Abstract In this section we describe the classification of unramified representations of connected reductive groups over nonarchimedean local fields. Along the way we discuss the Satake isomorphism and the Langlands dual group.

7.1 Unramified representations

By Flath’s theorem, if $\pi$ is an automorphic representation of $G(\mathbb{A}_F)$, then

$$\pi \cong \bigotimes_v \pi_v$$

where for almost every $v$ the representation $\pi_v$ is unramified in the sense that it contains a (unique) vector fixed under a hyperspecial subgroup $K \leq G(F_v)$. In this section we discuss the classification of unramified representations. It turns out that they can be explicitly parametrized in terms of conjugacy classes in the dual group of $G$ (see Theorem 7.2.1 and Theorem 7.5.1). This
fundamental fact will be used in §7.7 to state a version of the Langlands functoriality conjecture.

In this section we let $G$ be a reductive group over a non-archimedean local field $F$. Our purpose here is to study unramified representations. Recall that $G$ is unramified if $G$ is quasi-split and split over an unramified extension of $F$. In this case there exists a hyperspecial subgroup $K \leq G(F)$.

**Definition 7.1.** An irreducible representation $(\pi, V)$ of $G(F)$ is called unramified if $G$ is unramified and $V^K \neq 0$.

Notice that we do not assume that $V$ is admissible or even smooth. In fact, an unramified representation is automatically smooth and admissible (see Exercise 7.2). It is important to realize also that when we speak of unramified representations we should really speak of $K$-unramified representations. Hyperspecial subgroups of $G$ are not necessarily $G(F)$-conjugate, they are only conjugate under $G^{ad}(F)$ (see [Tit79, §2.5]). Here $G^{ad} := G/Z_G$ is the adjoint quotient of $G$. We will however follow tradition and suppress the dependence on $K$ (but see §12.5).

Let $K \leq G(F)$ be a hyperspecial subgroup. Then the subalgebra

$$C_c^\infty(G(F) \sslash K) \leq C_c^\infty(G(F))$$

is known as the unramified Hecke algebra of $G(F)$ (with respect to $K$). Let $f \in C_c^\infty(G(F) \sslash K)$ and let $\pi$ be unramified. Then $\pi(f)$ acts via a scalar on $V^K$. It is sensible to denote the scalar by $\text{tr} \pi(f)$ (see also §8.5). The map

$$C_c^\infty(G(F) \sslash K) \to \mathbb{C}
\quad f \mapsto \text{tr} \pi(f)$$

is called the Hecke character of $\pi$.

**Proposition 7.1.1** An unramified representation $\pi$ is determined up to isomorphism by its Hecke character.

**Proof.** Say that a representation $(\pi, V)$ of $G(F)$ is generated by a subspace $W \leq V$ if $V = \pi(G)W$.

There is an equivalence of categories

$$\{\text{representations of } G(F) \text{ generated by } V^K\} \xrightarrow{\sim} \{C_c^\infty(G(F) \sslash K)\text{-modules}\}
\quad V \mapsto V^K.$$

Here we have given the objects of the categories on both sides; morphisms are simply intertwining maps.

Any irreducible representation is generated by any nonzero subspace, so an unramified representation is generated by $V^K$. We deduce the proposition. □
7.2 Satake isomorphism

If we don’t know anything about \( C_c^\infty(G(F) \backslash K) \), then we could hardly regard Proposition 7.1.1 as useful. However, it turns out that \( C_c^\infty(G(F) \backslash K) \) has a simple description:

**Theorem 7.2.1 (Satake)** Assume that \( G \) is split. There is an isomorphism of algebras

\[
S : C_c^\infty(G(F) \backslash K) \rightarrow \mathbb{C}[\hat{T}]^{W(\hat{G}, \hat{T})(\mathbb{C})}
\]

where \( \hat{G} \) is the complex connected reductive algebraic group with root datum dual to that of \( G \) and \( \hat{T} \leq \hat{G} \) is a maximal torus.

To gain intuition for the Satake isomorphism, let us consider the special case of \( GL_n \). The Hecke algebra \( C_c^\infty(GL_n(F) \backslash GL_n(\mathcal{O}_F)) \), as a \( \mathbb{C} \)-module, has a basis given by

\[
\mathbb{I}_{\lambda} := \mathbb{I}_{GL_n(\mathcal{O}_F)} \left( \begin{array}{cccc} \varpi_{\lambda_1} & & & \\
 & \varpi_{\lambda_2} & & \\
 & & \ddots & \\
 & & & \varpi_{\lambda_n} \\
\end{array} \right)_{GL_n(\mathcal{O}_F)}
\]

with \( \lambda := (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \), \( \lambda_i \geq \lambda_{i+1} \) for all \( 1 \leq i \leq n - 1 \) (this follows from the theory of elementary divisors). As a \( \mathbb{C} \)-algebra it is generated by \( \mathbb{I}_{\lambda} \) with \( \lambda = (1^r) := (1, \ldots, 1, 0, \ldots, 0) \) (\( r \) ones and \( n - r \) zeros) for \( 1 \leq r \leq n \) and \( \lambda = ((-1)^n) := (-1, \ldots, -1) \).

On the generating set above the Satake isomorphism is given by

\[
S(\mathbb{I}_{(1^r)}) = q^{r(n-r)/2} \text{tr}(\wedge^r \mathbb{C}^n) \quad (7.1)
\]

where \( \mathbb{C}^n \) is the standard representation of \( GL_n(\mathbb{C}) \), \( \wedge^r \mathbb{C}^n \) is the \( r \)th exterior power of the standard representation, and \( \text{tr}(\wedge^r \mathbb{C}^n) \) is the trace of a diagonal matrix in \( GL_n(\mathbb{C}) \) acting on the given representation (see [Gro98, (3.14)]).

It is instructive to indicate various ways of rephrasing this basic result. By the Chevalley restriction theorem, the inclusion \( \hat{T} \hookrightarrow \hat{G} \) induces an isomorphism

\[
\text{Spec}(\mathbb{C}[\hat{T}]^{W(\hat{G}, \hat{T})}) = \hat{T}^{W(\hat{G}, \hat{T})} \rightarrow \hat{G}/\text{conj} = \text{Spec}(\mathbb{C}[\hat{G}]/\hat{G}) \quad (7.2)
\]

where the quotient on the right is the scheme theoretic quotient of \( \hat{G} \) by the action of conjugation (for more on scheme-theoretic quotients over fields [MFK94] is the canonical reference). In particular \( \hat{G}/\text{conj}(\mathbb{C}) \) is just the set of closed conjugacy classes in \( G(\mathbb{C}) \). The closed orbits are precisely the orbits of semisimple elements; a nice reference for this is [Ste65, Corollary 6.13]. Thus we have a sequence of isomorphisms
\[ \text{Hom}(C^\infty_c(G(F) \sslash K), \mathbb{C}) \xrightarrow{(S^{-1})^*} \text{Hom}(\mathbb{C}[\hat{T}]^W(\hat{G}, \hat{T}), \mathbb{C}) \longrightarrow G^{ss}(\mathbb{C})/\text{conj} \]

where the first map is pullback along the inverse of the Satake isomorphism. Here homomorphism means homomorphism of \(C\)-algebras, and \(G^{ss} \subset G\) is the closed subscheme consisting of semisimple elements. In Proposition 7.1.1 we saw that unramified representations were in bijection with Hecke characters, which are precisely elements of \(\text{Hom}(C^\infty_c(G(F) \sslash K), \mathbb{C})\). We have therefore proven the following corollary of the Satake isomorphism:

**Corollary 7.2.1** Assume that \(G\) is split. The composite isomorphism (7.3) induces a bijection between semi-simple conjugacy classes in \(\hat{G}(\mathbb{C})\) and isomorphism classes of irreducible unramified representations of \(G(F)\).

