Chapter 5
Representations of Totally Disconnected Groups

Abstract In this chapter our goal is to develop enough of the representation theory of locally compact totally disconnected groups (or td groups for short) to state a refined definition of an automorphic representation in the next chapter.

5.1 Totally disconnected groups

We start with the following definition, following \[Car79\]:

**Definition 5.1.** A topological group $G$ is **td** or of **td-type** if every neighborhood of the identity contains a compact open subgroup.

Our basic examples are given by the next lemma, which follows readily from the definition of the topology on the points of an algebraic group from §2.2:

**Lemma 5.1.1** Let $G$ be an algebraic group over a local nonarchimedean field $F$. Then $G(F)$ is of td-type. Similarly, if $F$ is a global field and $S$ is a finite set of places of $F$ including the archimedean places if $F$ is a number field, then $G(\mathbb{A}_F^S)$ is a td group. \qed

A td group is Hausdorff and locally compact, so the theory of §3.1 is applicable. A td-type group is also totally disconnected, which explains the terminology (see Exercise 5.1). Thus the topology on a td group is very different from that of a connected Lie group. In fact, in stark contrast to the defining property of td-type group, a connected Lie group satisfies has no small subgroups in a sense made precise in the following lemma:

**Lemma 5.1.2** Let $G$ be a connected Lie group. Then there is a neighborhood $U$ of the identity so that for all $g \in U$ there is an integer $N$ depending on $g$ such that $g^N \notin U$. 

97
Proof. It is a standard result that there are neighborhoods $V_0$ of 0 in $\text{Lie}(G)$ and $U_1$ of 1 in $G$ such that the exponential map $\exp : V_0 \to U_1$ is a diffeomorphism. Let $U = \exp \left( \frac{1}{2} V_0 \right)$. Then if $g = \exp \left( \frac{1}{2} v \right) \in U$ for some $v \in V_0$ one has

$$g^n = \exp \left( \frac{1}{2} v \right) \ldots \exp \left( \frac{1}{2} v \right) .$$

Choosing $n$ large enough that $\frac{n}{2} v \not\in V_0$ we deduce the result. \qed

Remark 5.1. This proof illustrates the difference between nonarchimedian and archimedian metrics, see Exercise 5.5.

Our aim in this chapter is to use the special topological properties of td groups to refine our understanding of their representation theory. We deal with the basic theory in this chapter, including the local Hecke algebra (see §3.1), the definition of admissible in this context (see §5.3, and Flath’s theorem on decomposition of admissible representations 5.7.1. The reader will notice that in this section we essentially make no use of the fact that the groups in question arise as the points of reductive groups in nonarchimedian local fields. The (deeper) part of the theory that requires this structure is relegated to Chapter 8.

Unless otherwise specified for the remainder of this chapter we let $G$ be a locally compact td group.

5.2 Smooth functions in td groups

The first step in understanding the representation theory of a td-group is to define what we mean by a smooth function:

Definition 5.2. A function $f \in C_c(G)$ is **smooth** if it is locally constant. The complex vector subspace of $C_c(G)$ consisting of smooth functions is denoted by $C^\infty_c(G)$.

It turns out that $C^\infty_c(G)$ is preserved under convolution, defined as in §3.3. Thus $C^\infty_c(G)$ is an algebra under convolution. It is known as the **Hecke algebra of $G$**. If $K \leq G$ is a compact open subgroup, then we let

$$C^\infty_c(G // K) \leq C^\infty_c(G)$$

denote the subalgebra of functions that are right and left $K$-invariant.

Lemma 5.2.1 Any element $f \in C^\infty_c(G)$ is in $C^\infty_c(G // K)$ for some compact open subgroup $K$. If $f \in C^\infty_c(G // K)$, then $f$ is a finite linear combination of elements of the form

$$\mathbb{1}_{K \gamma K}$$
for \( \gamma \in G \). Here \( 1_X \) denotes the characteristic function of a set \( X \).

