Abstract In this chapter we explain how general admissible representations are built up out of supercuspidal representations via the process of parabolic induction.

8.1 Introduction

Let $G$ be a connected reductive group over a nonarchimedean local field $F$, let $P \leq G$ be a parabolic subgroup, and let $P = MN$ be a Levi decomposition with $M \leq P$ a Levi subgroup and $N$ the unipotent radical of $P$. This notation will be in force throughout the chapter.

We recall from Proposition 3.5.1 that the modular quasicharacter

\[ \delta_P := \delta_{P(F)} : P(F) \rightarrow \mathbb{R}^\times \quad (8.1) \]

is

\[ \delta_P(p) := |\det(\text{Ad}_p(p) : p(F) \rightarrow p(F))|. \]

Let $(\sigma, V)$ be a smooth irreducible representation of $M(F)$. Using the modular character we define the parabolically induced representation (or simply induced representation) $I(\sigma) := \text{Ind}_P^G(\sigma)$ to be the (smooth) representation of $G(F)$ on the space of functions

\[ \text{Ind}_P^G(V) := \left\{ \text{locally constant } \varphi : G(F) \rightarrow V : \varphi(mng) = \delta_P(m)^{1/2} \sigma(m) \varphi(g) \right\} \]

for all $(m, n, g) \in M(F) \times N(F) \times G(F)$. Note that a function $\varphi : G(F) \rightarrow V$ is locally constant if and only if it is continuous with respect to the natural topology on $G(F)$ and the usual topology (or even the discrete topology) on $V$. The group $G(F)$ acts via right translation:

\[ I(\sigma)(g)\varphi(x) = \varphi(xg). \quad (8.2) \]
The factor of $\delta_p^{1/2}$ is present so that if $\sigma$ is unitarizable then $I(\sigma)$ is also unitarizable (see Proposition 8.2.1). We note that this procedure yields a functor

$$\text{Ind}_{P}^{G} : \text{Rep}_{\text{sm}} M(F) \longrightarrow \text{Rep}_{\text{sm}} G(F),$$

(8.3)

from smooth representations of $M(F)$ to smooth representations of $G(F)$ (in these categories objects are smooth representations and morphisms are equivariant maps, see §5.3). This functor is called **parabolic induction**.

In the previous chapter we gave the classification of unramified representations of $G(F)$. In this chapter we delve deeper into the representation theory of $G(F)$.

If $V$ is a smooth representation of $G(F)$ then its contragredient $V^\vee$ is defined as in §5.4. A function of the form

$$m(g) := m_{\varphi, \varphi^\vee}(g) := \langle \pi(g)\varphi, \varphi^\vee \rangle$$

(8.4)

for $(\varphi, \varphi^\vee) \in V \times V^\vee$ is called a **matrix coefficient** of $\pi$ (for the relationship between this notion and that of §4.7 see Exercise 8.1).

**Definition 8.1.** A **supercuspidal** (resp. **quasicuspidal**) representation of $G(F)$ is an admissible (resp. smooth) representation all of whose matrix coefficients are compactly supported modulo the center of $G(F)$.

The main result of this chapter is a theorem of Jacquet which states that every irreducible admissible representation is obtained as a subquotient of the parabolic induction of a supercuspidal representation for an appropriately chosen parabolic subgroup (see Corollary 8.3.3 for the precise statement). One can profitably compare this result with the Langlands classification from §4.8 and the classification of automorphic representations explained in Chapter 10. These results can all be viewed as manifestations of what Harish-Chandra called the “philosophy of cusp forms” [HC70]. Interpreted broadly, this is a slogan for the statement that all irreducible representations of reductive groups can all be obtained via parabolic induction from cuspidal representations of parabolic subgroups. Of course, the correct notion of cuspidal representation varies based on the context.

We will not give a full proof of Theorem 8.3.3, but we will develop much of the machinery used in the proof because it is useful in many contexts. We will also discuss the notion of a trace of an admissible representation, and in addition develop some of the basic properties of supercuspidal representations, including the fact that they admit coefficients (see Proposition 8.5.2), a fact that we will later use in our treatment of simple trace formulae.

The canonical reference for the topics covered in this chapter is the unpublished manuscripts of Casselman [Cas] and [Ber]. The paper [Car79] contains sketches of proofs and refers to [Cas] for the details. We have sometimes followed the exposition in [BH06] and [Mur], which fill in many details omitted in the previous two sources.
8.2 Parabolic induction

Let us now prove some basic facts about parabolic induction. Let \((\sigma, V)\) be a smooth representation of \(M(F)\).

We start with the following observation:

**Lemma 8.2.1** If \(K \leq G(F)\) is a compact open subgroup and \(\varphi \in \text{Ind}_{P}^{G}(V)^{K}\) then
\[
\varphi(g) \in V^{M(F) \cap gKg^{-1}}.
\]

**Proof.** Under the assumptions in the lemma if \(k \in g^{-1}M(F)g \cap K\) we have
\[
\varphi(g) = \varphi(gk) = \varphi(gkg^{-1}g) = \sigma(gkg^{-1})\varphi(g).
\]

\(\square\)

**Proposition 8.2.1** One has the following:

(a) If \(\sigma\) is admissible then \(I(\sigma)\) is admissible.

(b) If \(\sigma\) is unitary then \(I(\sigma)\) is unitarizable.

**Proof.** Assume that \((\sigma, V)\) is admissible. Let \(K \leq G(F)\) be a compact open subgroup. We are to show that \(\text{Ind}_{P}^{G}(V)^{K}\) is finite dimensional. An element \(\varphi \in \text{Ind}_{P}^{G}(V)\) is determined by its value on any set of representatives \(X\) for the double cosets in \(P(F) \backslash G(F)/K\).

This set of double cosets is finite by the Iwasawa decomposition, and hence \(X\) is finite. On the other hand, for any \(x \in X\) one has \(\varphi(x) \in V^{M(F) \cap xKx^{-1}}\) by Lemma 8.2.1, and this space is finite dimensional by admissibility. This proves (a).

For (b), let \((\cdot, \cdot)_{V}\) be an \(M(F)\)-invariant inner product on \(V\). Choose a maximal compact open subgroup \(K_{\text{max}} \leq G(F)\) so that the Iwasawa decomposition \(G(F) = P(F)K_{\text{max}}\) holds (see Theorem A.1.1), and normalize the Haar measures \(dg\) on \(G(F)\), \(dk\) on \(K_{\text{max}}\), and \(d_{\ell}p\) on \(P(F)\) so that \(dg = d_{\ell}pdk\) (see Exercise 3.5).

For \(\varphi_{1}, \varphi_{2} \in \text{Ind}_{P}^{G}(V)\) we define
\[
(\varphi_{1}, \varphi_{2}) := \int_{K} (\varphi_{1}(k), \varphi_{2}(k))_{V} dk.
\]

Since every element of \(\text{Ind}_{P}^{G}(V)\) is determined by its restriction to \(K\) this is a positive definite inner product. We must check that it is \(G(F)\)-invariant.