From the point of view of automorphic representation theory the fact that the Satake isomorphism is only valid for split groups is problematic. Indeed, suppose for the moment that \(G\) is a reductive group over a global field \(F\). Then for all but finitely many places \(v\) of \(F\) the group \(G_F\) is unramified by Proposition 2.4.1 and hence in particular is quasi-split. However, the group \(G_F\) can be nonsplit for infinitely many \(v\) (see Exercise 7.3).

Langlands was able to extend the Satake isomorphism to the quasi-split case. Historically this was important because it gave crucial hints as to the structure of the Langlands dual group, which he introduced at the same time [? ] (see also [Lan]). We will define the Langlands dual group in the next section and use it to extend the Satake isomorphism in §7.5.

### 7.3 The Langlands dual group

For the moment, let \(G\) be a connected reductive group over a global or local field \(F\) and let \(T \leq G\) be a maximal torus. To these data we associated in §1.8 a root datum \(\Psi(G, T) = (X^*(T), X_*(T), \Phi, \Phi^\vee)\). We remind the reader that \(X^*(T)\) and \(X_*(T)\) are the character groups, \(\Phi \subset X^*(T)\) is the set of roots of \(T\) in \(\mathfrak{g}\) and and \(\Phi^\vee \subset X_*(T)\) is the set of coroots. We also remind the reader that the root datum characterizes \(G_{F^{\text{sep}}}\) where \(F^{\text{sep}}\) is a separable closure of \(F\).

If \(G\) is split, then we set

\[ L^G := \hat{G}(\mathbb{C}) \times \text{Gal}(F^{\text{sep}}/F) \]

where \(\hat{G}\) is the complex dual group defined as in §1.8; it is the connected reductive group over \(\mathbb{C}\) with root datum \((X_*(T), X^*(T), \Phi^\vee, \Phi)\).

If \(G\) is not split then the \(L\)-group has a more complicated definition involving a Galois action on \(\hat{G}\) that records the fact that \(G\) is a nonsplit group over \(F\). To define it we must construct an action of \(\text{Gal}(F^{\text{sep}}/F)\) on \(\hat{G}(\mathbb{C})\).
It is obvious that the root datum should be involved in this process. Indeed, given a Galois action on \( G_{\text{sep}} \) we obtain one on \( \Psi(G_{\text{sep}}, T_{\text{sep}}) \) and hence tautologically we obtain one on \( \Psi(\hat{G}, \hat{T}) \). This is not enough, as we really need a Galois action on \( \hat{G}(\mathbb{C}) \). It would suffice to produce a splitting of the surjective map 
\[
\text{Aut}(\hat{G}(\mathbb{C})) \rightarrow \text{Aut}(\Psi(\hat{G}, \hat{T}))
\]
described in Proposition 7.3.1 below. What we actually do is to produce a refinement \( \Psi(\hat{G}, \hat{B}, \hat{T}) \) of \( \Psi(\hat{G}, \hat{T}) \) called a based root datum and a surjective map 
\[
\text{Aut}(\hat{G}(\mathbb{C})) \rightarrow \text{Aut}(\Psi(\hat{G}, \hat{B}, \hat{T})).
\]
We then provide an explicit description of splittings of this map via something known as a pinning, or épilange, of \( \hat{G} \). This then suffices to construct our desired action of \( \text{Gal}(F_{\text{sep}}/F) \) on \( \hat{G}(\mathbb{C}) \).

We now begin this process. We assume until otherwise specified that \( G \) is a split, connected reductive group over a separably closed field \( k \). Let \( \Delta \subset \Phi \) be a base for \( \Phi \) (see (1.15)). The set \( \Delta^\vee \subset \Phi^\vee \) of coroots dual to \( \Delta \) forms a base of \( \Phi^\vee \). With this in mind, a tuple 
\[
(X, Y, \Delta, \Delta^\vee)
\]
is called a **based root datum** if there is a root datum \( (X, Y, \Phi, \Phi^\vee) \) such that \( \Delta \subset \Phi \) is a maximal set of simple roots (with respect to some set of positive roots) and \( \Delta^\vee \) is the dual set. We note that \( \Delta \) spans \( \Phi \) and \( \Delta^\vee \) spans \( \Phi^\vee \) as \( \mathbb{Z} \)-modules, so there is no ambiguity in the notation (7.4). There is an obvious notion of isomorphism of root data; it is simply a pair of linear isomorphisms on the first two factors preserving the pairing and the sets of simple roots.

We let \( \Psi(G, B, T) := (X^*(T), X_*(T), \Delta, \Delta^\vee) \) be a choice of based root datum, and \( \Psi(G, T) \) the root datum it defines. The reason for the \( B \) in this notation is the following lemma:

**Lemma 7.3.1** The choice of a set of simple roots \( \Delta \subset \Phi \) is equivalent to the choice of a Borel subgroup \( B \subset G \) containing \( T \).

We have already mentioned a more general result in §1.9, but we did not prove it there. We pause here to sketch the proof. For each \( \alpha \in \Phi \) there is a unique subgroup \( N_\alpha \leq G \) normalized by \( T \) such that \( \text{Lie } N_\alpha = g_\alpha \). It is called the **root group** attached to \( \alpha \). For example if \( G = \text{GL}_n \) and \( \alpha = e_{ij} \) in the notation of (1.9) then the corresponding root group is \( I_n + R e_{ij} \).

**Proof of Lemma 7.3.1:** We just explain the bijection, leaving the details to the standard references mentioned at the beginning of Chapter 1. Given a choice of a set of simple roots \( \Delta \) the group 
\[
B = \langle T, \{N_\alpha\}_{\alpha \in \Delta} \rangle
\]
is a Borel subgroup. Conversely, given a Borel subgroup the set of roots of $T$ in $\text{Lie} B$ form a set of positive roots which provides us with a set of simple roots.

To better describe $\Psi(G,B,T)$ we recall the notion of a pinning:

**Definition 7.2.** A pinning of $G$ is a tuple

$$(B, T; \{X_\alpha\}_{\alpha \in \Delta})$$

where $T$ is a maximal torus and $B$ is a Borel subgroup containing it, $\Delta$ is the set of simple roots attached to $B$, and $X_\alpha \in \mathfrak{g}_\alpha - \{0\}$ for all $\alpha$.

We let $\text{Aut}(B, T; \{X_\alpha\}_{\alpha \in \Delta})$ be the group of automorphisms of $G$ that preserve $B$ and $T$ and the set $\{X_\alpha\}_{\alpha \in \Delta}$. The theorem of Chevalley and Demazure (Theorem 1.8.1) provides lifts of automorphisms of root data of $(G, T)$ to automorphisms of the pair $(G, T)$; using it we obtain an isomorphism

$$\text{Aut}(B, T; \{X_\alpha\}_{\alpha \in \Delta}) \xrightarrow{\sim} \text{Aut}(\Psi(G, B, T)) \quad (7.6)$$

where we use $\Delta$ to define $\Psi(G, B, T)$.