**Proof.** By local constancy of \( f \) for every \( \gamma \) in the support \( \text{supp}(f) \) we can choose a compact open subgroup \( K(\gamma) \leq G \) so that \( f \) is constant on \( \gamma K(\gamma) \). Then

\[
\bigcup_{\gamma} \gamma K(\gamma)
\]

is an open cover of \( \text{supp}(f) \), so it admits a finite subcover \( \bigcup_{i=1}^{n} \gamma_i K(\gamma_i) \) since \( f \) has compact support. Let \( K_\gamma := \bigcap_{i=1}^{n} K(\gamma_i) \); it is a finite intersection of compact open subgroups so it is itself a compact open subgroup. We see that \( f \) is right \( K_\gamma \)-invariant. Similarly we can find a compact open subgroup \( K_\ell \leq G \) such that \( f \) is left \( K_\ell \)-invariant, and letting \( K := K_\ell \cap K_\gamma \) we see that \( f \in C_c^\infty(G // K) \).

Assume \( f \in C_c^\infty(G // K) \).

\[
\bigcup_{\gamma \in \text{supp}(f)} K \gamma K
\]

is an open cover of \( \text{supp}(f) \), which therefore admits a finite subcover. The last claim in the lemma follows. \( \square \)

Note that \( C_c^\infty(G // K) \leq C_c^\infty(G // K') \) if \( K \leq K' \). Thus, in view of the lemma, to explicitly describe the convolution operation on \( C_c^\infty(G) \) it suffices to describe

\[
1_{K_\alpha K} * 1_{K_\beta K}
\]

for a fixed compact open subgroup \( K \leq G \) and \( \alpha, \beta \in G \). We can write

\[
K_\alpha K = \Pi_i K \alpha_i \quad \text{and} \quad K_\beta K = \Pi_j K \beta_j K
\]

where both sums are finite and \( \alpha_i, \beta_j \in G \). We then have

\[
1_{K_\alpha K} * 1_{K_\beta K} = \sum_{i,j} 1_{K \alpha_i \beta_j K}.
\]

(5.1)

**Example 5.1.** All compact open subgroups of \( \text{GL}_n(\mathbb{A}_Q^{\infty}) \) are of the form

\[
K_S \prod_{p \notin S} \text{GL}_n(\mathbb{Z}_p)
\]

for \( S \) a finite set of finite primes and \( K_S \) is a compact open subgroup of \( \text{GL}_n(\mathbb{Q}_S) \). The subgroup \( \text{GL}_n(\mathbb{Z}) = \prod_p \text{GL}_n(\mathbb{Z}_p) \leq \text{GL}_n(\mathbb{A}_Q^{\infty}) \) is a maximal compact open subgroup, and all maximal compact open subgroups are conjugate to this maximal compact open subgroup [Ser06, Chapter IV, Appendix 1]. Examples of nonmaximal compact open subgroups are given by the kernel of the reduction map

\[
\text{GL}_n(\mathbb{Z}) \to \text{GL}_n(\mathbb{Z}/m)
\]

in various degree.
for integers $m$.

If $\gamma \in G(\hat{A}_F^\infty)$, then $\gamma \in G(\hat{O}_F^S)$ for some finite set $S$ of places. If $K^\infty \leq G(\hat{A}_F^\infty)$ is a compact open subgroup, then upon enlarging $S$ we can assume that $K^S = G(\hat{O}_F^S)$. For such a choice of $S$ we have

$$1_{K^\infty \gamma K^\infty} = 1_{K^S \gamma K^S} \otimes 1_{K^S}$$

for some finite set of nonarchimedian places $S$. This reduces the study of nonarchimedian Hecke algebras to the study of the local Hecke algebras

$$C_c^\infty(G(F_v))$$

as $v$ varies over nonarchimedian places $v$ of $F$. This idea will be formalized in a very useful manner in §5.7 below.

### 5.3 Smooth and admissible representations

**Definition 5.3.** A representation of $G$ on a complex vector space $V$ is **smooth** if the stabilizer of any vector in $V$ is open in $G$.

