\mathrm{Gr}^{G'}_r V') \otimes_R (\mathrm{Gr}^{G''}_s V'').$$

There are two equivalent ways to induce a filtration on $\mathrm{Hom}_R(V', V'')$. The first is to use the isomorphism

$$\mathrm{Hom}_R(V', V'') \cong \mathrm{Hom}_R(V', R) \otimes_R V'',$$

the second is to define directly

$$\begin{aligned} &G_m \mathrm{Hom}_R(V', V'') \\ &\qquad := \{\phi \in \mathrm{Hom}_R(V', V'') : \phi(G'_n V') \subseteq G''_{m+n} V'' \text{ for all } n\}. \end{aligned}$$

If V' and V'' are two Λ-mixed Hodge-Tate structures, then $V' \otimes V''$ is the Λ mixed Hodge structure whose underlying Λ-module is $V'_\Lambda \otimes_\Lambda V''_\Lambda$ and whose Hodge and weight filtrations are the tensor product of those of V' and V''.

Similarly, $\mathrm{Hom}(V', V'')$ is the Λ-mixed Hodge-Tate structure whose underlying Λ-module is $\mathrm{Hom}_\Lambda(V'_\Lambda, V''_\Lambda)$. The Hodge and weight filtrations are both induced by those of V' and V''.

[15] The case of decreasing filtrations follows using the trick of negating indices: if $G_\bullet$ is an increasing filtration of V, then the filtration $G^\bullet$ defined by $G^m V = G_{-m}V$ is decreasing, and vice-versa.

Exercise 29. Show that

$$\Lambda(n) \otimes \Lambda(m) \cong \Lambda(n+m) \text{ and } \operatorname{Hom}(\Lambda(n), \Lambda(m)) \cong \Lambda(m-n).$$

The next exercise gives the relationship between the two different Hom sets in the category of mixed Λ-mixed Hodge-Tate structures.

Exercise 30. For a mixed Λ-Hodge-Tate structure V, define

$$\Gamma V := \operatorname{Hom}_{\text{Hodge}}(\Lambda(0), V).$$

This can be thought of the "invariants" or "global sections" of V. Show that

$$\operatorname{Hom}_{\text{Hodge}}(V_1, V_2) = \Gamma \operatorname{Hom}_\Lambda(V_1, V_2)$$

where V_1 and V_2 are Λ-mixed Hodge-Tate structures.

Theorem 22. *The category of $\mathbb{Q}$-mixed Hodge-Tate structures is an abelian tensor category.* □

3.6. **Periods, Moduli and Extensions.** There can be many mixed Hodge-Tate structures with the same weight graded quotients. These are parameterized by a moduli space whose coordinates are the periods.

The basic idea is quite simple. Suppose that V is a Λ-mixed Hodge-Tate structure. For simplicity, we suppose that V_Λ is torsion free and that

$$\operatorname{Gr}^W_{-2m} V_\Lambda \cong \Lambda(m)^{r_m}.$$

(This always holds when $\Lambda = \mathbb{Q}$.) Write

$$V_\mathbb{C} = \bigoplus_{m\in\mathbb{Z}} V^{m,m}$$

and chose a basis $e_1^{(m)}, \dots, e_{r_m}^{(m)}$ of $V^{-m,-m}$ such that the Λ-lattice underlying $\operatorname{Gr}^W_{-2m} V$ is

$$(2\pi i)^m \Lambda e_1^{(m)} \oplus \cdots \oplus (2\pi i)^m \Lambda e_{r_m}^{(m)}.$$

Define $W_{-2m}V_\Lambda = V_\Lambda \cap W_{-2m}V_\mathbb{C}$. Since $W_{-2m}V_\Lambda \to \operatorname{Gr}^W_{-2m} V_\Lambda$ is surjective, there is a basis

$$\{v_1^{(m)}, \dots, v_{r_m}^{(m)} : m \in \mathbb{Z}\}$$

of V_Λ where $v_j^{(m)} \in W_{-2m}V_\Lambda$ and

$$v_j^{(m)} \equiv (2\pi i)^m e_j^{(m)} \bmod W_{-2m-2}V_\Lambda.$$

Set

$$\mathbf{e}^{(m)} = (e_1^{(m)}, \dots, e_{r_m}^{(m)})^T \text{ and } \boldsymbol{v}m = (v_1^{(m)}, \dots, v_{r_m}^{(m)})^T.$$

The two bases are related by an upper triangular matrix:

$$
\begin{pmatrix} \vdots \\ \boldsymbol{v}0 \\ \boldsymbol{v}1 \\ \boldsymbol{v}2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \ddots & * & * & * & * \\ 0 & I_{r_0} & * & * & * \\ 0 & 0 & (2\pi i)I_{r_1} & * & * \\ 0 & 0 & 0 & (2\pi i)^2 I_{r_2} & * \\ 0 & 0 & 0 & 0 & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ \mathbf{e}^{(0)} \\ \mathbf{e}^{(1)} \\ \mathbf{e}^{(2)} \\ \vdots \end{pmatrix} \tag{12}
$$

The entries of the matrix relating the two bases are called the *periods* of V and the matrix is called the *period matrix* of V.