**Proposition 7.3.1** There is a split exact sequence

$$1 \longrightarrow \text{Inn}(G(k)) \xrightarrow{a} \text{Aut}(G) \xrightarrow{b} \text{Aut}(\Psi(G, B, T)) \longrightarrow 1. \quad (7.7)$$

The splittings are in bijection with pinnings $(B', T', \{X_\alpha\}_{\alpha \in \Delta})$ up to conjugation by $T'(k)$.

**Proof.** The arrow $a$ is the obvious injection. To describe $b$, note that for any automorphism $\phi \in \text{Aut}(G(k))$ there is a $g \in G(k)$ such that $\phi \circ \text{Ad}(g)$ preserves $B$ and $T$ since all Borel subgroups and tori are conjugate under $G(k)$. Thus $\phi \circ \text{Ad}(g)$ restricts to define an automorphism of the root datum $\Psi(G, B, T)$. It already preserves $B$, and by replacing $g$ by $tg$ for some $t \in T(k)$ we can assume that

$$\phi \circ \text{Ad}(g)$$

preserves the simple roots. Thus we obtain the map $b : \text{Aut}(G(k)) / \text{Inn}(G(k)) \rightarrow \text{Aut}(\Psi(G, B, T))$. It is surjective by Theorem 1.8.1. Moreover, by loc. cit. given an automorphism $\phi' \in \text{Aut}(\Psi(G, T))$ there is an automorphism $\phi \in \text{Aut}(G)$ such that $\phi(T) = T$ and $b(\phi) = \phi'$. If $\phi$ in addition preserves the simple roots then it follows that the lift is uniquely determined (see [Con14, §1.5] for more details). This proves that $\text{Im}(a) = \ker(b)$.

Since $\text{Aut}((B, T; \{X_\alpha\}_{\alpha \in \Delta}))$ is defined to be a subset of $G$, we see that a pinning defines a section of $\text{Aut}(G(k)) \longrightarrow \text{Aut}(\Psi(G, B, T))$.

On the other hand, if $g \in G(k)$, then the conjugate of this section may be thought of as $\text{Aut}(gBg^{-1}, gTg^{-1}, \{gX_\alpha g^{-1}\}_{\alpha \in \Delta})$, which is also a pinning of $G_k$. □
With this preparation complete we can now define the Langlands dual group. Assume that $G$ is a connected reductive group over a local or global field $F$. Choose a Borel subgroup $B_{F^\text{sep}} \leq G_{F^\text{sep}}$ and a maximal torus $T_{F^\text{sep}} \leq B_{F^\text{sep}} \leq G_{F^\text{sep}}$. We obtain a based root datum

$$\Psi(G, B, T) := \Psi(G_{F^\text{sep}}, B_{F^\text{sep}}, T_{F^\text{sep}}).$$

Using the exact sequence of Proposition 7.3.1 the homomorphism $\text{Gal}(F^\text{sep}/F) \to \text{Aut}(G_{F^\text{sep}})$ giving $G$ its $F$-structure yields an action of $\text{Gal}(F^\text{sep}/F)$ on the based root datum. Now

$$(X_*(T)_{F^\text{sep}}, X^*(T)_{F^\text{sep}}, \Delta^\vee, \Delta)$$

is again a based root datum with associated root datum $\Psi(\widehat{G}, \widehat{T})$, so we have a Borel subgroup $\widehat{B} \leq \widehat{G}$ such that

$$(X_*(T)_{F^\text{sep}}, X^*(T)_{F^\text{sep}}, \Delta^\vee, \Delta) = \Psi(\widehat{G}, \widehat{B}, \widehat{T}). \quad (7.8)$$

We therefore obtain an action of $\text{Gal}(F^\text{sep}/F)$ on $\Psi(\widehat{G}, \widehat{T})$.

Via a choice of pinning of $\widehat{G}$ we obtain a section of the map

$$\text{Aut}(\widehat{G}) \to \text{Aut}(\Psi(\widehat{G}, \widehat{B}, \widehat{T}))$$

and hence a map $\text{Gal}(F^\text{sep}/F) \to \text{Aut}(\widehat{G})$. We define the **Langlands dual group of** $G$ or the **$L$-group of** $G$ to be the semidirect product

$$L^G := \widehat{G}(\mathbb{C}) \rtimes \text{Gal}(F^\text{sep}/F)$$

with respect to this action. Notice that the action is canonical up to conjugation by $\widehat{G}(\mathbb{C})$ by Proposition 7.3.1.

A morphism of $L$-groups

$$L^H \to L^G$$

is simply a homomorphism commuting with the projections to $\text{Gal}(F^\text{sep}/F)$ such that its restriction to the neutral components is induced by a map of algebraic groups $\widehat{H} \to \widehat{G}$. Morphisms are also sometimes called $L$-maps.

Assume for the moment that $F$ is a global field. For every place $v$ upon choosing an embedding $F^\text{sep} \to F_v^\text{sep}$ we obtain an embedding

$$\text{Gal}(F_v^\text{sep}/F_v) \to \text{Gal}(F^\text{sep}/F),$$

unique up to conjugacy. This induces a morphism

$$L^{G_{F_v}} \to L^{G_F}. \quad (7.9)$$

which allows us to relate local and global $L$-groups.
Occasionally one works with modifications of the $L$-group. For example, we can replace $\text{Gal}(F^{\text{sep}}/F)$ by its quotient $\text{Gal}(E/F)$ where $E/F$ is a finite degree Galois extension such that $G_E$ splits. In some applications one wants to modify or enlarge $\text{Gal}(F^{\text{sep}}/F)$ (see [Mok15]). In general we can replace $\text{Gal}(F^{\text{sep}}/F)$ by any topological group $\Gamma$ admitting a continuous homomorphism $\Gamma \to \text{Gal}(F^{\text{sep}}/F)$ that surjects onto $\text{Gal}(E/F)$ for any field $E$ such that $G_E$ is split; all of the constructions above have obvious analogues in this level of generality.

It is useful to give some examples of $L$-groups in this nonsplit situation. First let $G = \text{GL}_n$ over a field $F$ and let $E/F$ be a field extension (of finite degree). One has

$$L \text{Res}_{E/F} \text{GL}_n \cong \text{Gal}(F^{\text{sep}}/F) \rtimes \text{GL}_n(\mathbb{C})^{\{\sigma: E \to F^{\text{sep}}\}}$$

(7.10)

where $\text{Gal}(F^{\text{sep}}/F)$ acts via permuting the factors. For a slightly more complicated example let $M/F$ be a quadratic extension and let $\sigma \in \text{Gal}(M/F)$. For an $F$-algebra $R$, consider the quasi-split unitary group

$$U(R) := \left\{ g \in \text{GL}_n(M \otimes_F R) : \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \sigma(g)^{-t} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} = g \right\}.$$ 

One has

$$L U \cong \text{Gal}(F^{\text{sep}}/F) \times \text{GL}_n(\mathbb{C})$$

(7.11)

where $\text{Gal}(F^{\text{sep}}/F)$ acts via its quotient $\text{Gal}(M/F)$, which acts on $\text{GL}_n(\mathbb{C})$ via the isomorphism $g \mapsto g^{-t}$. We mention that there is an $L$-map

$$L U \to L \text{Res}_{M/F} \text{GL}_n$$

$$g \rtimes \sigma \mapsto (g, g^{-t}) \rtimes \sigma.$$ 

### 7.4 The Parabolic subgroups of $L$-groups

Let $G$ be a connected reductive group over a local or global field $F$. In Chapter 12 we will require the notion of a parabolic subgroup of an $L$-group. It is most natural to discuss this immediately after discussing the $L$-group, but the reader should feel free to skip this section and then refer back to it as needed.