Equivalently, $V$ is smooth if and only if

$$V = \bigcup_{K \leq G} V^K$$

where the superscript denotes the subspace of fixed vectors and where the union is over all compact open subgroups $K \leq G$.

In this definition we do not assume that the representation $V$ is continuous, or for that matter even give a topology on $V$. In fact, $V$ is smooth if and only if the representation is continuous if we give $G$ its usual topology and $V$ the discrete topology. We note that the smooth representations of a given td group form a category where morphisms are simply $G$-equivariant $\mathbb{C}$-linear maps. The category is even abelian.

It is sometimes useful to rephrase the condition of smoothness in terms of the associated representation of the algebra $C_c^\infty(G)$. For this we recall that a module $M$ for an algebra $A$ is **nondegenerate** if every element of $M$ can be written as a finite sum

$$a_1 m_1 + \cdots + a_n m_n$$

for some $(a_i, m_i) \in A \times M$. Of course, this is trivially true if $A$ contains an identity, but we will be applying this concept when $A = C_c^\infty(G)$, which does not have an identity. However, it has approximate identities in the following sense. For each compact open subgroup $K \leq G$ let

$$1_{K \leq G} = 1_{K^S} \otimes 1_{K^S}$$
5.3 Smooth and admissible representations

\[ e_K := \frac{1}{\text{meas}(K)} \mathbb{1}_K. \]

Then \( e_K \) acts as the identity on

\[ C_c^\infty(G \// K) = e_K C_c^\infty(G)e_K \]

and on \( V^K := e_K V \). This observation is in fact the key to the proof of the following lemma, which we leave as an exercise:

**Lemma 5.3.1** There is an equivalence of categories between non-degenerate \( C_c^\infty(G) \)-modules and smooth representations of \( G \).

Using this equivalence we prove the following irreducibility criterion:

**Proposition 5.3.1** A smooth \( G \)-module \( V \) is irreducible if and only if \( V^K \) is an irreducible \( C_c^\infty(G \// K) \)-module for all compact open subgroups \( K \leq G \).

**Proof.** Suppose \( V \) is reducible, that is, \( V = V_1 \oplus V_2 \) as \( C_c^\infty(G) \)-modules, where \( V_1 \) and \( V_2 \) are nonzero. Then

\[ V^K = V_1^K \oplus V_2^K \]

as \( C_c^\infty(G \// K) \)-modules for all compact open subgroups \( K \leq G \), and for sufficiently small \( K \) the spaces \( V_1^K \) and \( V_2^K \) are nonzero by smoothness. Conversely, suppose \( V \) is irreducible, and suppose that

\[ V^K = V_1 \oplus V_2 \]

as \( C_c^\infty(G \// K) \)-modules for some compact open subgroup \( K \leq G \). Here \( V_1 \) and \( V_2 \) are two subspaces of \( V^K \) (necessarily fixed by \( K \)). Then

\[ V_1 = C_c^\infty(G \// K) V_1 = e_K C_c^\infty(G)e_K V_1 = e_K e_K V_1 = V^K. \]

Thus \( V_2 = 0 \).

We now come to the most important definition of the chapter:

**Definition 5.4.** A representation \( V \) of \( G \) is **admissible** if it is smooth and \( V^K \) is finite dimensional for every compact open subgroup \( K \leq G \). A representation of \( C_c^\infty(G) \) is **admissible** if it is nondegenerate and \( e_K V \) is finite dimensional for all compact open subgroups \( K \).

It is immediate that a representation \( V \) of \( G \) is admissible if and only if its associated \( C_c^\infty(G) \)-module is admissible.