The set of all mixed Hodge-Tate structures whose weight graded quotients are isomorphic to

$$
V_o = \bigoplus_{m\in\mathbb{Z}} \Lambda(m)^{r_m}
$$

via a fixed isomorphism is the quotient $G(\Lambda)\backslash M(\mathbb{C})$ of the set $M(\mathbb{C})$ of complex upper triangular matrices of the form

$$
\begin{pmatrix} \ddots & * & * & * & * \\ 0 & I_{r_0} & * & * & * \\ 0 & 0 & (2\pi i)I_{r_1} & * & * \\ 0 & 0 & 0 & (2\pi i)^2 I_{r_2} & * \\ 0 & 0 & 0 & 0 & \ddots \end{pmatrix}
$$

by the subgroup $G(\Lambda)$ of $GL_r(\Lambda)$ consisting of matrices of the form

$$
\begin{pmatrix} \ddots & * & * & * & * \\ 0 & I_{r_0} & * & * & * \\ 0 & 0 & I_{r_1} & * & * \\ 0 & 0 & 0 & I_{r_2} & * \\ 0 & 0 & 0 & 0 & \ddots \end{pmatrix}.
$$

Here $r = \sum_m r_m$, the rank of V. The coset of the identity matrix corresponds to the *split* mixed Hodge-Tate structure

$$
V_o = \bigoplus_{m\in\mathbb{Z}} \Lambda(m)^{r_m}.
$$

Exercise 31. Prove this.

In the case when $V_o = \Lambda(0) \oplus \Lambda(1)$, the moduli space is

$$\begin{aligned}\operatorname{Ext}^1_{\text{Hodge}}(\Lambda, \Lambda(1)) &= G(\Lambda)\backslash M(\mathbb{C}) \\ &= \begin{pmatrix} 1 & \Lambda \\ 0 & 1 \end{pmatrix} \Big\backslash \begin{pmatrix} 1 & \mathbb{C} \\ 0 & 2\pi i \end{pmatrix} \\ &\cong \mathbb{C}/2\pi i \Lambda \\ &\cong \mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda.\end{aligned}$$

Exercise 32. Show that

$$\operatorname{Ext}^1_{\text{Hodge}}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong \begin{cases} 0 & n \geq 0 \\ \mathbb{C}/(2\pi i)^n \mathbb{Z} & n > 0. \end{cases}$$

More generally, show that

$$\operatorname{Ext}^1_{\text{Hodge}}(\mathbb{Z}(0), \mathbb{Z}(n)) \cong \begin{cases} \mathbb{Q}/\mathbb{Z} & n > 0; \\ 0 & n = 0; \\ \mathbb{C}/(2\pi i)^n \mathbb{Z} & n > 0. \end{cases}$$

Example 23. The mixed Hodge-Tate structure in $\operatorname{Ext}^1_{\text{Hodge}}(\mathbb{Z}, \mathbb{Z}(1))$ corresponding to $x \in \mathbb{C}^*$ can be constructed as follows. Let e_0 and e_1 be the standard basis of $\mathbb{C}^2$. Set $V^{-m,-m} = \mathbb{C}e_m$. Define the Hodge and weight filtrations on $V_{\mathbb{C}}$ using formula (10). The integral lattice $V_{\mathbb{Z}}$ is spanned by v_0 and v_1 where

$$\begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1 & \log x \\ 0 & 2\pi i \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \end{pmatrix}$$

When $x \neq 1$, this can be constructed geometrically as $H_1(\mathbb{C}^*, \{0, x\})$. This group is freely generated by the classes of σ and γ illustrated in Figure 1.

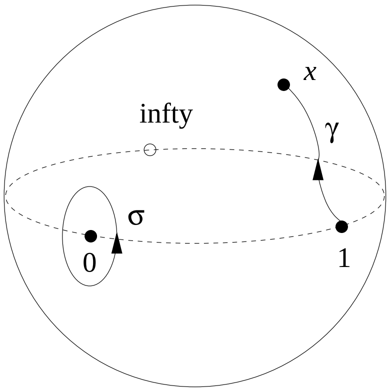


FIGURE 1. Generators of $H_1(\mathbb{C}^*, \{1, x\})$

Its dual $H^1(\mathbb{C}^*, \{0, x\})$ is spanned by the boundary map

$$\partial : H_1(\mathbb{C}^*, \{0, x\}) \to \widetilde{H}(\{0, 1\}; \mathbb{Z}) \cong \mathbb{Z}([x] - [1]) \cong \mathbb{Z}$$

and by $\int dz/z : H_1(\mathbb{C}^*, \{0, x\}) \to \mathbb{C}$. Let $e_0, e_1 \in H_1(\mathbb{C}^*, \{0, x\}; \mathbb{C})$ be dual to $\partial, dz/z$. The period matrix of this mixed Hodge-Tate structure with respect to $v_0 = \gamma, v_1 = \sigma$ and e_0, e_1 is

$$\begin{pmatrix} 1 & \int_\gamma \frac{dz}{z} \\ 0 & \int_\sigma \frac{dz}{z} \end{pmatrix} = \begin{pmatrix} 1 & \log x \\ 0 & 2\pi i \end{pmatrix}.$$