We follow the discussion in [Bor79]. A parabolic subgroup of $L$ is a closed subgroup $Q$ such that $\hat{Q} \cap \hat{G}(\mathbb{C})$ is the complex points of a parabolic subgroup of $\hat{G}$ and the restriction of the canonical map $\hat{L} \to \hat{G}$ to $Q$ is surjective. Clearly

$$L B := \hat{B}(\mathbb{C}) \rtimes \text{Gal}(F^{\text{sep}}/F)$$

(7.12)
is a parabolic subgroup, where $\hat{B}$ is the group of (7.8). A parabolic subgroup of $L^G$ is **standard** if it contains $L^B$. As in the case of algebraic groups, every parabolic subgroup of $L^G$ is conjugate under $\hat{G}(\mathbb{C})$ to one and only one standard parabolic subgroup.

We now isolate a subset of **relevant** parabolic subgroups of $L^G$. This will allow us to define a dual parabolic subgroup $L^P$ to any proper parabolic subgroup $P \leq G$. We will also explain how this duality behaves with respect to Levi subgroups.

For every field extension $F^{\text{sep}} \geq k \geq F$ we let $P(k)$ denote the set of parabolic subgroups of $G_k$. Let $(P/G)(k)$ denote the set of $G(F^{\text{sep}})$-conjugacy classes of parabolic subgroups in $G(F^{\text{sep}})$ that are fixed by $\text{Gal}(F^{\text{sep}}/k)$. There is a natural map

$$P(k) \longrightarrow (P/G)(k)$$

which is surjective if $G$ is quasi-split.

Similarly, let $L^P$ denote the set of parabolic subgroups of $L^G$ and let $L^P/L^G$ denote the set of $L^G$-conjugacy classes of parabolic subgroups of $L^G$. The bijection $\Delta \leftrightarrow \Delta^\vee$ yields a bijection

$$(P/G)(k) \longleftrightarrow L^P/L^G.$$  

Note that we view $(P/G)(k)$ above as a group scheme opposed to the fact that $L^P/L^G$ is a group.

We say that a parabolic subgroup of $L^G$ is **relevant** if its $L^G$-conjugacy class is in the image of the composite

$$P(k) \longrightarrow (P/G)(k) \longrightarrow L^P/L^G.$$  

(7.13)

If $P \in P(k)$ we denote by $L^P$ the standard parabolic subgroup in the class that is the image of $P$ under the map (7.13).

Let $Q$ be a parabolic subgroup of $L^G$. The unipotent radical $N$ of $Q^\circ$ is normal in $Q$ and will also be called the unipotent radical of $Q$. Then $Q$ is the semidirect product of $N$ by the normalizer in $Q$ of any Levi subgroup $M^\circ$ of $Q^\circ$. Those normalizers will be called the Levi subgroups of $Q$.

For given parabolic subgroup $P \leq G$, a unique $G(F^{\text{sep}})$-conjugate $gP_{F^{\text{sep}}}g^{-1}$ of $P_{F^{\text{sep}}}$ contains $B_{F^{\text{sep}}}$. Since the parabolic subgroups of $G_{F^{\text{sep}}}$ containing $B$ are in bijection with subsets of $\Delta$ as explained in Theorem 1.9.2 we can therefore associate a subset $J(P) \subseteq \Delta$ to $P$. Given a Levi subgroup $M \leq P$, there is a $g' \in G(F^{\text{sep}})$ such that $g'Mg'^{-1}$ has based root datum

$$(X^\ast(T), X_\ast(T), J(P), J(P)^\vee)$$

and hence a Levi subgroup of $L^P$ with based root datum

$$(X_\ast(T), X^\ast(T), J(P)^\vee, J(P)).$$
The semidirect product of this Levi subgroup of $LP^o$ with $\text{Gal}(F^\text{sep}/F)$ is then a Levi subgroup of $LP$ denoted by $LM$. A Levi subgroup of a parabolic subgroup $LP$ of $LG$ is **relevant** if $LP$ is.

### 7.5 The Satake isomorphism for quasi-split groups

Let $G$ be an unramified connected reductive group over a local field $F$ and let $K \leq G(F)$ be a hyperspecial subgroup. We wish to describe Langlands’ extension of the Satake isomorphism to this setting. Since $G$ is unramified we can take the $L$-group to be

$$\hat{G}(\mathbb{C}) \rtimes \text{Fr}^Z$$

where $\text{Fr} \in \text{Gal}(F^\text{sep}/F)$ is a choice of Frobenius element. In the split case, the Satake transform could be phrased in terms of the quotient

$$\hat{G}(\mathbb{C})/\text{conj}.$$ 

In the nonsplit case, the appropriate quotient is

$$\hat{G} \rtimes \text{Fr}/\sim$$

where the action is given on $\mathbb{C}$-algebras $R$ by

$$\hat{G}(R) \times \hat{G}(R) \times \text{Fr} \rightarrow \hat{G}(R) \times \sigma$$

$$(g, (h \times \text{Fr})) \mapsto gh\text{Fr}(g)^{-1} \times \text{Fr}.$$ 

This action can also be described as the action of $\hat{G}$ on itself via $\text{Fr}$-conjugacy, a notion that plays a key role in the twisted trace formula (see §???? below).

We now describe the analogue of the Chevalley restriction theorem (7.2) in this setting. Given a maximal torus $T \leq G$, its dual $\hat{T}$ is a maximal torus of $\hat{G}$, and we can consider its Weyl group $W(\hat{G}, \hat{T})$. Now

$$W(G,T)(F) \leq \text{Aut}(\Psi(G,T)) = \text{Aut}(\Psi(\hat{G}, \hat{B}, \hat{T}))$$

so we can consider $W(G,T)(F)$ as a subgroup of $W(\hat{G}, \hat{T})$ which is a quotient of the normalizer $N_{\hat{G}}(\hat{T})(\mathbb{C})$. Let $F_N \leq N_{\hat{G}}(\hat{T})(\mathbb{C})$ denote the inverse image of $W(G,T)(F)$ under the quotient map $N_{\hat{G}}(\hat{T})(\mathbb{C}) \rightarrow W(\hat{G}, \hat{T})(\mathbb{C})$. The analogue of the Chevalley restriction theorem in this setting is

$$\hat{T} \rtimes \text{Fr}/F_N \rightarrow \hat{G} \rtimes \text{Fr} / \sim$$

(see [Bor79, Lemma 6.5]). With this isomorphism in mind the following theorem is natural:
Theorem 7.5.1 (Langlands and Satake) There is an isomorphism
\[ S : C^\infty_c(G(F) \backslash K) \rightarrow \mathbb{C}[\hat{T} \rtimes \text{Fr}]^N. \]

As before, we can rephrase this as an isomorphism
\[
\text{Hom}(C^\infty_c(G(F) \backslash K), \mathbb{C}) \rightarrow (\hat{T} \rtimes \text{Fr}/_F N)(\mathbb{C}) \rightarrow (\hat{G} \rtimes \text{Fr})^{\text{Fr-ss}}(\mathbb{C})/\sim \quad (7.15)
\]
where \((\hat{G} \rtimes \text{Fr})^{\text{Fr-ss}}\) is the subscheme of elements that are Fr-semisimple; a concise way to phrase this condition is that the orbit under the action of \(\hat{G}\) is closed. Thus we obtain the following analogue of Corollary 7.2.1:

Corollary 7.5.1 Assume that \(G\) is unramified over \(F\). The composite isomorphism (7.15) induces a bijection
\[
\left\{ (\hat{G} \rtimes \text{Fr})^{\text{Fr-ss}}(\mathbb{C})/\sim \right\} \leftrightarrow \left\{ \text{isomorphism classes of irreducible unramified representations of } G(F) \right\}.
\]

By way of terminology the Fr-semisimple conjugacy class attached to an isomorphism class of unramified representations is called its Langlands class. In the split case, the eigenvalues of a representative of the conjugacy class are called its Satake parameters.