Choose a nonnegative \(f \in C^{\infty}(G(F))\) such that
\[
\int_{P(F)} f(px) d_{\ell}p = 1
\]
for all \( x \in G(F) \) (see Exercise 8.3). For \( x \in G(F) \) one has

\[
\int_{K_{\text{max}}} (\varphi_1(kx), \varphi_2(kx))_V dk = \int_{K_{\text{max}}} (\varphi_1(kx), \varphi_2(kx))_V \int_{P(F)} f(pkx) d\ell_p dk
\]

\[
= \int_{P(F) \times K_{\text{max}}} (\varphi_1(pkx), \varphi_2(pkx))_V f(pkx) \delta_p^{-1}(p) d\ell_p dk \quad \text{(see Exercise 3.3)}
\]

\[
= \int_{G(F)} (\varphi_1(gx), \varphi_2(gx))_V f(gx) dg
\]

\[
= \int_{G(F)} (\varphi_1(g), \varphi_2(g))_V f(g) dg.
\]

This expression is independent of \( x \), so we deduce the invariance of our inner product.

\[\square\]

**Proposition 8.2.2** If \( \sigma \) is smooth then \( I(\sigma^\vee) \cong I(\sigma)^\vee \).

**Proof.** Fix a maximal compact open subgroup \( K_{\text{max}} \leq G(F) \) so that the Iwasawa decomposition \( G(F) = P(F)K_{\text{max}} \) holds (see Theorem A.1.1).

Let \( \varphi \in \text{Ind}_{P}^{G}(V) \) and \( \varphi^\vee \in \text{Ind}_{P}^{G}(V^\vee) \). We then have a pairing

\[
(\varphi, \varphi^\vee) := \int_{K_{\text{max}}} \langle \varphi(k), \varphi^\vee(k) \rangle dk \quad (8.5)
\]

where the pairing on the right is the pairing between \( V \) and \( V^\vee \). Arguing as in the proof of part (b) of Proposition 8.2.1 we deduce that this pairing is \( G(F) \)-invariant.

Since \( \varphi^\vee \) is smooth the linear form \( \langle \cdot, \varphi^\vee \rangle \) on \( V \) is smooth and we therefore obtain a \( \mathbb{C} \)-linear map

\[
\text{Ind}_{P}^{G}(V^\vee) \rightarrow \text{Ind}_{P}^{G}(V)^\vee \quad (8.6)
\]

\[
\varphi^\vee \mapsto \langle \cdot, \varphi^\vee \rangle.
\]

It is an intertwining map because (8.5) is \( G(F) \)-equivariant. We must show that it is bijective. By smoothness it suffices to show that it induces an isomorphism

\[
\text{Ind}_{P}^{G}(V^\vee)^K \rightarrow \text{Ind}(V)^{\vee K}
\]

for all compact open subgroups \( K \leq G(F) \).

To do this we first describe a basis of \( \text{Ind}(V)^K \). We let \( X \) be a set of representatives for the double cosets in \( P(F) \backslash G(F) / K \). Since \( P(F) \backslash G(F) \) is compact \( X \) is a finite set. For each \( x \in X \) let \( B_x \) be a basis of \( V^{P(F) \cap gKg^{-1}} \).
By Lemma 8.2.1, $\varphi(g) \in V^M(F) \cap xKx^{-1}$. For each $w \in B_x$ let $\varphi_{x,w}$ be the function supported on $P(F)xK$ such that

$$\varphi_{x,w}(mnxk) = \sigma(m)w$$

for $(m, n, k) \in M(F) \times N(F) \times K$. Then

$$\{ \varphi_{x,w} : x \in X, w \in B_x \}$$

is a $C$-basis of $I(\sigma)^K$. Now for each $x$ let $B_x^\vee$ be the basis of $V^\vee M(F) \cap xKx^{-1}$ dual to $B_x$. We define $\varphi_{x,w}^\vee$ by replacing $V$ with $V^\vee$ in the construction above, and we obtain a basis

$$\{ \varphi_{x,w}^\vee : x \in X, w^\vee \in B_x^\vee \}$$

for $\text{Ind}^G_P(V^\vee)^K$. It is clear that (8.7) and (8.8) are a basis and dual basis with respect to the pairing (8.5).

It is often useful to introduce a variant of the parabolic induction functor. We recall briefly the Harish-Chandra map (4.5) discussed in §4.8. We define a map

$$H_M : M(F) \rightarrow a_M := \text{Hom}(X^*(M)_F, \mathbb{R})$$

via

$$\langle H_M(m), \lambda \rangle = \log |\lambda(m)|$$

for $\lambda \in X^*(M)_F$. For each $\lambda \in a^*_M \mathbb{C}$ we then obtain a quasi-character

$$m \mapsto e^{\langle H_M(m), \lambda \rangle}.$$ 

It is clearly unramified. We then define

$$I(\sigma, \lambda) := I(\sigma \otimes e^{\langle H_M(\cdot), \lambda \rangle}).$$

The point of introducing this extra notation is that it makes clear that the induced representation $I(\sigma, \lambda)$ is part of a continuous family, indexed by $a^*_M \mathbb{C}$, of representations. In practice this can be very useful. We employed this construction in the special case where $P$ is a Borel subgroup and $\sigma$ is trivial in our construction of the unramified principal series in §7.6.

### 8.3 Jacquet modules

We now define left-adjoints to the functors $\text{Ind}^G_P$. Let $(\pi, V)$ be a smooth representation of $G(F)$. Set
\[ V(N) := \langle (\pi(n) - \varphi : \varphi \in V, n \in N(F) \rangle \quad \text{and} \quad V_N := V/V(N). \]  

(8.11)

The space \( V_N \) is referred to as the space of coinvariants. We note that since \( M(F) \) normalizes \( N(F) \) it normalizes \( V(N) \) and hence \( M(F) \) acts on \( V_N \). We can therefore define

\[ N : \text{Rep}_{\text{sm}}(G(F)) \rightarrow \text{Rep}_{\text{sm}}(M(F)) \]

(8.12)

\[ (\pi, V) \mapsto (\pi_N := \pi|_{M(F)} \otimes \delta_{\frac{1}{2}}, V_N). \]

The representation \( (\pi_N, V_N) \) is referred to as the \textbf{Jacquet module} of \( (\pi, V) \) (with respect to \( N \)).

\textbf{Proposition 8.3.1} Restriction is left adjoint to induction; in other words for smooth representations \( V \) of \( G(F) \) and \( W \) of \( M(F) \) one has a bijection

\[ \text{Hom}_{G(F)}(V, \text{Ind}_{P}^{G}(W)) \cong \text{Hom}_{M(F)}(V_N, W) \]

that is functorial in \( V \) and \( W \).

\textbf{Proof.} There is an \( M(F) \)-equivariant map

\[ \text{ev}_1 : \text{Ind}_{P}^{G}(W) \rightarrow W \]

\[ \varphi \mapsto \varphi(1) \]

where \( 1 \in G(F) \) is the identity; it is clearly surjective. Thus we have a \( M(F) \)-equivariant map

\[ \text{ev}_1 \circ (\cdot) : \text{Hom}_{G(F)}(V, \text{Ind}_{P}^{G}(W)) \rightarrow \text{Hom}_{M(F)}(V_N, W) \]

given by composition with \( \text{ev}_1 \). Here we are using the fact that \( N \) acts trivially on \( \text{Ind}_{P}^{G}(W) \) and thus any \( G(F) \)-equivariant map \( V \rightarrow \text{Ind}_{P}^{G}(W) \) factors through \( V_N \). To construct the inverse, suppose we are given an \( M(F) \)-intertwining map \( \Phi : V_N \rightarrow W \). We define

\[ \tilde{\Phi} : V \rightarrow \text{Ind}_{P}^{G}(W) \]

\[ \varphi \mapsto (g \mapsto \Phi(g \cdot \varphi)). \]

The map \( \Phi \mapsto \tilde{\Phi} \) is inverse to \( \text{ev}_1 \circ (\cdot) \). \( \square \)

One would hope that the functor \( N \) preserves admissibility, and this is indeed the case:

\textbf{Theorem 8.3.1 (Jacquet)} The functor \( N \) takes admissible representations to admissible representations. \( \square \)

Jacquet also proved the following elegant characterization of quasicuspidal representations using these functors:
Theorem 8.3.2 (Jacquet) A smooth irreducible representation $(\pi, V)$ of $G(F)$ is quasicuspidal if and only if $V_N = 0$ for all parabolic subgroups $P \leq G$.