Jacquet and Langlands introduced this definition in their classic work [JL70], though some indication of the definition in the real case appeared in work of Harish-Chandra [HC53]. The importance of the definition is that it isolates exactly the correct category of representations to study, and at the
same time eliminates extraneous topological assumptions (e.g. the presence of a Hilbert or pre-Hilbert space structure on $V$). One indication that this is the correct category is the following theorem, which is proven as part of Theorem ?? below:

**Theorem 5.1.** Let $F$ be a nonarchimedian field. Assume that $G$ is the $F$-points of a reductive algebraic group over $F$. Then an irreducible smooth representation of $G$ is admissible.

Another indication that admissible representations are the correct category to study is the fact that unitary representations of $G$ give rise to admissible representations. To explain this we require some preparation. Recall that in the archimedian setting of §4.4 we passed from Hilbert representations to their subspace of $K$-finite vectors ($K$ a maximal compact subgroup of the relevant group) in order to define a notion of admissibility for them. We require a similar process here, but it is slightly simpler. Given a Hilbert space representation $V$ of $G$ we let

$$V_{\text{sm}} := \bigcup_K V^K \leq V$$

where the union is over all compact open subgroups $K \leq G$. Then $V_{\text{sm}} \leq V$ is evidently a smooth subrepresentation of $G$. We say that the Hilbert representation $V$ is admissible if $V_{\text{sm}}$ is admissible. We note that in this case $V_{\text{sm}}$ is a pre-Hilbert representation.

**Lemma 5.3.2** Let $V$ be a Hilbert representation of $G$. The subspace $V_{\text{sm}} \leq V$ is dense. Moreover, if $V$ and $W$ are Hilbert representations and $V_{\text{sm}} \to W_{\text{sm}}$ is a continuous $G$-intertwining morphism then it extends uniquely to a $G$-intertwining continuous morphism $V \to W$. If the morphism $V_{\text{sm}} \to W_{\text{sm}}$ is an isomorphism, then so is $V \to W$.

We leave the proof as an exercise (see Exercise 5.9). Thus, by the lemma, we can detect isomorphisms of Hilbert representations using the underlying representations in the smooth category. We also have the following analogue of Theorem 4.4.1:

**Proposition 5.3.2** If $V_1, V_2$ are Hilbert representations of $G$ and their spaces of smooth vectors $V_{1\text{sm}}$ and $V_{2\text{sm}}$ are equivalent then $V_1$ and $V_2$ are unitarily equivalent.

We give the proof of Proposition 5.3.2 in §5.4. Finally, we have the following nonarchimedian analogue of Theorem 4.1:

**Theorem 5.2 (Harish-Chandra, Bernstein).** Let $F$ be a nonarchimedian field and let $G$ be the $F$-points of a reductive algebraic group over $F$. Then all irreducible unitary representations of $G$ are admissible. $\square$
The proof was reduced to a conjectural estimate on the dimensions of the $V^K$ in [HC70] (which invoked earlier work of Godement, see loc. cit.). This estimate was proven in [Ber74].

Taken together, Lemma 5.3.2, Proposition 5.3.2 and Theorem 5.2 tell us that the study of unitary irreducible representations of $G$ that are the $F$-points of reductive groups over $F$ is equivalent to the study of preunitary admissible representations of this type of group.

5.4 Contragredients

In this section we use the proof of Proposition 5.3.2 as a convenient excuse for introducing the useful concepts of the contragredient representation of a given smooth representation of a td group $G$.

Let $(\pi, V)$ be a smooth representation of $G$. Then there is an action of $G$ on the space of $\mathbb{C}$-linear functionals $\text{Hom}(V, \mathbb{C})$ on $V$ given by

$$G \times \text{Hom}(V, \mathbb{C}) \rightarrow \text{Hom}(V, \mathbb{C})$$

$$(g, \lambda) \mapsto \lambda \circ \pi(g^{-1}).$$

The smooth dual $V^\vee \subseteq \text{Hom}(V, \mathbb{C})$ is the subspace of smooth linear functionals, this is precisely the set of $\lambda$ that are fixed by some compact open subgroup $K$. The action (5.2) preserves $V^\vee$ and affords a smooth representation $(\pi^\vee, V)$ of $G$.