We will not prove either Theorem 7.2.1 or Theorem 7.5.1, though we will say more about the definition of the map \(S\) in a moment. The standard references are [Car79, Theorem 4.1], [Bor79, §7], [Sat63], which we follow. There is a categorified version of the Satake correspondence due to Mirković and K. Vilonen that is known as the geometric Satake correspondence, see [MV07]. This result is an important tool in the geometric Langlands program. Moreover it has now been extended to mixed characteristic by work of Zhu and Bhatt-Scholze.

Let \(B \leq G\) be a Borel subgroup containing a maximal torus \(T\) of \(G\) and let \(K\) be a hyperspecial subgroup. We let \(N\) denote the unipotent radical of \(B\). We say that \(K\) is in good position with respect to \((B,T)\) if the Iwasawa decomposition
\[ G(F) = KB(F) \]
holds and
\[ B(F) \cap K = (K \cap T(F))(K \cap N(F)). \]
We also can and do assume that $K_T := K \cap T(F)$ is a maximal compact subgroup of $T$. In fact, we given $B$ and $T$ we can always choose $K$ so that these assumptions hold (see Theorem 7.2).

Using the Iwasawa decomposition $G(F) = KB(F) = KT(F)N(F)$ we have a decomposition of measures

\[ dg = dkd_r b = \delta_B(t)dkdtn \]

(see Proposition 3.2.1); we always assume that $dk(K) = 1$. We define a $C$-linear map

\[ C^\infty_c(G(F)/K) \rightarrow C^\infty_c(T(F)/K_T) \]

\[ f \mapsto f^B \]

where

\[ f^B(t) := \delta_B(t)^{1/2} \int_{N(F)} f(tn)dn. \]

The function $f^B$ is known as the constant term of $f$ along $B$.

The following lemma is implied by Lemma 8.6.1 and Proposition 8.7.2 below:

**Lemma 7.5.1** The map $f \mapsto f^B$ is an algebra homomorphism and has image in $C^\infty_c(T(F)/K_T)^{W(G,T)}(F)$. \hfill $\Box$

Let $T_s \leq T$ be the greatest $F$-split torus in $T$. Thus we can write $T = T_sT_a$ where $T_a$ is anisotropic and $T_s \cap T_a$ is finite. Let $K_s = K \cap T_s(F)$; it is a maximal compact subgroup of $T_s(F)$. We note that

\[ W(G,T)(F) = W(G,T_s)(F) = W(G,T_s)(F_{\text{sep}}). \]

**Lemma 7.5.2** There is a $W(G,T)(F) \leq W(G,T)(F_{\text{sep}}) = W(\hat{G},\hat{T})(\mathbb{C})$-equivariant isomorphism

\[ C^\infty_c(T(F)/K_T)^{W(G,T)(F)} \rightarrow \mathbb{C}[\hat{T}_s]^{W(G,T)(F)}. \]

**Proof.** We note that $T_a(F)$ is compact, so via restriction we obtain an isomorphism

\[ C^\infty_c(T(F)/K_T)^{W(G,T)(F)} \rightarrow C^\infty_c(T_s(F)/K_s)^{W(G,T)(F)}. \]

We note that there is an isomorphism of $\mathbb{C}$-algebras

\[ C^\infty_c(T_s(F)/K_s) \rightarrow \text{Hom}(X^*(T_s)_F, \mathbb{C}) = \mathbb{C}[X_s(T_s)_F] \]

\[ \mathbb{1}_{T_s} \mapsto (\chi \mapsto \text{ord}_v(\chi(t))). \]
It is not hard to see that this isomorphism is \( \mathbb{C}[X_*(T_s)_F] = \mathbb{C}[X_*(\widehat{T}_s)_C] = \mathbb{C}[\widehat{T}_s] \)
we deduce the lemma. \( \square \)

**Lemma 7.5.3** The inclusion \( T_s \rightarrow T \) induces a bijection
\[
(\widehat{T} \rtimes \Fr)(\mathbb{C})/\sim \rightarrow \widehat{T}_s(\mathbb{C})/W(G,T)(F)
\]
where on the left the quotient is modulo the conjugation action of \( F_N \).

**Proof.** Let \( \nu : \widehat{T} \rightarrow \widehat{T}_s \) be the map induced by the inclusion \( T_s \rightarrow T \) via duality.

As mentioned after [Bor79, Lemma 6.4] every element of \( F_N \) is of the form \( ws \) where \( w \in W(G,T)(F) \) and \( u \in \widehat{T}(\mathbb{C}) \). Moreover
\[
(wu)^{-1}(t \rtimes \Fr)wu = u^{-1}\Fr(u)w^{-1}tw \rtimes \Fr.
\]
Since \( \widehat{T}_s \) is the fixed points of \( \Fr \) on \( \widehat{T} \), we see that we obtain a map
\[
\nu : (\widehat{T} \rtimes \Fr)(\mathbb{C}) \rightarrow \widehat{T}_s(\mathbb{C})
\]
\[
(t \rtimes \Fr) \mapsto \nu(t)
\]
that is equivariant with respect to the action of \( F_N \) (which acts via its quotient \( W(G,T_s)(F) \) on the right hand side). It is clearly surjective. To prove injectivity, assume that \( t, t' \in \widehat{T}(\mathbb{C}) \) and
\[
\nu(t \rtimes \Fr) = w^{-1}\nu(t \rtimes \Fr)w
\]
for some \( w \in W(G,T)(F) \). Then \( \nu(t) = \nu(w^{-1}t'w) \) since \( w \) is fixed by \( \Fr \). Thus \( t = xw^{-1}t'w \) for some \( x \in \ker(\nu) \). By Hilbert’s theorem 90 all \( x \in \ker(\nu) \) are of the form \( u^{-1}\Fr(u) \), and we deduce that \( t \rtimes \Fr = (wu)^{-1}(t' \rtimes \Fr)wu. \) \( \square \)

Combining lemmas 7.5.1, 7.5.2 and the map of coordinate rings induced by Lemma 7.5.3 we see that we have constructed the Satake morphism
\[
\mathcal{S} : C_c^\infty(G(F) \parallel K) \rightarrow C_c^\infty(T(F)/K_T)^{W(G,T)(F)} \rightarrow \mathbb{C}[\widehat{T} \rtimes \Fr]^{F,N}, \quad (7.18)
\]
though we have not proved that it is injective or surjective. The first map in this factorization of the Satake isomorphism is a special case of a parabolic descent map. In general, any construction that relates objects on the group \( G \) to objects on Levi subgroups of its parabolic subgroups (in this case the Levi subgroup \( T \) of the Borel subgroup \( B \)) is known as parabolic descent. We will use this idea in the next section to give a more explicit parametrization of unramified representations.
7.6 The Principal series

We now explain how to explicitly realize unramified representations. Let $G$ be an unramified connected reductive group with Borel subgroup $B \leq G$ and maximal torus $T \leq B$. We assume that $B$ and $T$ are chosen so that the Iwasawa decomposition

$$G(F) = B(F)K$$

holds and that $K_T := T(F) \cap K$ is a maximal compact open subgroup of $T(F)$; it is always possible to arrange this by Lemma A.5.1. Let $N \leq B$ be the unipotent radical of $B$.