It is because of these theorems that $V_N$ called the Jacquet module. Strictly speaking Jacquet wrote up his results only for the case of $\text{GL}_n$ [Jac71], but the proof carries over to the general case and was worked out in [Cas].

We will not prove theorems 8.3.1 or 8.3.2, referring the reader to [Cas, §3-4]. In the remainder of this section we use theorems 8.3.1 and 8.3.2 to prove a subrepresentation theorem for admissible representations over nonarchimedean local fields (see Theorem 8.3.3).

Before proving Theorem 8.3.3 we prove a warm-up result. Recall that a supercuspidal representation is an admissible quasicuspidal representation.

Proposition 8.3.2 A quasicuspidal irreducible representation is supercuspidal.

Proof. Let $(\pi, V)$ be a quasicuspidal irreducible representation of $G(F)$. Fix a nonzero $\varphi \in V$. For all compact open subgroups $K \leq G(F)$ one has

$$V^K = \pi(e_K)V = \langle \pi(e_K)\pi(g)\varphi : g \in G(F) \rangle,$$

where

$$e_K := \frac{1}{\text{meas}(K)} \mathbb{1}_K$$

is the idempotent attached to $K$. We must show $V^K$ is finite dimensional. Since $\pi$ admits a central quasi-character (see Exercise 5.6) if $V^K$ is not finite dimensional then there exists \( \{g_n : n \in \mathbb{Z}_{>0}\} \subseteq G(F) \), all inequivalent modulo $Z_G(F)$, such that $\pi(e_K)\pi(g_n)\varphi$ are linearly independent. Let $W \subseteq V^K$ be a $\mathbb{C}$-vector subspace space such that

$$V^K = W \oplus \langle \pi(e_K)\pi(g_n)\varphi : g_n \in G(F), n \in \mathbb{Z}_{>0} \rangle.$$

As $V = V^K \oplus \ker \pi(e_K)$, we can define $\varphi^\vee \in \text{Hom}(V, \mathbb{C})$ such that

$$\langle \pi(e_K)\pi(g_n)\varphi, \varphi^\vee \rangle = n$$

for all $n$ and $\varphi^\vee |_{W \oplus \ker \pi(e_K)} = 0$. Then $\varphi^\vee$ is fixed by $K$, hence is smooth, and hence is an element of $V^\vee$. On the other hand, by construction the support of the matrix coefficient $\langle \pi(g)\varphi, \varphi^\vee \rangle$ is not compact modulo the center $Z_G(F)$. This implies the proposition. □

Combined with Proposition 8.3.2, Theorem 8.3.1 and 8.3.2 allow us to deduce the following concrete manifestation of the philosophy of cusp forms:

Theorem 8.3.3 If $(\pi, V)$ is a smooth irreducible representation of $G(F)$ then one has the following:
(a) There exists a parabolic subgroup $P = MN \leq G$, a supercuspidal representation $(\sigma, W)$ of $M(F)$ and a nonzero intertwining map $V \hookrightarrow \text{Ind}_P^G(W)$.

(b) The representation $\pi$ is admissible.

Proof. The first assertion implies the second, as induction preserves admissibility by Proposition 8.2.1. For the first, we proceed by induction on the dimension of $G$ (as an $F$-algebraic group, say). If $G$ has dimension 1 then it is a torus, and so the result is trivial.

Assume that for all proper parabolic subgroups $P$ there is no nonzero intertwining map $V \to \text{Ind}_P^G(W)$ where $(\sigma, W)$ is a smooth irreducible representation of $M(F)$. Applying Frobenius reciprocity (Proposition 8.3.1) we see that

$$\text{Hom}_{M(F)}(V_N, W) = 0$$

for all parabolic subgroups $P = MN$ and all smooth representations $\sigma$ of $M(F)$, which implies $\pi$ is supercuspidal by Theorem 8.3.2.

Now assume that there is a proper parabolic subgroup $P < G$, a Levi subgroup $M \leq P$, a smooth representation $(\sigma, W)$ of $M(F)$, and a nonzero (hence injective) intertwining map $V \to \text{Ind}_P^G(W)$. By Frobenius reciprocity, there is a nonzero intertwining map

$$V_N \to W$$

of representations of $M(F)$ so we can apply our inductive hypothesis to deduce that there is a parabolic subgroup $Q \leq M$ with Levi subgroup $M_Q$ and a supercuspidal representation $(\rho, U)$ of $M_Q(F)$ and a nonzero intertwining map

$$V_N \to W \to \text{Ind}_Q^M(U).$$

Applying Frobenius reciprocity again we obtain a nonzero intertwining map

$$V \to \text{Ind}_P^G \circ \text{Ind}_Q^M(U) \cong \text{Ind}_{Q_N}^G(U)$$

(see Exercise 8.4 for the last isomorphism). \qed

8.4 The Bernstein-Zelevinsky classification

Theorem 8.3.3 provides the first step towards a classification of all irreducible admissible representations of $G(F)$ in terms of supercuspidal representations. The remaining task is to understand isomorphisms between subquotients of induced representations.

It is useful to start by stepping backwards and recalling the Langlands classification in the nonarchimedean setting. This classifies irreducible admissible representations in terms of tempered representations and is the analogue in the current setting of Theorem 4.8.2.
Fix a minimal parabolic subgroup $P_0 \leq G$ and call a parabolic subgroup **standard** if it contains $P_0$. Let $M(F)^1 := \ker H_M$ where $H_M$ is defined as in (8.9). Let $(\sigma, V)$ be an irreducible admissible representation of $M(F)^1$. We extend it trivially to $M(F)$. Given $\lambda \in a^*_M$ we can form the induced representation $I(\sigma, \lambda)$ as in (8.10).

The following is the analogue of Theorem 4.8.1 in this setting:

**Theorem 8.4.1** If $\sigma$ is unitary and tempered and $\lambda$ is in the positive Weyl chamber then $I(\sigma, \lambda)$ admits a unique irreducible quotient $J(\sigma, \lambda)$. \hfill \Box

The representation $J(\sigma, \lambda)$ is known as the **Langlands quotient** of $I(\sigma, \lambda)$.

**Theorem 8.4.2** Every irreducible admissible representation of $G(F)$ is isomorphic to some $J(\sigma, \lambda)$. Moreover if we insist that the parabolic subgroup defining $J(\sigma, \lambda)$ is standard, fix a Levi decomposition of each standard parabolic subgroup, and stipulate that $\lambda$ is in the positive Weyl chamber then every irreducible representation of $G(F)$ is isomorphic to a $J(\sigma, \lambda)$ that is unique up to replacing $\sigma$ by another representation of $M(F)^1$ equivalent to $\sigma$. \hfill \Box

For the proofs of these theorems we refer to [BW00, §XI.2]. See also [?].

In view of these theorems to classify the admissible irreducible representations of $G(F)$ it suffices to classify the irreducible tempered representations of $G(F)$. This work is largely complete in the case when $G$ was a classical groups, see [Mg02], [MgT02] and [Jan14] for example. The situation for general linear groups is arguably the simplest and we will discuss it in this section. The results are due to Bernstein and Zelevinsky [BZ77] and [Zel80], and the theory is known as the Bernstein-Zelevinsky classification.