**Definition 5.5.** The contragredient representation $(\pi^\vee, V^\vee)$ is the representation of $G$ on the smooth dual $V^\vee$ given above.

We note that $\vee$ is a contravariant functor from the category of admissible representations of $G$ to itself.

We now define a variant of the contragredient representation. For any $\mathbb{C}$-vector space $V$

$$\nabla := \mathbb{C} \otimes_{(\lambda \mapsto \bar{\lambda})} \mathbb{C} \otimes V.$$ 

This has the same underlying additive group as $V$, but with complex conjugation defined by $(\lambda, v) = \bar{\lambda} \cdot v$ (the bar denoting complex conjugation). The representation

$$(\pi^*, V^*) := (\pi^\vee, \nabla^\vee)$$

is known as the **Hermitian contragredient**. The map $*$ again defines a contravariant functor from the category of admissible representations of $G$ to itself. If $V$ is admissible then there are natural isomorphisms

$$V \rightarrow (V^\vee)^\vee \quad \text{and} \quad V \rightarrow (V^*)^*$$

(see Exercise 5.11).
The following lemma, which is [? , Lemma D.6.3], is easily seen to imply Proposition 5.3.2:

**Lemma 5.4.1** Let $(\pi, V)$ be an admissible representation of $G$. There is a canonical bijection between the set of $G$-invariant, definite, Hermitian scalar products $(\cdot, \cdot)$ on $V$ and the set of isomorphisms

$$\iota : V \rightarrow V^*$$

of smooth representations such that $\iota^* = \iota$. If $(\pi, V)$ is irreducible any two $G$-invariant, definite (resp. positive definite) Hermitian scalar products on $V$ are $\mathbb{R}^\times$ (resp. $\mathbb{R}_{>0}$) multiples of each other.

Here we use the canonical isomorphism $(V^*)^* = V$ to regard $\iota^*$ as a map from $V$ to itself.

**Proof.** The last claim follows from Schur’s lemma and the first claim. Suppose that we have a $G$-invariant Hermitian inner product $(\cdot, \cdot)$ on $V$ ($\mathbb{C}$-linear in the first variable and $\mathbb{C}$-antilinear in the second). We then obtain an $G$-equivariant morphism

$$V \rightarrow V^*$$

$$\varphi_0 \mapsto (\varphi \mapsto (\varphi, \varphi_0)). \quad (5.3)$$

It is clearly injective. Now if $K \leq G$ is a compact open subgroup, then $\dim_C((V^*)^K) = \dim_C((V^\vee)^K) = \dim_C(V^K)$. Thus for all compact open subgroups $K \leq G$ we have that $(5.3)$ induces an isomorphism

$$V^K \congto (V^*)^K.$$  

Hence by the smoothness of $V$ and $V^*$ the injection $(5.3)$ is an isomorphism. Since $(\varphi_1, \varphi_2) = (\varphi_2, \varphi_1)$ we deduce that $\iota = \iota^*$. Conversely if $\iota : V \rightarrow V^*$ is an isomorphism of admissible representations satisfying $\iota = \iota^*$, we can associate to $\iota$ the $G$-invariant Hermitian inner product

$$(\varphi_1, \varphi_2) := \iota(\varphi_2)(\varphi_1).$$

$\Box$

### 5.5 The spherical Hecke algebra

Let $G$ be a connected reductive group over $F$, where $F$ is a nonarchimedean local field. Recall from §2.4 that $G$ is **unramified** if $G$ is quasi-split and split over an unramified extension of $F$. In this case there is a unique $G(F)$-conjugacy class of hyperspecial subgroups $K$ (see §2.4).
5.5 The spherical Hecke algebra

Definition 5.6. If $G$ is unramified and $K \leq G(F)$ is a hyperspecial subgroup, then
\[ C_c^\infty(G(F) \sslash K) \]
is known as the spherical Hecke algebra.

Here $C_c^\infty(G(F) \sslash K)$ is the subalgebra of functions invariant on the left and right under $K$.