The most interesting period is the composite

$$\mathbb{Z} \xrightarrow{\gamma} H_1(\mathbb{C}^*, \{0, x\}; \mathbb{C}) \xrightarrow{dz/z} \mathbb{C}$$

which takes 1 to $\int_\gamma dz/z = \log x$.

3.7. **Periods of $\pi_1(\mathbb{P}^1(\mathbb{C}) - S, x)$ and its Dual.** The mixed Hodge-Tate structure on $\mathbb{Z}\pi_1(\mathbb{P}^1(\mathbb{C}) - S, x)/J^{n+1}$ is the dual of the mixed Hodge-Tate structure constructed in Section 3.3. This can be described concretely using the isomorphism

$$\widehat{\Theta} : \mathbb{C}\pi_1(\mathbb{P}^1(\mathbb{C}) - S, x)/J^{n+1} \to A/I^{n+1}$$

given by Proposition 11. The basis of monomials

$$X_{j_1} X_{j_2} \dots X_{j_r} \quad j_k \in \{1, 2, \dots, N\},\ 0 \le r \le n$$

of A/I^{n_1} is dual to the basis

$$\int w_{j_1} w_{j_2} \dots w_{j_r} \quad j_k \in \{1, 2, \dots, N\},\ 0 \le r \le n$$

of $L_n \operatorname{Ch}(H^0(\Omega^1_{\mathbb{P}^1}(\log S)))$. The monomials of degree r span $A^{-r,-r}$, and the Hodge and weight filtrations on

$$A/I^{n_1} \cong \mathbb{C}\pi_1(\mathbb{P}^1(\mathbb{C}) - S, x)/J^{n+1}$$

are defined using the formulas in (11). The weight filtration is rationally defined as

$$W_{-2m}\mathbb{C}\pi_1(\mathbb{P}^1(\mathbb{C}) - S, x)/J^{n+1} = J^m/J^{n+1} \quad 0 \le m \le n.$$

The periods of $\mathbb{Z}\pi_1(\mathbb{P}^1(\mathbb{C}) - S, x)/J^{n+1}$ (and its dual) are matrix entries:

$$\mathbb{Z} \xrightarrow{\quad c \quad} \mathbb{C}\pi_1(\mathbb{P}^1(\mathbb{C}) - S, x)/J^{n+1} \xrightarrow{\int w_{j_1} \dots w_{j_r}} \mathbb{C}$$

where $c \in \mathbb{Z}\pi_1(\mathbb{P}^1(\mathbb{C}) - S, x)/J^{n+1}$ and $j_k \in \{1, \dots, N\}$. That is, the periods are complex numbers of the form $\int_c w_{j_1} w_{j_2} \dots w_{j_r}$.

3.8. **Hodge Theory for the Fundamental Groupoid.** The same method can be used to define a mixed Hodge-Tate structure on

$$H_0(P_{x,y}(\mathbb{P}^1(\mathbb{C}) - S))/J_{x,y}^{n+1}.$$

Define Hodge and weight filtrations on A/I^{n+1} as above. Transfer these to $H_0(P_{x,y}(\mathbb{P}^1(\mathbb{C}) - S))/J_{x,y}^{n+1}$ via the isomorphism

$$\widehat{\Theta} : H_0(P_{x,y}(\mathbb{P}^1(\mathbb{C}) - S))/J_{x,y}^{n+1} \to A/I^{n+1}$$

described in Remark pathDR. Its periods are simply $\mathbb{Z}$-linear combinations of values of iterated integrals $\int w_{j_1} \dots w_{j_r}$ over paths from x to y.

The extension corresponding to $x \in \mathbb{C}^*$ can also be realized naturally using path spaces.

Exercise 33. For any manifold M and points $x, y \in M$, there is a natural augmentation $H_0(P_{x,y}M; R) \to R$ defined by taking each $\gamma \in \pi_0(M; x, y)$ to 1. Show that there is a natural isomorphism $H_1(M; R) \cong J_{x,y}/J_{x,y}^2$. Deduce that there is a natural exact sequence

$$0 \to H_1(M; R) \to H_0(P_{x,y}M; R)/J_{x,y}^2 \to R \to 0.$$

Show that if $U = \mathbb{P}^1(\mathbb{C}) - S$, then, for each $n \geq 1$, the augmentation $H_0(P_{x,y}M)/J_{x,y}^n \to \mathbb{Z}(0)$ is a morphism of mixed Hodge-Tate structures.