We start with the classification of representations that are essentially square integrable. Let $n > 1$, let $a|n$, and let $\sigma$ be an irreducible supercuspidal representation of $GL_{n/a}(F)$. The external tensor product $\sigma^a := \sigma \otimes a$ can then be viewed as a (supercuspidal) representation of $GL_{n/a}(F)$, which we view as a Levi subgroup of the standard parabolic subgroup of $GL_n$ of type $(n/a, \ldots, n/a)$ (see Example 1.10 for our conventions regarding standard parabolic subgroups).

There is then a unique irreducible subquotient $J(\sigma^a, \lambda_a)$ of $I(\sigma^a, \lambda_a)$ where

$$\lambda_a := \left( \frac{a - 1}{2}, \frac{a - 3}{2}, \ldots, \frac{a - 3}{2}, \frac{a - 1}{2} \right) \in X^*(M)_F \otimes \mathbb{C}.$$  

This subquotient is essentially square-integrable. Conversely, all irreducible essentially square integrable representations arise in this manner:

**Theorem 8.4.3 (Bernstein)** Every irreducible (admissible) essentially square-integrable representation $\pi$ of $GL_n(F)$ is isomorphic to $J(\sigma^a, \lambda_a)$ for a pair $(\sigma, a)$ where $a|n$ and $\sigma$ is an irreducible supercuspidal representation of $GL_{n/a}(F)$. The parameter $a$ is uniquely determined by $\pi$, and $\sigma$ is determined by $\pi$ up to isomorphism. \hfill \Box
The only reference for the proof appears to be [Ber, Proposition 42]; the theorem is stated in [Zel80, §9.1]. There is a sketch of the proof in [JS83, §1.2].

To proceed we need the notion of linked representations. Following [BZ77] one calls any set of supercuspidal representations of $GL_r(F)$ for some $r$ of the form

$$\{\sigma \otimes e^{H_{GL_r}(-)}, \sigma \otimes e^{H_{GL_r}(-)}, \ldots, \sigma \otimes e^{H_{GL_r}(-)}\}$$

for some integer $d$ a segment. Two segments are linked if neither is included in the other and their union is a segment.

To each $J(\sigma^a, \lambda_a)$ we associate the segment

$$\{\sigma \otimes e^{H_M(-), \frac{a-1}{2}}, \ldots, \sigma \otimes e^{H_M(-), \frac{-a-1}{2}}\}.$$  

We say that $J(\sigma^a, \lambda_a)$ and $J(\sigma^{a'}, \lambda_{a'})$ (8.13) are linked if their associated segments are linked.

Now assume $\sum_{i=1}^J a_i = n$ and $a_i | n_i$ for all $i$. If $P$ is the standard parabolic of type $(n_1, \ldots, n_j)$ then for any collection of square-integrable representations $J(\sigma_{n_i}^a, \lambda_{a_i})$ of $GL_{n_i}(F)$ as above we can form the induced representation

$$\pi = \text{Ind}(J(\sigma_{n_1}^a, \lambda_{a_1}) \otimes \cdots \otimes J(\sigma_{n_j}^a, \lambda_{a_j}), 0).$$ (8.14)

In the case where $J(\sigma_{n_i}^a, \lambda_{a_i})$ is linked with $J(\sigma_{n_j}^{a'}, \lambda_{a_j})$ if and only if $i = j$ we say that none of the $J(\sigma_{n_i}^a, \lambda_{a_i})$ are linked.

The following theorem is [Zel80, Theorem 9.7(a)]:

**Theorem 8.4.4** If none of the $J(\sigma_{n_i}^a, \lambda_{a_i})$ are linked then the representation $\pi$ in (8.14) is irreducible. □

A representation $\pi$ is irreducible and generic in the sense of §11.2 if and only if it is isomorphic to a $\pi$ as in (8.14) above where none of the $J(\sigma_{n_i}^a, \lambda_{a_i})$ are linked [Zel80, Theorem 9.7].

**Theorem 8.4.5** An irreducible representation of $GL_n(F)$ is tempered if and only if it is of the form (8.14) where all of the $J(\sigma_{n_i}^a, \lambda_{a_i})$ are square integrable. The $J(\sigma_{n_i}^a, \lambda_{a_i})$ are uniquely determined up to reordering indices.

To clarify the assumptions in the theorem, we note that the $J(\sigma_{n_i}^a, \lambda_{a_i})$ are always essentially square integrable, so the assumption of square integrability in the theorem amounts to assuming that the central character of $J(\sigma_{n_i}^a, \lambda_{a_i})$ is unitary.

**Proof.** In [JS83, §1.2] one finds an argument that proves that an irreducible pre-unitary tempered representation is generic. In fact any tempered representation is pre-unitary (see Definition 4.6), so any irreducible tempered
representation is generic. Now since the \( J(\sigma^a_i, \lambda_a) \) are all square integrable none of them are linked by Exercise 8.15. Thus we can then use \([Zel80, Theorem 9.7(b)]\) to conclude the assertion of the theorem.

Combining the Langlands classification of Theorem 8.4.2 and the subsequent description of tempered representations given by Theorem 8.4.5 we can give an analogue of the Langlands classification in terms of square-integrable representations. We defer a discussion of this to §10.5.

8.5 Traces, characters, coefficients

Let \( \pi \) be an admissible representation of \( G(F) \). Then for all \( f \in C^\infty_c(G(F)) \) one has an operator

\[
\pi(f) : V \rightarrow V.
\]

There is a compact open subgroup \( K \leq G(F) \) such that \( f \in C^\infty_c(G(F) / K) \), and hence \( \pi(f) \) induces an operator

\[
\pi(f) : W \rightarrow W
\]

for any finite dimensional subspace \( V^K \leq W \leq V \). We define the trace of \( \pi(f) \) by

\[
\text{tr} \, \pi(f) := \text{tr} \, \pi(f)|_W
\]

for any such \( W \). The notion of the trace of a representation will play a crucial role in the trace formula in later chapters.

In this nonarchimedian setting, a distribution on \( G(F) \) is simply a complex linear functional on \( C^\infty_c(G(F)) \). Thus the trace map

\[
\text{tr} \, \pi : C^\infty_c(G(F)) \rightarrow \mathbb{C}
\]

is a distribution on \( G(F) \), called the character of \( \pi \). Of course this distribution depends on a choice of Haar measure. Let \( G^\text{reg} \leq G \) denote the (open) subscheme consisting of regular semisimple elements. The notion of a regular semisimple element will be discussed in more detail in §17.1 below. For the moment we point out that for fields \( E/F \)

\[
G^\text{reg}(E) := \{ \gamma \in G(E) : C_\gamma \leq G_E \text{ is a maximal torus} \}.
\]

The following is a fundamental and deep result:

**Theorem 8.5.1 (Harish-Chandra)** The distribution \( \text{tr} \, \pi \) is represented by a locally constant function \( \Theta_\pi \) with support in \( G^\text{reg}(F) \).

(see \([HC99]\) for the proof and \([Kot05]\) for a detailed exposition).
In other words there is a locally constant function $\Theta_\pi$ on $G^{\text{reg}}(F)$ such that
\[
\text{tr } \pi(f) = \int_{G(F)} \Theta_\pi(g)f(g)dg
\]
for all $f \in C_c^\infty(G(F))$. This result tells us that we can almost regard $\text{tr } \pi$ as a function.

The following is a version of linear independence of characters adapted to this setting:

**Proposition 8.5.1 (Linear independence of characters)** If $\pi_1, \ldots, \pi_n$ is a finite set of admissible irreducible representations such that $\pi_i \cong \pi_j$ implies $i = j$, then the distributions $\text{tr } \pi_i$ are linearly independent.