We recall from Proposition 3.5.1 that the modular character

$$\delta_B := \delta_{B(F)} : B(F) \rightarrow \mathbb{C}$$

is

$$\delta_B(b) := |\det(\text{Ad}(b))|.$$ 

The last ingredient we need to define unramified principal series representations is unramified quasi-characters.

**Definition 7.3.** A quasi-character $\chi : T(F) \rightarrow \mathbb{C}^\times$ is unramified if it is trivial on $K_T$.

Let us explicitly describe unramified quasi-characters. As in the previous section, we let $T_s \leq T$ be the maximal split torus and choose an anisotropic torus $T_a \leq T$ such that $T_a T_s = T$ and $T_a \cap T_s$ is finite. We remind the reader that $X^*(T)_F$ is the abelian group of homomorphisms $T \rightarrow \mathbb{G}_m$; this is in general a proper subgroup of the group $X^*(T)_{F_{\text{sep}}}$ of homomorphisms $T_{F_{\text{sep}}} \rightarrow \mathbb{G}_m$. In fact

$$X^*(T)_F = X^*(T_s)_F.$$ 

In analogy with (10.1) we define a map

$$H_T : T(F) \longrightarrow a_T := \text{Hom}(X^*(T)_F, \mathbb{R})$$

via

$$e^{(H_T(t), \chi)} = |\chi(t)|$$

for $\chi \in X^*(T)$. For each

$$\lambda \in a^*_T \subseteq X^*(T)_F \otimes \mathbb{C}$$

we then obtain a quasi-character

$$t \mapsto e^{(H_T(t), \lambda)}.$$
It is clearly unramified. Moreover, it depends only on the image of $\lambda$ under the map

$$X^*(T)_F \otimes \mathbb{C} \longrightarrow X^*(T)_F \otimes \mathbb{C}^\times = \hat{T}_s(\mathbb{C})$$

induced by the map $x \mapsto e^x$ (for the last equality see Exercise 7.5). Thus for every element of $\hat{T}_s(\mathbb{C})$ we obtain an unramified quasi-character of $T(F)$. Taking $T = G$ in the following lemma we see that all unramified quasi-characters arise in this manner:

**Lemma 7.6.1** One has natural $F \cdot N$-equivariant bijections

$$\text{Hom}(T(F)/K_T, \mathbb{C}^\times) \leftarrow \hat{T}_s(\mathbb{C}) \longrightarrow (\hat{T}(\mathbb{C}) \times \text{Fr})^F N$$

$$e^\lambda \longmapsto (t \mapsto e^{(H_T(t),\lambda)})$$

provided that we let $F \cdot N$ act via its quotient $W(G,T)(F) \leq W(G,T)(F^{\text{sep}}) = W(\hat{G}, \hat{T})(\mathbb{C})$ on the left two groups.

**Proof.** For the first isomorphism we start by noting that the inclusion $T_s \rightarrow T$ induces a $W(G,T)(F)$-equivariant isomorphism

$$T_s(F)/K_s \cong T(F)/K_T,$$

where $K_s := K \cap T_s(F)$ is a maximal compact subgroup of $T_s(F)$. Thus we are immediately reduced to the case where $T$ is split.

We henceforth assume $T_s = T$. The map

$$\hat{T}_s(\mathbb{C}) \longrightarrow \text{Hom}(T_s(F)/K_s, \mathbb{C}^\times)$$

$$e^\lambda \longmapsto (t \mapsto e^{(H_T(t),\lambda)})$$

is $W(G,T)(F)$-equivariant, so we need only check it is an isomorphism. By choosing an isomorphism $T_a \cong \mathbb{G}_m$, we are reduced to observing that

$$\mathbb{C}^\times \rightarrow \text{Hom}(F^\times / \mathcal{O}_F^\times, \mathbb{C}^\times)$$

$$q^\lambda \longmapsto (t \mapsto q^{\lambda \log |t|})$$

is a bijection (here we represent an element of $\mathbb{C}^\times$ as $q^\lambda$ for some $\lambda \in \mathbb{C}$). This completes the proof of the first bijection. The second bijection has already been proven in Lemma 7.5.3. $\square$

For $\lambda \in \mathfrak{a}_{\mathcal{C}}^+\mathbb{C}$ we define the representation

$$I(\lambda) := \text{Ind}_B^G(t \mapsto e^{(H_T(t),\lambda)})$$

to be the (smooth) representation of $G(F)$ on the space of functions.
\[ \{ \varphi \in C^\infty(G(F)) : \varphi(tng) = \delta_B(t^{1/2}e^{i[H_T(t),\lambda]}g) \text{ for all } (t,n,g) \in T(F) \times N(F) \times G(F) \} \]

Here \( G(F) \) acts via right translation:

\[ I(\lambda)(g)(x) := \varphi(xg). \]  

(7.20)

This is an example of an induced representation, see §8.2 for the general construction. The factor of \( \delta_B^{1/2} \) is present so that if \( \lambda \in i\mathfrak{a}_T^* \) then \( I(\lambda) \) is pre-unitarizable; this assertion is part of the following special case of Proposition 8.2.1 later:

**Proposition 7.6.1** The representations \( I(\lambda) \) are admissible and satisfy

\[ I(\lambda)^\vee \cong I(-\lambda). \]

If \( \lambda \in i\mathfrak{a}_T^* \) then \( I(\lambda) \) is pre-unitarizable.

We note that the condition \( \lambda \in i\mathfrak{a}_T^* \) is equivalent to the statement that the quasi-character \( t \mapsto e^{i[H_T(t),\lambda]} \) is unitary (i.e. is a character).

**Definition 7.4.** An unramified principal series representation of \( G(F) \) is a representation isomorphic to \( I(\lambda) \) for some \( \lambda \in \mathfrak{a}_T^* \).

This definition is traditional, but it is a little misleading. Indeed, unramified principal series representations need not be irreducible, but we do have the following lemma:

**Lemma 7.6.2** There is a unique line in \( I(\lambda) \) fixed by \( K \).

**Proof.** A function in \( I(\lambda) \) is uniquely determined by its restriction to \( K \). □

If \( I(\lambda) \) is reducible, the situation is slightly more complicated, but not insufferably so. To describe what happens, let us start by remarking that \( W(G,T)(F) \) acts on \( X_\lambda(T)_F \), hence \( \mathfrak{a}_T^* \) by duality. It therefore makes sense to talk of \( I(w(\lambda)) \) for \( \lambda \in \mathfrak{a}_T^* \) and \( w \in W(G,T)(F) \). We say that \( \lambda \in \mathfrak{a}_T^* \) is **regular** if \( w(\lambda) = \lambda \) for \( w \in W(G,T)(F) \) implies that \( w \) is the identity. This is equivalent to \( \lambda \) lying in an open Weyl chamber in \( \mathfrak{a}_T^* \). With this action of \( W(G,T)(F) \) in mind we are ready to state a result known as the unramified subquotient theorem:

**Theorem 7.6.1** Let \( \lambda, \lambda' \in \mathfrak{a}_T^* \). Then one has the following:

(a) The representations \( I(\lambda) \) and \( I(\lambda') \) are isomorphic if and only if \( \lambda = w(\lambda') \) for some \( w \in W(G,T)(F) \).