Show that, for all $x \in \mathbb{C}^*$,

$$0 \to H_1(\mathbb{C}^*) \to H_0(P_{x,y}\mathbb{C}^*)/J_{1,x}^2 \to \mathbb{Z}(0) \to 0$$

is an exact sequence of mixed Hodge structures that represents the element x of

$$\mathrm{Ext}^1_{\mathrm{Hodge}}(\mathbb{Z}, H_1(\mathbb{C}^*)) \cong \mathrm{Ext}^1_{\mathrm{Hodge}}(\mathbb{Z}, \mathbb{Z}(1)) \cong \mathbb{C}^*.$$

Here we are identifying $H_1(\mathbb{C}^*)$ with $\mathbb{Z}(1)$ as in Exercise 18.

3.9. **Periods in the Presence of Arithmetic.** When a mixed Hodge-Tate structure V arises from a variety defined over a subfield K of $\mathbb{C}$ there is typically a K structure on the vector space $V_{\mathbb{C}}$. This is easy to see in the case of the mixed Hodge-Tate structure on the dual of the truncated fundamental group when the variety is an open subset of $\mathbb{P}^1$.

Suppose that $U = \mathbb{P}^1(\mathbb{C}) - S$ and that $S = \{a_1, \dots, a_N, \infty\}$, where each $a_j \in K$. Then each $w_j = dz/(z - a_j)$ is defined over K and

$$H^0(\Omega^1_{\mathbb{P}^1_{\mathbb{C}}}(\log S)) = H^0(\Omega^1_{\mathbb{P}^1_K}(\log S)) \otimes_K \mathbb{C}$$

This K-structure on the logarithmic differentials extends to the iterated integrals: set

$$V^{\mathrm{DR}}_{\mathbb{C}} = L_n \,\mathrm{Ch}(H^0(\Omega^1_{\mathbb{P}^1_{\mathbb{C}}}(\log S)))$$

and
$$V^{\mathrm{DR}} = L_n \operatorname{Ch}(H^0(\Omega^1_{\mathbb{P}^1_K}(\log S))).$$
Then $V^{\mathrm{DR}}_{\mathbb{C}} = V^{\mathrm{DR}} \otimes_K \mathbb{C}$. This gives the iterated integrals a natural K structure where each
$$\int w_{j_1} w_{j_2} \dots w_{j_r} \in V^{\mathrm{DR}}$$
and the Hodge and weight filtrations are defined over K — that is, they are induced by filtrations of V^{DR}.

In these lectures, the most important case is where $S = \{0, 1, \infty\}$. In this case, $K = \mathbb{Q}$. The periods of the mixed Hodge-Tate structure on
$$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}\pi_1(U, x)/J^{n+1}, \mathbb{Z})$$
measure the difference between the two rational structures on it:
$$\operatorname{Hom}(\mathbb{Z}\pi_1(U, x)/J^{n+1}, \mathbb{Q}) \hookrightarrow V^{\mathrm{DR}}_{\mathbb{C}} \hookleftarrow V^{\mathrm{DR}}.$$

3.10. **Postscript.** Mixed Hodge-Tate structures are examples of mixed Hodge structures. A good introduction to Deligne's theory of mixed Hodge structures can be found in Carlson's paper [1]. Once you understand the basic ideas, Deligne's paper [4] is an excellent source. The paper [11] contains an introduction to the mixed Hodge structure on the fundamental group of a smooth complex algebraic variety in terms of iterated integrals. Details of the case where this mixed Hodge structure is Tate can be found in [15].

4. Limit Mixed Hodge Structures and the Drinfeld Associator

4.1. **Preamble.** It is useful to begin with an informal discussion of "mixed Tate motives over $\operatorname{Spec}\mathbb{Z}$." Motives over $\operatorname{Spec}\mathbb{Z}$ should arise as invariants (cohomology, homotopy, etc.) of varieties (and stacks) defined over $\mathbb{Z}$ that have good reduction at every prime number. Obvious examples include the projective spaces $\mathbb{P}^N_{\mathbb{Z}}$ and the moduli stacks of curves $\mathcal{M}_{g,n}$.

Here we are interested in open subsets of the projective line:

$$U_{\mathbb{Z}} = \mathbb{P}^1_{\mathbb{Z}} - S := \operatorname{Spec}\mathbb{Z}[z, t_1, \dots, t_N]/\big((z-a_j)t_j - 1 : j = 1, \dots, N\big),$$

where $S = \{a_1, \dots, a_N, \infty\}$ and each $a_j \in \mathbb{Z}$. This has good reduction at the prime p if the cardinality of S mod p equals that of S. It is clear that U has good reduction at all primes if and only if $S = \{0,1,\infty\}$.[16]

Now take $U = \mathbb{P}^1 - \{0,1,\infty\}$. In order to consider the fundamental group of U, we need a base point x. If we choose $x \in \mathbb{Z} - \{0,1\}$, then the pair (U, x) has bad reduction at the prime p whenever $p|x(x-1)$ as then the base point reduces to 0 or 1, which are not in $U(\mathbb{F}_p)$.