**Proof.** We use admissibility to reduce the assertion to a finite-dimensional setting. Fix a compact open subgroup $K \leq G(F)$ such that $V^K_i \neq 0$ for all $i$. This implies that $\{V^K_i\}$ is a finite family of finite dimensional $\mathbb{C}$-vector spaces with an action of $C_c^\infty(G(F) \sslash K)$. They are all simple, that is irreducible, for this action. Moreover, they are pairwise nonisomorphic as $C_c^\infty(G(F) \sslash K)$-modules by an analogue of the argument in the proof of Proposition 7.1.1. Let
\[
A := \text{Im} \left( C_c^\infty(G(F) \sslash K) \longrightarrow \prod_i \text{End}_\mathbb{C}(V^K_i) \right).
\]

Then $A$ is a finite-dimensional $\mathbb{C}$-algebra and the $\{V^K_i\}$ are a finite family of finite-dimensional nonisomorphic simple $A$-modules. Hence the traces $\text{tr } \pi_i|_{C_c^\infty(G(F) \sslash K)}$ are linearly independent. As a reference for this last statement we give [GW09, Lemma 4.1.18]. A simpler argument that depends on the fact that $C_c^\infty(G(F) \sslash K)$ has a unit is given as Exercise 8.9. \qed

Thus traces can be used to distinguish between a finite set of representations. In particular, if $\{\pi_1, \ldots, \pi_n\}$ is a finite set of pairwise nonisomorphic irreducible representations then linear independence of characters implies that we can find an $f \in C_c^\infty(G(F))$ such that
\[
\text{tr } \pi_i(f) = 0 \iff i \neq 1.
\]

For a refinement of this result at the level of operators see Exercise 8.10.

One can ask for more. Let $\pi$ be an admissible irreducible representation. A **coefficient** of $\pi$ is a smooth function $f_\pi \in C_c^\infty(G(F))$ such that $\text{tr } \pi(f_\pi) \neq 0$ and $\text{tr } \pi(f_\pi) = 0$ for $\pi' \not\cong \pi$. Thus if a coefficient for $\pi$ exists, we can use it to isolate $\pi$ among any set of irreducible admissible representations, finite or not. For general $\pi$, it is not necessarily true that such functions $f$ exist. To put this in perspective, it is useful to recall the Heisenberg uncertainty principle, namely that the Fourier transform of a compactly supported smooth function on a real vector space cannot again be compactly supported. Thus if we replaced $G(F)$ with the nonreductive group $\mathbb{R}$ there would be no way to...
construct coefficients. However, we are not in this setting, and in certain circumstances we can construct coefficients:

**Proposition 8.5.2** Assume that $Z_G(F)$ is compact, and let $(\pi, V)$ be an irreducible supercuspidal representation of $G(F)$. Then for all $f \in C^\infty_c(G(F))$ there exists a unique $f_\pi \in C^\infty_c(G(F))$ such that

$$\pi(f_\pi) = \pi(f) \text{ and } \pi'(f_\pi) = 0 \text{ if } \pi' \not\simeq \pi.$$ 

As an immediate consequence, we see that coefficients exist for supercuspsidals. The proof we give depends on our original definition of supercuspidal representations, that is, they are admissible representations whose matrix coefficients are compactly supported modulo the center.

To prove Proposition 8.5.2 we collect some observations on the space of endomorphisms of a representation. For any smooth representation $(\pi, V)$ of $G(F)$ the $\mathbb{C}$-vector space $\text{End}(V) := \text{Hom}_\mathbb{C}(V, V)$ is naturally a $G(F) \times G(F)$-module, where one copy of $G$ acts via precomposition and the other via postcomposition. The action is given explicitly by $(g_1, g_2).A = \pi(g_1) \circ A \circ \pi(g_2^{-1})$. We let

$${\text{End}}_{\text{sm}}(V) \leq \text{End}(V)$$

denote the subspace consisting of **smooth endomorphisms**, that is, endomorphisms that are left and right invariant by a compact open subgroup $K \leq G$. This is a smooth representation of $G(F) \times G(F)$.

The usual isomorphism $V \otimes \text{Hom}_\mathbb{C}(V, \mathbb{C}) \rightarrow \text{End}(V)$ given on pure tensors by

$$(\varphi \otimes \varphi^\vee) \mapsto (\varphi \mapsto \langle \varphi_0, \varphi^\vee \rangle \varphi)$$

is $G(F) \times G(F)$-equivariant and upon restriction induces an isomorphism $V \otimes V^\vee \rightarrow \text{End}_{\text{sm}}(V)$ (here all of the tensor products are over $\mathbb{C}$). Thus the action of $G(F) \times G(F)$ on $\text{End}_{\text{sm}}(V)$ can be reasonably denoted by $\pi \otimes \pi^\vee$ (for more details on product representations see Theorem 5.7.2 above). This is a special case of an exterior tensor product; unfortunately, it seems that the notation $\otimes$ is used more commonly for this than $\boxtimes$. If $(\pi, V)$ is admissible, then $\text{End}_{\text{sm}}(V)$ is also admissible (see Exercise 8.13).

One also has an intertwining map

$$\beta : (\pi \otimes \pi^\vee, \text{End}_{\text{sm}}(V)) \rightarrow (\rho, C^\infty_c(G(F)))$$

given by
Here $\rho$ acts via $\rho(g_1, g_2)(f)(h) = f(g_1^{-1}hg_2)$. We note that for each $A \in \text{End}_{\text{sm}}(V)$ the function $\beta(A)$ is a sum of matrix coefficients. Indeed, if we choose a compact open subgroup $K \leq G(F)$ such that $A$ is fixed on the left and right under $K$, choose a basis $B$ of $V^K$ and a dual basis $B^\vee$ of $V^\vee^K$ then
\[
\beta(A)(g) = \sum_{\varphi \in B} \langle \pi(g) \circ A \varphi, \varphi^\vee \rangle.
\] (8.17)

Proof of Proposition 8.5.2: Since $Z_G(F)$ is compact by assumption and $\pi$ is supercuspidal, we have $\beta(\text{End}_{\text{sm}}(V)) \leq C^\infty_c(G(F))$. Since there are $\varphi \in V$ and $\varphi^\vee \in V^\vee$ such that $\langle \varphi, \varphi^\vee \rangle = \langle \pi(1)\varphi, \varphi^\vee \rangle \neq 0$, we have that $\beta$ is not identically zero. Thus since the exterior tensor product $\pi \otimes \pi^\vee$ is irreducible $\beta$ is an embedding.

Consider
\[
\beta'(\rho, C^\infty_c(G(F))) \rightarrow (\pi \otimes \pi^\vee, \text{End}_{\text{sm}}(V))
\] (8.18)
\[f \mapsto \pi(f).\]

Then $\beta' \circ \beta$ is an endomorphism of the irreducible representation $\text{End}_{\text{sm}}(V)$ of $G(F) \times G(F)$. Hence $\beta' \circ \beta$ is scalar by Schur’s lemma (see Exercise 5.5), say $\beta' \circ \beta = \lambda \text{Id}$ for some $\lambda \in \mathbb{C}$. We will now show that $\lambda \neq 0$ and that we can take
\[f_\pi := \lambda^{-1} \beta \circ \beta'(f).
\]
First,
\[\pi(\beta \circ \beta'(f)) = \beta' \circ \beta \circ \beta'(f) = \lambda \beta'(f) = \lambda \pi(f).
\]
To show that $\lambda \neq 0$, note that we can find $f \in C^\infty_c(G(F))$ such that $\pi(f) \neq 0$ (take, for example, $f$ to be the characteristic function of a sufficiently small compact open subgroup). Thus $\beta'(f) = \pi(f) \neq 0$, and since $\beta$ is an embedding, we deduce that $\beta \circ \beta'(f) \neq 0$.