(b) Every unramified representation is isomorphic to a subrepresentation of an \( I(\lambda) \), and every \( I(\lambda) \) admits a unique unramified subquotient.
(c) Assume that \( \lambda \in i\mathfrak{a}_T \). Under this assumption the representation \( I(\lambda) \) is irreducible if and only if \( \lambda \) is regular.

\( \square \)

For the proof of the theorem one can refer to [Car79, §3-§4]. These results are due to Casselman, and one can also find proofs of them in his unpublished notes [Cas]. For \( \lambda \in \mathfrak{a}_T^* \) we let \( J(\lambda) \) be the unique unramified subquotient of \( I(\lambda) \). The reason for the apparent asymmetry in item (b) is that \( J(\lambda) \) is not necessarily an irreducible subrepresentation of \( I(\lambda) \), merely a subquotient. It is however an irreducible subrepresentation of \( I(w(\lambda)) \) for some \( w \in W(G,T)(F) \).

By the theorem, every unramified representation is isomorphic to a \( J(\lambda) \) where \( \lambda \) is unique up to the action of \( W(G,T)(F) \). This is consonant with Corollary 7.5.1, the parametrization of unramified representations afforded by the Satake isomorphism, and it is useful to explicitly relate Corollary 7.5.1 and Theorem 7.6.1:

**Proposition 7.6.2** Let \( f \in C_c^\infty(G(F) \backslash K) \). For \( \lambda \in \mathfrak{a}_T^* \) one has

\[
\text{tr} J(\lambda)(f) = \mathcal{S}(f)(e^{\lambda}).
\]

**Proof.** One has

\[
\text{tr} I(\lambda)(f) = \text{tr} e^{\lambda} f^B
\]

by Proposition 8.6.1, and by unwinding the definition of the Satake isomorphism one obtains

\[
\text{tr} e^{(H_T(\cdot),\lambda)} f^B = \mathcal{S}(f)(e^\lambda).
\]

Thus to complete the proof it suffices to show that \( \text{tr} I(\lambda)(f) = \text{tr} J(\lambda)(f) \). Invoking Theorem 7.6.1 we see that upon replacing \( \lambda \) by \( w\lambda \) for some \( w \in W(G,T)(F) \) we can assume that \( J(\lambda) \) is a subrepresentation of \( I(\lambda) \).

There is a unique line \( \mathbb{C}\varphi_0 \) in the space of \( I(\lambda) \) fixed by \( K \) by Lemma 7.6.2. It follows that \( I(\lambda)(f) \) acts via the scalar \( \text{tr} I(\lambda)(f) \) on \( \mathbb{C}\varphi_0 \). On the other hand there is an equivariant map

\[
J(\lambda) \longrightarrow I(\lambda). \tag{7.21}
\]

Since \( J(\lambda) \), being unramified, has a unique spherical line this line its image must be \( \mathbb{C}\varphi_0 \) under (7.21). \( \square \)

### 7.7 Weak global L-packets

Let \( G \) be a reductive group over a global field \( F \).

**Definition 7.5.** We say that two irreducible admissible representations \( \pi_1, \pi_2 \) of \( G(\mathbb{A}_F) \) are **weakly globally L-indistinguishable** if \( \pi_1^S \cong \pi_2^S \) for
some finite set of places $S$ of $F$. An equivalence class of $L$-indistinguishable admissible representations is called a weak global L-packet. If an adelic $L$-packet contains an automorphic representation, we say that it is a weak automorphic $L$-packet.

Here when we say an admissible representation of $G(\mathbb{A}_F)$ we mean an admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$-module in the archimedian case and an admissible $G(\mathbb{A}_F^\infty)$-module in the function field case. Later on we will define a notion of local $L$-packets that complements this definition. The definition of local $L$-packets will be used to give a refinement of the notion of a global $L$-packet; this is why the adjective weak is used above.

For the remainder of this section, we will omit the word global to save ink. It may seem odd that we include merely admissible, and not automorphic, representations in a weak automorphic $L$-packet, but this turns out to be convenient for technical reasons. In practice one describes the weak $L$-packet of an automorphic representation, and then asks which representations in the weak global $L$-packet are automorphic.

To describe weak $L$-packets in more detail we set some notational conventions. If $S$ is a finite set of places of $F$ including the infinite places such that $G_v$ is unramified for $v \notin S$ we say that $G$ is unramified outside of $S$. Likewise if $\pi$ is an admissible representation of $G(\mathbb{A}_F)$ then we say $\pi$ is unramified outside of $S$ if $\pi_v$ is unramified for $v \notin S$.

The Satake isomorphism provides a convenient way of thinking about weak $L$-packets. Assume that $G$ is unramified outside of $S$. Notice that if $\pi$ is an admissible representation of $G(\mathbb{A}_F)$ unramified outside of $S$ then it defines a character $\text{tr } \pi^S : C^\infty_c(\mathbb{G}^S(\mathbb{A}_F) / K^S) \to \mathbb{C}$ where $K^S \leq \mathbb{G}^S(\mathbb{A}_F)$ is a hyperspecial subgroup, i.e. $K_v \leq G(F_v)$ is hyperspecial for all $v \notin S$. By the Satake isomorphism, giving this character is equivalent to giving, for each $v \notin S$, an element $c_v \in ^L G^v \rtimes \text{Fr}_v / \sim$, where $\text{Fr}_v$ is a choice of Frobenius element at $v$ and the quotient is with respect to the conjugation action of $^L G^v$; the element $c_v$ will be $\text{Fr}_v$-semisimple. Let

$$\Pi_w(G) : = \{ \text{weak L-packets of } G(\mathbb{A}_F) \}$$

$$\Pi_w, \text{aut}(G) : = \{ \text{weak automorphic L-packets of } G(\mathbb{A}_F) \}$$

and

$$c(G) : = \{ (c_v) \in \prod_{v \notin S} ^L G^v \rtimes \text{Fr}_v / \sim : c_v \text{ is } \text{Fr}_v\text{-semisimple for all } v \notin S \} / \sim$$

where we say that $\prod_{v \notin S} c_v \sim \prod_{v \notin S} c'_v$ if and only if $c_v = c'_v$ for almost every $v \notin S$ (i.e. all $v$ outside of a finite set). There is no need to be specific about the set of places $S$ in the notation here because we can always enlarge $S$ by a finite set. Any weak $L$-packet has a representative that is unramified outside of $S$, so we have a map
7.7 Weak global $L$-packets

$$c : \Pi_w(G) \to c(G)$$

(7.23)

defined in the obvious manner. We denote by $c(\Pi)$ the image of an $L$-packet $\Pi$ under this map. We refer to it as the **Langlands class or Satake parameter** of $\Pi$. If $G(F_v)$ has a unique $G(F_v)$-conjugacy class of hyperspecial subgroups for almost every $v$ then by the Satake isomorphism the map $c$ is bijective. In general hyperspecial subgroups are only conjugate under $G^{ad}(F_v)$ [Tit79, §2.5] so the map can have infinite fibers.