This forces us to consider "asymptotic base points." These are tangent vectors of $\mathbb{P}^1_{\mathbb{Z}}$ at $\{0,1,\infty\}$ that are non-zero at each prime p, such as

$$\overrightarrow{01} := \partial/\partial z \in T_0\mathbb{P}^1 \text{ and } \overrightarrow{10} := -\partial/\partial z \in T_1\mathbb{P}^1.$$

The tannakian category of $\mathbb{Q}$-mixed Tate motives over $\operatorname{Spec}\mathbb{Z}$ does exist via the works of Voevodsky [22], Levine [18, 17] and Goncharov [8]. Deligne and Goncharov [6] have shown that the direct system (9) of the $\operatorname{Hom}_{\mathbb{Q}}(\mathbb{Q}\pi_1(\mathbb{P}^1 - \{0,1,\infty\}, \overrightarrow{01})/J^{n+1}, \mathbb{Q})$ is a directed system in this category.

The topological and Hodge theoretic aspects of $\pi_1(\mathbb{P}^1 - \{0,1,\infty\}, \overrightarrow{01})$ and $\pi_0(P_{\overrightarrow{01},\overrightarrow{10}}(\mathbb{P}^1 - \{0,1,\infty\}))$ will be discussed in the rest of this section.

4.2. **Asymptotic Base Points.** Suppose that $C' = C - S$, where C is a Riemann surface and S is a discrete subset. Suppose that $\vec{v}$ is a non-zero tangent vector of C at $P \in S$. Deligne [5] introduced the idea of the fundamental group of C with the *asymptotic base point* $\vec{v}$. It is isomorphic to the standard fundamental group of C'. Intuitively, it is $\pi_1(C', x)$ where x is infinitesimally close to P in the direction of $\vec{v}$.

It will be convenient to define $P_{\vec{v},\vec{w}}C'$ where $\vec{v} \in T_PC$, $\vec{w} \in T_QC$ and $P, Q \in S$. This is the set γ of piecewise smooth paths in C that begin at P with tangent vector $\vec{v}$ and end at Q with tangent vector $-\vec{w}$. The path is also required to satisfy $\gamma(]0,1[) \subset C'$. The set of

[16] Note that in this case, $U = \mathcal{M}_{0,4}$.

homotopy classes of such paths will be denoted by $\pi_0(P_{\vec{v},\vec{w}}C')$ or by $\pi(C';\vec{v},\vec{w})$. The fundamental group of C' with base point $\vec{v}$ is defined to be $\pi_0(P_{\vec{v},\vec{v}}C')$.

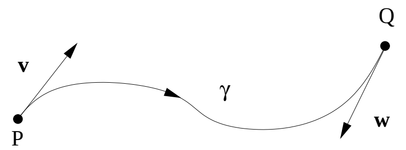


FIGURE 2. An element of $P_{\vec{v},\vec{w}}C'$

For non-zero tangent vectors $\vec{u},\vec{v},\vec{w}$ of C at points of S, composition of paths

$$\pi_0(P_{\vec{u},\vec{v}}C') \times \pi_0(P_{\vec{v},\vec{w}}C') \to \pi_0(P_{\vec{u},\vec{w}}C')$$

is well defined. Consequently, $\pi_1(C',\vec{v})$ is a group.

Exercise 34. Show that if $x \in C'$, then $\pi_1(C',\vec{v})$ is isomorphic to $\pi_1(C',x)$ by an isomorphism unique up to an inner automorphism.

Alternatively, one may define $P_{\vec{v},\vec{w}}C'$ by replacing C by the real oriented blowup

$$\widetilde{C} := \mathrm{Bl}^{\mathbb{R}+}_{P,Q} C$$

of C at P and Q and then removing $S' := S - \{P,Q\}$. The vectors $\vec{v},\vec{w}$ determine points $[\vec{v}],[\vec{w}]$ in the boundary of $\widetilde{C}-S'$, which consists of the exceptional circles that lie over P and Q. One can then define $P_{\vec{v},\vec{w}}C'$ to be $P_{[\vec{v}],[\vec{w}]}(\widetilde{C}-S')$.

In this section we will consider the path spaces $P_{\vec{v},\vec{w}}(\mathbb{P}^1-\{0,1,\infty\})$ where $\vec{v},\vec{w} \in \{\overrightarrow{01},\overrightarrow{10}\}$.