Second, let $(\pi', V')$ be a smooth irreducible representation of $G(F)$, and let $\varphi' \in V'$ be a non-zero vector. Let
\[
\gamma : (\rho|_{G(F) \times 1}, C^\infty_c(G(F))) \rightarrow (\pi', V')
\] (8.19)
\[f \mapsto \pi'(f)\varphi'.
\]
As a representation of $G(F)$ one has that $\text{End}_{\text{sm}}(V)|_{G(F) \times 1}$ is isomorphic to a direct sum of copies of $\pi$, and thus the same is true of $\gamma(\beta(\text{End}_{\text{sm}}(V)))$. Thus $\gamma(\beta(\text{End}_{\text{sm}}(V))) = 0$ unless $\pi' \cong \pi$ (since whenever the former is nonzero we...
obtain an intertwining operator between $\pi'$ and $\pi$. It follows that $\pi'(f_\pi) = 0$ if $\pi' \not\cong \pi$. This completes the proof of the proposition.

\section{Parabolic descent of representations}

Parabolic descent is a term for the process by which one passes from objects (say, representations) on a group to objects on a Levi subgroup of a parabolic subgroup. In this section we give two examples of this phenomenon. First we show that the character of a parabolically induced representation is easily related to the character of the inducing representation (Proposition 8.6.1).

Let $P \leq G$ be a parabolic subgroup with Levi decomposition $P(F) = M(F)N(F)$ and let $K \leq G(F)$ be a maximal compact subgroup.

\begin{definition}
A maximal compact subgroup $K \leq G(F)$ is said to be in good position with respect to $(P, M)$ if $G(F) = P(F)K$ and $P(F) \cap K = (M(F) \cap K)(N(F) \cap K)$.
\end{definition}

Given a parabolic subgroup $P$ we can always find a maximal compact subgroup in good position with respect to $P$. We can even assume that

$$K_M := M(F) \cap K$$

is a maximal compact open subgroup (see Theorem A.4.2); we will also assume this in what follows.

We assume throughout this section that Haar measures are chosen so that

$$dg = \delta_P(m)dkdn$$  \hspace{1cm} (8.20)

where $dk(K) = dn(N(F) \cap K) = 1$ (see Exercise 8.2).

The constant term of $f \in C_c^\infty(G(F))$ along $P$ is the function

$$f^P(m) := \delta_P^{1/2}(m) \int_{N(F)} f(mn)dn.$$  \hspace{1cm} (8.21)

This is equal to (7.17) in the special case considered in §7.5.

Let $T \leq M$ be a maximal split torus, and let

$$C_c^\infty(G(F); K) := \{ f \in C_c^\infty(G(F)) : f(k^{-1}xk) = f(x) \text{ for all } k \in K \}.$$

We note that $C_c^\infty(G(F); K)$ is precisely the image of the map

$$C_c^\infty(G(F)) \rightarrow C_c^\infty(G(F); K)$$

$$f \mapsto f_K$$  \hspace{1cm} (8.22)
where \( f_K(x) := \int_K f(k^{-1}xk)dk \). The space \( C_c^\infty(G(F); K) \leq C_c^\infty(G(F)) \) is a subalgebra under convolution, and one trivially has

\[
\text{tr} \pi(f) = \text{tr} \pi(f_K).
\]  

(8.23)

**Proposition 8.6.1** The constant term gives a map

\[
C_c^\infty(G(F); K) \to C_c^\infty(M(F); K_M)
\]

\[
f \mapsto f^P.
\]

If \((\sigma, V)\) is an admissible representation of \(M(F)\), then

\[
\text{tr} I(\sigma)(f) = \text{tr} \sigma(f^P).
\]

**Remark 8.1.** The archimedean analogue of this proposition is valid as well. One reference is [Kna02, (10.23)].

We follow the proof of [Lau96, Lemma 7.5.7] in the following:

**Proof.** Since \( K \) is compact, \( \delta_P(k) = 1 \) for all \( k \in M(F) \cap K \). It follows that \( f^P \in C_c^\infty(M(F); K_M) \) as claimed.

We now prove the equality of traces. Consider the space \( \text{Ind}^G_P(V)|_K \) of locally constant functions

\[ \varphi : K \to V \]

such that \( \varphi(mnk) = \sigma(m)\varphi(k) \) for \((m, n, k) \in K_M \times (N(F) \cap K) \times K\). There is a representation \( I(\sigma|_K) \) of \( K \) on this space given by

\[
I(\sigma|_K)(k)\varphi(g) = \varphi(gk).
\]

(8.24)

By the Iwasawa decomposition restriction of functions to \( K \) induces an isomorphism

\[
\text{Ind}^G_P(V) \to \text{Ind}^G_P(V)|_K
\]

(8.25)

that intertwines \( \text{Res}^G_K(I(\sigma)) \) with \( I(\sigma|_K) \). Since this is an isomorphism, \( I(\sigma)(f) \) induces an endomorphism of \( \text{Ind}^G_P(V)|_K \), and \( \text{tr} I(\sigma)(f) \) is equal to the trace of this endomorphism on \( \text{Ind}^G_P(V) \). Define

\[
\Phi_{k,k'}(m) := \delta_P^{1/2}(m) \int_{N(F)} f(k^{-1}mnk')dn
\]

for \( k, k' \in K \) and \( m \in M(F) \). Then if \( \varphi \in \text{Ind}^G_P(V) \) one has

\[
I(\sigma)(f)\varphi(k) = \int_{G(F)} f(g)\varphi(kg)dg
\]

\[
= \int_{G(F)} f(k^{-1}g)\varphi(g)dg
\]
8.6 Parabolic descent of representations

\[
\int_M (\mathcal{F}) \int_N (\mathcal{F}) \int_K f(k^{-1} mnk') \varphi(mnk') dk' dnm
\]

\[
= \int_M (\mathcal{F}) \int_N (\mathcal{F}) \int_K f(k^{-1} mnk') \delta_P(m)^{1/2} \sigma(m) \varphi(k') dk' dnm
\]

\[
= \int_K \sigma(\Phi_{k,k'}) \varphi(k') dk'.
\]

It follows that

\[
\text{tr} I(\sigma)(f)
\]

is the trace of the endomorphism \( \varphi \mapsto (k \mapsto \int_K \sigma(\Phi_{k,k'}) \varphi(k') dk') \). This endomorphism has trace

\[
\int_K \text{tr} \sigma(\Phi_{k,k}) dk.
\]

On the other hand

\[
f^P = \int_K \Phi_{k,k} dk.
\]

The lemma follows. \( \square \)

As an addendum, we prove the following lemma:

**Lemma 8.6.1** The constant term provides an algebra homomorphism

\[
C^\infty_c(G(F) \parallel K) \longrightarrow C^\infty_c(M(F) \parallel K_M).
\]

We used this lemma in the special case where \( G \) is unramified and \( P = B \) in \S 7.5:

**Proof.** Let \( f_1, f_2 \in C^\infty_c(G(F) \parallel K) \). One has

\[
(f_1 \ast f_2)^P(m) = \delta_P(m)^{1/2} \int_{N(F)} (f_1 \ast f_2)(mn) dn
\]

\[
= \delta_P(m)^{1/2} \int_{N(F)} \left( \int_{G(F)} f_1(mng^{-1}) f_2(g) dg \right) dn.
\]