We are now in a position to discuss a naïve form of the Langlands functoriality conjecture. Let

$$r : {}^L H \to {}^L G$$

be an $L$-map. This gives rise to a map

$$c(H) \to c(G).$$

We can now state a weak form of the Langlands functoriality conjecture:

**Conjecture 7.7.1 (Langlands)** Given a weak global $L$-packet $\Pi$ of automorphic representations of $H(\mathbb{A}_F)$ there is a global $L$-packet $r(\Pi)$ of automorphic representations of $G(\mathbb{A}_F)$ such that $c(r(\Pi)) = r(c(\Pi))$.

One can also phrase this as the existence of a top arrow making the following diagram commute:

$$\begin{array}{ccc}
\Pi_{w,\text{aut}}(H) & \xrightarrow{3\Pi} & \Pi_{w,\text{aut}}(G) \\
\downarrow c & & \downarrow c \\
 c(H) & \xrightarrow{r} & c(G)
\end{array}$$

The fact that the class $c(\Pi)$ is only defined up to a finite set of places is an irritant, but it is not as horrible as it first appears. To explain, we will require some results described in more detail in §10.6. Let $n_1, \ldots, n_d$ be a collection of positive integers with $\sum_{i=1}^d n_i = n$. Let $P = MN \leq \text{GL}_n$ be the parabolic subgroup of type $(n_1, \ldots, n_d)$ with its standard Levi decomposition (see Example 1.10). For each $i$ let $\pi_i$ be a cuspidal automorphic representation of $A_{\text{GL}_{n_i}} \backslash \text{GL}_{n_i}(\mathbb{A}_F)$ and let $\lambda \in a_M^*$. We can then form the induced representation

$$I(\pi_1, \ldots, \pi_n, \lambda)$$

as in §10.3. In general it may have several irreducible subquotients, all of which turn out to be automorphic. However it always has a canonical subquotient that will be described in more detail in §10.6. Automorphic representations equivalent to this canonical subquotient are known as **isobaric automorphic representations**. In particular, if $I(\pi_1, \ldots, \pi_n, \lambda)$ is irreducible then it is isobaric; this holds in particular if $n_1 = n$ which is to say that the representation is cuspidal.

We now record the following foundational fact from [JS81b] [JS81a]:
Theorem 7.7.1 (Jacquet and Shalika) Every weak automorphic L-packet in $\Pi(\text{GL}_n)$ contains a unique isobaric element. □

This tells us that, at least for $\text{GL}_n$, admissible representations that are automorphic are very special; they are determined by their local factors at any cofinite set of places of $F$ (see Exercise 7.8).

On the other hand, if we fix a finite set of places $S$ of $F$ and choose an analytically well-behaved irreducible admissible representation $\pi_S$ of $A_G \backslash G(F_S)$, then $\pi_S$ is almost always the local factor at $S$ of an automorphic representation. To make this precise, recall the notion of a tempered representation from Definition 4.6 and the Fell topology on the unitary dual from §3.8. For a proof of the following theorem see [Clo07, §3.3] and the references therein:

Theorem 7.7.2 (Burger, Li and Sarnak) Suppose that $\pi_S$ is a unitary irreducible tempered representation of the semisimple group $G$ over $F$. Then there is an automorphic representation $\pi'$ of $G(\mathbb{A}_F)$ such that $\pi'_S$ in the closure of the point $\pi_S$ is in the unitary dual of $G(F_S)$. □

This theorem can be interpreted as saying that almost every irreducible tempered representation of $A_G \backslash G(F_S)$ is the local factor of an automorphic representation.

Exercises

7.1. Let $S_2$ act on $\mathbb{C}[t_1^\pm, t_2^\pm]$ by letting the nontrivial element act by $t_1 \mapsto t_2$. Define a linear map

$$S': C_\infty^c(\text{GL}_2(F) \backslash \text{GL}_2(\mathcal{O}_F)) \rightarrow \mathbb{C}[t_1^\pm, t_2^\pm]^{S_2}$$

by setting, for each $n \in \mathbb{Z}$, $k \in \mathbb{Z}_{\geq 0}$,

$$S' \left( \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \mathbf{1}_{(i+n,k-i+n)} \right) = (t_1 t_2)^n \text{Sym}^k(t_1, t_2).$$

Show that this is an algebra isomorphism and deduce that $S = S'$.

7.2. Let $G$ be a td-group and let $K \leq G$ be a compact open subgroup. A representation $(\pi, V)$ of $G$ is said to be generated by a subspace $W \leq V$ if $\pi(G)W = V$. Prove that there is an equivalence of categories

$$\{\text{representations of } G \text{ generated by } V^K\} \xrightarrow{\sim} \{C_\infty^c(G \backslash K)-\text{modules}\}$$

$$V \mapsto V^K.$$

In particular, deduce that every representation of $G$ generated by $V^K$ is smooth and admissible.
7.3. Let $E/F$ be a quadratic extension of number fields with Galois automorphism $\sigma$. For any $F$-algebra $R$, let
\[
U(R) := \{ g \in \text{GL}_2(E \otimes_F R) : (1 \ 1) \sigma(g)^{-t} (1 \ 1) = g \}.
\]
This is a unitary group in two variables over $F$. Prove that $U$ is quasi-split over $F$, $U_{F_v}$ is split at every place $v$ of $F$ where $E/F$ is split, and $U_{F_v}$ is nonsplit at every place $v$ of $F$ where $E/F$ is nonsplit.

7.4. Prove the isomorphisms (7.10) and (7.11).

7.5. Let $T$ be a torus over $\mathbb{C}$. Prove that the set valued functors on the category of $\mathbb{C}$ algebras are all naturally equivalent: For $\mathbb{C}$-algebra $R$,
\[
R \mapsto \tilde{T}_s(R),
R \mapsto X^*(T) \otimes \mathbb{C} R,
R \mapsto \text{Hom}(X_s(T), R).
\]

7.6. Let $F$ be a local field and let $T$ be a torus over $F$. If $T(F)$ is compact, then $T$ is anisotropic.

7.7. Let $G$ be an affine algebraic group over a local field $F$ and let $V$ be a topological $\mathbb{C}$-vector space (possibly of infinite dimension). Prove that a map
\[
G(F) \to V
\]
is continuous with respect to the natural topology on $G(F)$ and the given topology on $V$ if and only if it is locally constant. Conclude that the map $G(F) \to V$ is continuous if and only if it is continuous when we give $V$ the discrete topology.

7.8. Let $\pi$ be a cuspidal automorphic representation of $A_{\text{GL}_n} \backslash \text{GL}_n(\mathbb{A}_F)$ for some global field $F$. Show that if $S$ is a finite set of places of $F$ and $\pi'_S \not\sim \pi_S$ is an admissible representation of $\text{GL}_n(F_S)$ then
\[
\pi'_S \otimes \pi^S
\]
is never an isobaric automorphic representation of $A_{\text{GL}_n} \backslash \text{GL}_n(\mathbb{A}_F)$. 
References


[Lan] 136