Our goal is to compute the periods of $\mathbb{Z}\pi_1(\mathbb{P}^1-\{0,1,\infty\},\overrightarrow{01})/J^{n+1}$. If we do this naively — by taking the limit of the periods of

$$\mathbb{Z}\pi_1(\mathbb{P}^1-\{0,1,\infty\},x)/J^{n+1}$$

as $x \to 0$ along $\overrightarrow{01}$ — we find that the periods diverge.

Exercise 35. Suppose that $t_o,\epsilon \in \mathbb{R}$ and $\gamma \in \pi_1(\mathbb{P}^1-\{0,1,\infty\},t_o)$, where $0<\epsilon<t_o<1$. Choose $\sigma_1 \in \pi_1(\mathbb{P}^1-\{0,1,\infty\},t_o)$ whose winding number about 0 is 0 and about 1 is 1. Use the change of base point formula (Exercise 8) to show that

$$\int_{[\epsilon,t_o]\sigma_1[t_o,\epsilon]} w_0w_1 = 2\pi i\log\epsilon + \text{ bounded term}$$

as $\epsilon \to 0$.

It is necessary to regularize or renormalize the periods when taking the limit. The resulting mixed Hodge-Tate structure is then a *limit mixed Hodge structure*.

4.3. **Some ODE.** First, consider the ordinary differential equation

$$tv'(t) = Bv(t)$$

defined for $t \in \mathbb{C}^*$, where $B \in \mathbb{M}_N(\mathbb{C})$ and $v(t) \in \mathbb{C}^N$. This is the prototypical system of differential equations with a regular singular point at $t = 0$. The general solution is

$$v(t) = t^B v_0,$$

where $v_0 \in \mathbb{C}^N$ and t^B is the $GL_N(\mathbb{C})$-multivalued function on $\mathbb{C}^*$ defined by

$$t^B = e^{B \log t}.$$

Note that, when t^B is analytically continued around the unit circle, it becomes

$$e^{B(\log t + 2\pi i)} = e^{2\pi i B} t^B.$$

From this it follows that every solution $v = t^A v_0$ is multiplied on the left by $e^{2\pi i B}$ when it is analytically continued around the unit circle.

The following result is more general. A proof can be found in [23, Chapt. II].

Lemma 24. *Suppose that V is a finite dimensional complex vector space and that $B : \Delta \to \operatorname{End} V$ is holomorphic. If no two eigenvalues of $B_0 := B(0)$ differ by a non-zero integer (e.g., if B_0 is nilpotent), then there is a unique holomorphic function $P : \Delta \to \operatorname{Aut} V$ with $P(0) = \mathrm{id}_V$ such that each (local) solution $v : \Delta^* \to V$ of the differential equation*

$$tv'(t) = B(t)v(t)$$

is of the form

$$v(t) = P(t)\, t^{B_0} v_0.$$

Corollary 25. *If B_0 is nilpotent, then*

$$\lim_{t \to 0} t^{-B_0} v(t) = v_0$$

where the limit is taken along any angular ray.

Proof. We have

$$t^{-B_0} v(t) = t^{-B_0} P(t)\, t^{B_0} v_0.$$

If $B_0^{k+1} = 0$, then there is a constant C such that

$$\|t^{B_0}\| \text{ and } \|t^{-B_0}\| \le C\big(\log 1/|t|\big)^k$$

when $0 < |t| \le R$, for some $R > 0$. Writing $P(t) = I + \sum_{n\ge 1} P_n t^n$, we have

$$\|t^{-B_0}P(t)t^{B_0} - I\| \le 2C|t|\big(\log 1/|t|\big)^k \sum_{n=1}^{\infty} \|P_n\||t|^{n-1}$$

which goes to zero along each angular ray. □

Remark 26. Note that $t^{-B_0}v(t)$ is not, in general, single-valued on the punctured disk even though its limit as $t \to 0$ along each radial ray exists.

To see why this is relevant to understanding the asymptotics of periods of iterated integrals, we need to study how iterated integrals vary when the end point of the path is moved.

Suppose that M is a manifold and that $\gamma \in PM$. For $a, b \in [0,1]$, denote by γ^t the path defined by

$$\gamma_a^b(t) = \gamma\big(ta + (1-t)b\big)$$

This is the segment of γ that starts at $\gamma(a)$ and ends at $\gamma(b)$.

Exercise 36. Show that

$$\frac{d}{dt}\bigg|_{t=a} \int_{\gamma_t^b} w_1 \dots w_r = -\langle w_1, \gamma'(a)\rangle \bigg(\int_{\gamma_a^b} w_2 \dots w_r\bigg).$$

and that

$$\frac{d}{dt}\bigg|_{t=b} \int_{\gamma_a^t} w_1 \dots w_r = \bigg(\int_{\gamma_a^b} w_1 \dots w_{r-1}\bigg)\langle w_r, \gamma'(b)\rangle.$$

The significance of these formulas is that they show that iterated integrals satisfy ordinary differential equations.