Using Exercise 8.2, we rewrite this as

\[
\delta_P^{1/2}(m) \int_{N(F)} \left( \int_{K \times M(F) \times N(F)} \delta_P(g') f_1(gn(k'm'n')^{-1}) f_2(k'm'n') dk' dm' dn' \right) dn
\]

\[
= \delta_P^{1/2}(mm') \int_{N(F)} \left( \int_{K \times M(F) \times N(F)} f_1(mm'^{-1}nn'^{-1}) f_2(m'n') dk' dm' dn' \right) dn
\]

\[
= \delta_P^{1/2}(mm') \int_{M(F)} \left( \int_{N(F)} f_1(mm'^{-1}) dn \right) \left( \int_{N(F)} f_2(m'n') dn' \right) dm'
\]

\[
= f_1^P \ast f_2^P(m).
\]
This proves our assertion that the constant term is an algebra morphism. □

8.7 Parabolic descent of orbital integrals

In our discussion of the Satake isomorphism in §7.5 we claimed that the constant term map has image in Weyl invariant functions on the maximal torus. We prove this in the current section and use it as an opportunity to discuss descent of orbital integrals.

We keep the notation of the previous section; thus $P = MN$ is a parabolic subgroup of $G$, $K \leq G(F)$ is a maximal compact subgroup in good position with respect to $P$, and $K_M := M(F) \cap K$ is maximal in $K$.

Let $\gamma \in G(F)$ and let $G_\gamma$ be the centralizer of $\gamma$:

$$G_\gamma(R) = \{ g \in G(R) : g^{-1}\gamma g = \gamma \}.$$ (8.26)

We use the same notation when $G$ is replaced by $M$ if $\gamma$ is contained in $M(F)$.

We say that $\gamma \in G(F)$ is closed if the orbit $O(\gamma)$ of $\gamma$ under conjugation is closed as a subscheme of $G$. In characteristic zero, this is equivalent to $\gamma$ being semisimple [Ste65, Corollary 6.13, Proposition 6.14], and the same is true in positive characteristic provided that one takes the correct notion of semisimple; a good reference is [Lau96, §4.3] (which has a different convention regarding closed orbits). For a closed element the neutral component of the stabilizer $G_\gamma$ is reductive. For more details (including proofs and references) we refer to §17.1.

Assume that $\gamma \in G(F)$ is semisimple. Then $G_\gamma(F)$ is unimodular, and upon choosing a Haar measure $dg_\gamma$ on $G_\gamma(F)$ we can form a right $G(F)$-invariant Radon measure $\frac{dg}{dg_\gamma}$ on the quotient $G_\gamma(F) \backslash G(F)$ (see Theorem 3.2.1). We then define the orbital integral

$$O_\gamma(f, dg_\gamma) := \int_{G_\gamma(F) \backslash G(F)} f(g^{-1}\gamma g) \frac{dg}{dg_\gamma}.$$ (8.27)

We will check that this orbital integral is absolutely convergent as a special case of Proposition 17.3.1 below. If the measure is understood it is often omitted from notation.

Let $H \leq G$ be a connected subgroup. For $F$-algebras $R$ and $h \in H(R)$ it is convenient to define

$$D_{H,G}(h) := \det \left( I - \operatorname{Ad}(h)^{-1} : \mathfrak{h}(F) \backslash \mathfrak{g}(F) \to \mathfrak{h}(F) \backslash \mathfrak{g}(F) \right)$$ (8.27)

where (as usual) $\mathfrak{h} := \operatorname{Lie} H$ and $\mathfrak{g} := \operatorname{Lie} G$ and where $I$ is the identity map.

There is a dual relationship between orbital integrals and characters that can be made precise in many ways; one of the most profound is the absolute
trace formula (see §18.3). The following proposition is consonant with this duality:

**Proposition 8.7.1** Assume that \( \gamma \in G(F) \) is closed and \( G_\gamma \leq M \) (so \( G_\gamma = M_\gamma \)). If \( D_{M\setminus G}(\gamma) \neq 0 \) then one has

\[
O_\gamma(f, dg_\gamma) = |D_{M\setminus G}(\gamma)|^{-1/2}O_\gamma(fP, dg_\gamma).
\]

Here we normalize the Haar measure \( dg \) on \( G(F) \) as in (8.20). The following proof is taken from [Lau96, Proposition 4.3.11]:

**Proof.** Upon taking a change of variables \( n \mapsto mnm^{-1} \) we see that the normalization of the Haar measure in (8.20) is equivalent to

\[
d(mnk) = dmdndk.
\]

Thus one has

\[
O_\gamma(f, dg_\gamma) = \int_{M_\gamma(F)\setminus M(F)} \int_{N(F)} \int_K \frac{f(k^{-1}m^{-1}\gamma mnk)dkdn}{\frac{dm}{J(\gamma)m_\gamma}}.
\]

For each \( m \in M(F) \) there is a morphism of affine \( F \)-schemes \( N \rightarrow N \) given on points in an \( F \)-algebra \( R \) by

\[
N(R) \rightarrow N(R)
n \mapsto (m^{-1}m)^{-1}n^{-1}(m^{-1}\gamma m)n.
\]

We claim that this morphism is an isomorphism. The image is an orbit of the identity under an action of the unipotent group \( N \), and is therefore closed in \( N \) by a theorem of Rosenlicht [Ros61, Theorem 2]. On the other hand the morphism is injective at the level of points. Indeed if \( \gamma n^{-1}\gamma n = \gamma n'^{-1}\gamma n' \) for \( n, n' \in N(R) \) then \( n'n^{-1} \in G_\gamma(R) \) which implies \( n'n^{-1} = I \) by our assumption that \( G_\gamma \leq M \). The same is true if we replace \( \gamma \) by \( m^{-1}\gamma m \) and injectivity follows. In particular the stabilizer of the identity element is trivial. We concluded using Lemma ?? that the morphism is an isomorphism as claimed.

The Jacobian depends only on \( \gamma \) and is equal to

\[
J(\gamma) = |\det (I - \text{Ad}(\gamma) : n(F) \rightarrow n(F))|.
\]

We note that

\[
|D_{M\setminus G}(\gamma)| = \delta_P(\gamma)J(\gamma)^2.
\]

Therefore we can change variables and deduce that the integral above is

\[
\int_{M_\gamma(F)\setminus M(F)} \int_{N(F)} \int_K \frac{f(k^{-1}m^{-1}\gamma mnk)dkdn}{J(\gamma)m_\gamma}.
\]
and deduce the proposition.

\[ \square \]

Remark 8.2. This is the first time we have used the usual change of variables formula in the nonarchimedean setting. A general discussion of integration on manifolds over nonarchimedean fields, including the change of variables formula, is in [Igu00, Chapter 7]. We refer especially to [Igu00, Proposition 7.4.1] and [Igu00, §7.6].

We now prove the following proposition, used in §7.5:

**Proposition 8.7.2** Assume that \( G \) is quasi-split with Borel subgroup \( B \), that \( M \leq B \) is a maximal torus and that \( T \leq M \) is its maximal split subtorus. The constant term induces a homomorphism

\[
C_c^\infty(G(F) / K) \longrightarrow C_c^\infty(T(F)/K_T)^{W(T,G)(F)}.
\]

In the case at level \( M \), Proposition 8.7.2 is a refinement of Lemma 8.6.1.