Now let $U = \mathbb{P}^1(\mathbb{C}) - S$, where $S = \{a_1, \dots, a_N, \infty\}$. Suppose that $\lambda, \mu \in \mathbb{C}^*$. Set

$$\vec{v} = \lambda\frac{\partial}{\partial z} \in T_{a_j}\mathbb{P}^1 \text{ and } \vec{w} = \mu\frac{\partial}{\partial z} \in T_{a_k}\mathbb{P}^1$$

where z is the natural holomorphic coordinate on $\mathbb{P}^1(\mathbb{C}) - \{\infty\}$. Set

$$A = \mathbb{C}\langle\langle X_1, \dots, X_N\rangle\rangle.$$

We will now define a "regularized" mapping

$$\widehat{\Theta}^{\mathrm{reg}}_{\vec{v},\vec{w}} : P_{\vec{v},\vec{w}}\, U \to A.$$

We do this using the A-valued iterated integral

$$T = 1 + \sum_{r>0} \sum_{(j_1,\dots,j_r)} \int w_{j_1} \dots w_{j_r} X_{j_1} \dots X_{j_r}$$

and its truncations $T_n := T \bmod I^{n+1}$. Here, as usual, $w_j = d\log(z - a_j)$.

Every element of $\pi(U; \vec{v}, \vec{w})$ is represented by a path of the form

$$\gamma(t) = \begin{cases} a_j + t\lambda & 0 \leq t_o \\ \alpha(t) & t_o \leq t \leq 1 - t_o \\ a_k + (1-t)\mu & 1 - t_o \leq t \leq 1 \end{cases}$$

where $0 < t_o < 1/2$ and $\alpha : [t_o, 1 - t_o] \to U$ is a piecewise smooth path.

For t in a neighbourhood of 0, set

$$v_n(t) = \langle T_n, \gamma_t^{t_o} \rangle \in A/I^{n+1} \text{ and } v(t) = \lim_{n\to\infty} v_n(t) = \langle T, \gamma_t^{t_o} \rangle \in A.$$

Exercise 37. Show that $v_n(t)$ satisfies the differential equation

$$v_n'(t) = B(t) v_n(t)$$

where

$$B(t) = \text{ left multiplication by } \sum_{s=1}^{N} \frac{-\lambda X_s}{(a_j - a_s) + t\lambda}.$$

This has a pole at $t = 0$ with residue left multiplication by $-X_j$, which is a nilpotent endomorphism of A/I^{n+1}. Deduce that

$$\lim_{t\to 0} t^{X_j} v_n(t) \in A/I^{n+1}$$

exists. By taking limits in the I-adic topology, deduce that

$$\lim_{t\to 0} t^{X_j} v(t) \in A$$

exists.

By taking inverses or else giving a similar argument, show that

$$\lim_{t\to 0} \langle T, \gamma_{t_o}^{t} \rangle \, t^{-X_k} \in A$$

exists.

We shall use the notation of the previous exercise in the definition of the regularization of $\widehat{\Theta}$.

Definition 27. For $\vec{v} \in T_{a_j}\mathbb{P}^1$ and $\vec{w} \in T_{a_k}\mathbb{P}^1$, define

$$\widehat{\Theta}^{\mathrm{reg}}_{\vec{v},\vec{w}} : P_{\vec{v},\vec{w}}\, U \to A$$

by

$$\widehat{\Theta}^{\mathrm{reg}}_{\vec{v},\vec{w}}(\gamma) = \lim_{\substack{\epsilon\to 0\\ \delta\to 0}} \epsilon^{X_j} \langle T, \gamma_\epsilon^{1-\delta} \rangle \, \delta^{-X_k} = \lim_{t\to 0} t^{X_j} \langle T, \gamma_t^{1-t} \rangle \, t^{-X_k}$$

Exercise 38. Suppose that $\vec{u} \in T_{a_\ell}\mathbb{P}^1$. Show that if $\alpha \in P_{\vec{u},\vec{v}}U$ and $\beta \in P_{\vec{v},\vec{w}}U$, then

$$\widehat{\Theta}^{\mathrm{reg}}_{\vec{u},\vec{w}}(\alpha\beta) = \widehat{\Theta}^{\mathrm{reg}}_{\vec{u},\vec{v}}(\alpha)\widehat{\Theta}^{\mathrm{reg}}_{\vec{v},\vec{w}}(\beta).$$

Exercise 39. Show that

$$\widehat{\Theta}^{\mathrm{reg}}_{\vec{v},\vec{w}} : H_0(P_{\vec{v},\vec{w}}U;\mathbb{C})^\wedge \to A.$$

is an isomorphism, and that $\widehat{\Theta}^{\mathrm{reg}}_{\vec{v},\vec{w}}(J^n_{\vec{v},\vec{w}}) = I^n$ for all $n \geq 1$.

The limit mixed Hodge-Tate structure on $H_0(P_{\vec{v},\vec{w}}U)/J^{n+1}$ is now easily defined using $\widehat{\Theta}^{\mathrm{reg}}_{\vec{v},\vec{w}}$. One first defines the Hodge and weight filtrations on A/I^{n+1} in the standard way, and then transfers these to Hodge and weight filtrations on $H_0(P_{\vec{v},\vec{w}}U)/J^{n+1}$ using the isomorphism

$$\widehat{\Theta}^{\mathrm{reg}}_{\vec{v},\vec{w}} : H_0(P_{\vec{v},\vec{w}}U;\mathbb{C})/J^{n+1} \to A/I^{n+1}.$$

The periods of this limit MHS associated to $c \in H_0(P_{\vec{v},\vec{w}}U;\mathbb{C})/J^{n+1}$ are the coefficients of $\widehat{\Theta}^{\mathrm{reg}}_{\vec{v},\vec{w}}(c) \in A/I^{n+1}$.

Exercise 40. Suppose that $\vec{v} = \lambda\partial/\partial z \in T_{a_j}U$. Suppose that $\sigma_j \in P_{\vec{v},\vec{v}}U$ is the loop $\alpha\gamma\alpha^{-1}$ obtained by joining a sufficiently small, positively oriented circle γ centered at a_j to a_j by a line segment α from a_j to the point on γ of the form $a_j + t\lambda$ where $t > 0$. Show that

$$\widehat{\Theta}^{\mathrm{reg}}_{\vec{v},\vec{v}}(\sigma_j) = e^{2\pi i X_j} \in A.$$

4.4. **The Limit Mixed Hodge Structure on the Fundamental Groupoid of $\mathbb{P}^1 - \{0,1,\infty\}$.** We now consider the fundamental groupoid of $\mathbb{P}^1 - \{0,1,\infty\}$ with objects the two tangent vectors $\overrightarrow{01} \in T_0\mathbb{P}^1$ and $\overrightarrow{10} \in T_1\mathbb{P}^1$. This is generated by the paths

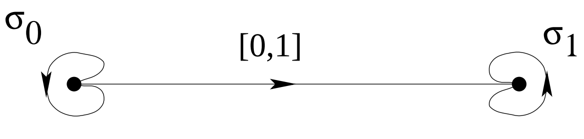


FIGURE 3

where $\sigma_0 \in P_{\overrightarrow{01},\overrightarrow{01}}(\mathbb{P}^1 - \{0,1,\infty\})$ and $\sigma_1 \in P_{\overrightarrow{10},\overrightarrow{10}}(\mathbb{P}^1 - \{0,1,\infty\})$.

Set

$$\Phi(X_0,X_1) = \widehat{\Theta}^{\mathrm{reg}}_{\overrightarrow{01},\overrightarrow{10}}([0,1]) = \lim_{t\to 0} t^{X_0}T([t,1-t])t^{X_1} \in A.$$

This is known as the *Drinfeld associator* and was first constructed in [7]. It has many remarkable properties, one of which is that the coefficients of $\Phi(X_0, X_1)$ are multiple zeta numbers. This should not be surprising as its coefficients are convergent iterated integrals on the unit interval. This fact was observed without proof by Drinfeld [7] and proved by Le and Murakami [16] where one can find an explicit formula for $\Phi(X_0, X_1)$.

Exercise 41. Prove the formulas

$$t^{X_0} = \langle 1 + \int w_0 X_0 + \int w_0 w_0 X_0^2 + \cdots, [1, t]\rangle$$

and

$$t^{-X_1} = \langle 1 + \int w_1 X_1 + \int w_1 w_1 X_1^2 + \cdots, [0, 1-t]\rangle$$

and use them to find an expression for the Drinfeld associator whose coefficients are iterated integrals of the form $\int_{[0,1]} w_1 \dots w_0$. Deduce that the coefficients of $\Phi(X_0, X_1)$ are multiple zeta numbers and that all multiple zeta numbers occur.

Theorem 28. *The periods of the limit mixed Hodge-Tate structure on* $\mathbb{Q}\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, \overrightarrow{01})^\wedge$ *is precisely* $\mathrm{MZN}_\mathbb{C}$.

Proof. This follows immediately from the facts that $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, \overrightarrow{01})$ is generated by the paths σ_0 and $[0, 1]\sigma_1[1, 0]$ and the facts that

$$\widehat{\Theta}_{\overrightarrow{01},\overrightarrow{01}}(\sigma_0) = e^{2\pi i X_0} \text{ and } \widehat{\Theta}_{\overrightarrow{01},\overrightarrow{01}}(\sigma_1) = e^{-2\pi i X_1}$$

and the fact that the coefficients of $\Phi(X_0, X_1)$ are multiple zeta numbers. $\square$

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