To prove the proposition we require the following proposition:

**Proposition 8.7.3** With assumptions as in Proposition 8.7.2 for any \( \gamma \in M(F) \)

\[
|D_{M,G}(\gamma)| \neq 0
\]

if and only if \( G_\gamma = M \). The set of elements of \( M(F) \) where \( G_\gamma = M \) is dense in \( M(F) \) in the natural topology.

By way of terminology, an element \( \gamma \in M(F) \) such that \( G_\gamma = M \) is called regular.

**Proof.** Clearly \( M \leq G_\gamma \) because \( M \) is connected and commutative. Since \( I - \text{Ad}(\gamma) \) acts by zero on the subspace

\[
m \setminus g_\gamma \leq m \setminus g
\]

we deduce that \( |D_{M,G}(\gamma)| \neq 0 \) if and only if \( m = g_\gamma \), which is to say that \( G_\gamma = M \). Here \( m = \text{Lie } M \) and \( g_\gamma = \text{Lie } G_\gamma \).

To deduce that the set of elements of \( M(F) \) where \( G_\gamma = M \) is dense in \( M(F) \) in the natural topology we remark that the set of points where a nonzero polynomial on a (nonarchimedean) analytic manifold does not vanish is necessarily dense. \[ \square \]

Remark 8.3. There are surprisingly few textbook treatments of analytic manifolds in the nonarchimedean setting; one is [Ser06] and some additional topics are in [Igu00]. Perhaps the reason is that more sophisticated treatments involving rigid analytic spaces, Berkovich spaces, etc. are often needed and have drawn more attention.

We now prove Proposition 8.7.2:
8.7 Parabolic descent of orbital integrals

Proof of Proposition 8.7.2: Every element of \( W(T, G)(F) \) has a representative \( w \in N_G(T)(F) \), the normalizer of \( T \) in \( G(F) \) (see above [Spr09, 16.1.3]). Thus it suffices to show that for \( f \in C_c^\infty(G(F) \backslash K) \) one has

\[
f^B(w^{-1}tw) = f^B(t)
\]

for \( t \in T(F) \). We will in fact show the stronger statement that

\[
f^B(w^{-1}\gamma w) = f^B(\gamma)
\]

for all \( \gamma \in M(F) \).

The set of \( \gamma \in M(F) \) such that \( G_{\gamma}^0 = M \) is dense by Proposition 8.7.3, and for such a \( t \) one has \( |D_{M \backslash G}(t)| \neq 0 \). Let \( dm \) be the Haar measure on \( M(F) \) used in (8.20). We then have

\[
O_\gamma(f, dm) = |D_{M \backslash G}(\gamma)|^{-1/2}O_t(f^B, dm) = |D_{M \backslash G}(\gamma)|^{-1/2}f^B(\gamma)
\]

by Proposition 8.7.1. Here the last equality follows since \( M(F) \) is commutative. It is clear that \( O_{w^{-1}\gamma w}(f, dm) = O_\gamma(f, dm) \), so at least for \( \gamma \in M(F) \) such that \( G_{\gamma}^0 = M \) we deduce (8.28). This set is dense in \( M(F) \) by Proposition 8.7.3. It is clear that \( f^B \) is continuous, so we deduce (8.28) for all \( \gamma \in M(F) \).

\[
\square
\]

Exercises

In all of these exercises \( G \) is a connected reductive group over a local field \( F \) and \( P \leq G \) is a parabolic subgroup with a fixed Levi subgroup \( M \leq P \) and unipotent radical \( N \).

8.1. Let \( (\pi, V) \) be a unitary representation of \( G(F) \) on a Hilbert space \( V \). Let \( V_{sm} \leq V \) be the subspace of smooth vectors (see Exercise 5.2). Construct an isomorphism \( V_{sm} \cong (V^{sm})' \). Using this isomorphism, show that a matrix coefficient in the sense of (8.4) is a matrix coefficient in the sense of §4.7.

8.2. Assume that \( K \leq G(F) \) is a maximal compact subgroup in good position with respect to \( (P, M) \). Show that we can normalize the Haar measures \( dg, dk, dm, dn \) on \( G(F), K, M(F), \) and \( N(F) \), respectively, so that

\[
dg(kmn) = \delta_P(m)dkdmn
\]

where \( dk(K) = dn(N(F) \cap K) = 1 \).

8.3. Show that we can choose a function \( f \in C^\infty(G(F)) \) such that
\[ \int_{P(F)} f(px) d_x p = 1 \]

for all \( x \in G(F) \).

8.4. Let \( Q \leq M \) be a parabolic subgroup. There is a natural transformation of functors

\[ \text{Ind}_P^G \circ \text{Ind}_Q^M \cong \text{Ind}_{QN}^G \]

The analogous statement for Jacquet modules is true as well.

8.5. The functor \( \text{Ind}_P^G \) is exact (that is, it preserves exact sequences) and additive (that is, takes direct sums to direct sums).

8.6. The functor \( \pi \mapsto \pi_N \) is exact (sends exact sequences to exact sequences) and additive (sends direct products to direct products).

8.7. Let \( (\pi, V) \) be a smooth representation of \( G(F) \). An element \( \varphi \in V \) is in \( V(N) \) if and only if \( \int_{K_N} \pi(n) \varphi dn = 0 \) for some compact subgroup \( K_N \leq N(F) \).

8.8. Prove that \( G^{\text{reg}} \), defined as in (8.16), is an open subscheme of \( G \).

8.9. Assume that \( A \) is a finite-dimensional (not necessarily commutative) \( \mathbb{C} \)-algebra with unit and that \( V_1, \ldots, V_n \) are finite-dimensional pairwise non-isomorphic simple nonzero \( A \)-modules. Prove that the functions

\[ A \longrightarrow \mathbb{C} \]

\[ a \mapsto \text{tr}(a : V_i \rightarrow V_i) \]

are linearly independent over \( \mathbb{C} \).

8.10. Let \( (\pi_1, V_1), \ldots, (\pi_n, V_n) \) be a finite set of irreducible admissible representations of \( G(F) \) such that \( \pi_i \cong \pi_j \) if and only if \( i = j \). Let \( K \leq G(F) \) be a compact open subgroup such that \( V_i^K \neq 0 \). Show that there exists an \( f \in C_c^\infty(G(F) \setminus K) \) such that \( \pi_i(f)|_{V_i^K} \) is the identity and \( \pi_i(f) = 0 \) if \( 1 < i \leq n \).

8.11. Without using the Bernstein-Zelevinsky classification, prove that an unramified irreducible admissible representation of \( \text{GL}_n(F) \) is tempered if and only if its Satake parameters have complex norm 1.

8.12. Suppose that \( \pi \) is a supercuspidal representation of \( G(F) \). Show that there exists a function \( f_\pi \in C_c^\infty(G(F)) \) such that \( \text{tr} \pi(f_\pi) = 1 \) and if \( \text{tr} \pi'(f_\pi) \neq 0 \) for some irreducible admissible representation \( \pi' \) of \( G(F) \) then \( \pi \cong \pi' \otimes \chi \) for some character \( \chi : Z_G(F) \rightarrow \mathbb{C} \).

8.13. Let \( (\pi, V) \) be an admissible representation of a \( \text{td} \) group \( G \). Prove that the space of smooth endomorphisms \( \text{End}_{\text{sm}}(V) \) is an admissible representation of \( G \times G \).

8.15. In the notation of §8.4, show that if $J(\sigma, \lambda_a)$ and $J(\sigma', \lambda_{a'})$ are both square integrable (not just essentially square integrable) then they are not linked.
References


[Cas] W. Casselman. Introduction to the theory of admissible representations of $p$-adic reductive groups. 158, 163