The following is arguably the most important fact about this algebra:

**Theorem 5.5.1** The spherical Hecke algebra $C_c^\infty(G(F) \sslash K)$ is commutative.

One way to prove this is via the Satake isomorphism (see Theorem 7.2.1). In special cases this can also be proven using a trick due to Gelfand (see Exercise 5.12). Let us describe the algebra in more detail in a special case:

**Example 5.2.** Let $G = \text{GL}_n$, viewed as a group over $\mathbb{Q}_p$. A hyperspecial subgroup is $\text{GL}_n(\mathbb{Z}_p)$. The spherical Hecke algebra in this case is
\[ C_c^\infty(\text{GL}_n(\mathbb{Q}_p) \sslash \text{GL}_n(\mathbb{Z}_p)). \]

If we set
\[ \lambda = (\lambda_1, \ldots, \lambda_n), \quad \lambda_1 \geq \cdots \geq \lambda_n, \]
and let
\[ \lambda(p) = (\lambda_1, \ldots, \lambda_n)(p) := \text{diag}(p^{\lambda_1}, \ldots, p^{\lambda_n}), \]
then a basis is given by
\[ \{ \mathbb{1}_{\text{GL}_n(\mathbb{Z}_p)}(\lambda(p)) \text{GL}_n(\mathbb{Z}_p) \} \lambda. \]

The Smith normal form for matrices over $\mathbb{Q}_p$, from the theory of elementary divisors, gives the decomposition
\[ \text{GL}_n(\mathbb{Q}_p) = \text{GL}_n(\mathbb{Z}_p)T(\mathbb{Q}_p)\text{GL}_n(\mathbb{Z}_p) \]
and it follows from this that the set above is a basis for the spherical Hecke algebra.

**Definition 5.7.** Assume that $G$ is unramified with hyperspecial subgroup $K \leq G(F)$. An irreducible admissible representation $(\pi, V)$ of $G(F)$ is unramified, or spherical if $V^K \neq 0$.

The following consequence of Theorem 5.5.1 is often referred to as uniqueness of the spherical vector, although it is more correct to say that it establishes the uniqueness of the spherical line:

**Corollary 5.5.1** Assume that $G$ is unramified and that $(\pi, V)$ is an irreducible admissible unramified representation of $G(F)$. Then $\dim \mathbb{C} V^K = 1$. 
Proof. Since $V$ is irreducible, $V^K$, if nonzero, must be an irreducible representation of the commutative algebra $C_c^\infty (G(F) \parallel K)$ by Proposition 5.3.1. \hfill \Box

5.6 Restricted tensor products of modules

In the representation theory of compact or algebraic groups one very quickly reduces the representation theory of products to representation theory of the individual factors, thus simplifying problems significantly. We now explain how one can do this in the context of automorphic representations. The problem is that $G(\mathbb{A}_F)$ is a restricted direct product

$$G(\mathbb{A}_F) \cong \prod_v G(F_v),$$

not a direct product, so we should not expect representations of $G(\mathbb{A}_F)$ to decompose into a direct external product of representations. What one does instead is define a notion of restricted direct products of representations. We make this precise in the current section, and then prove in Theorem 5.7.1 that automorphic representations to indeed factor into restricted direct products indexed by the places in $F$.

We start by defining a restricted direct product of vector spaces. Let $\Xi$ be a countable set, let $\Xi_0 \subset \Xi$ be a finite subset, let $\{W_v : v \in \Xi\}$ be a family of $\mathbb{C}$-vector spaces and for each $v \in \Xi \setminus \Xi_0$ let $w_{0v} \in W_v - 0$. For all sets

$$\Xi_0 \subseteq S \subseteq \Xi$$

of finite cardinality set $W_S := \prod_{v \in S} W_v$. If $S \subseteq S'$ there is a map

$$W_S \rightarrow W_{S'}$$

$$\otimes_{v \in S} \varphi_v \mapsto \otimes_{v \in S} \varphi_v \otimes (\otimes_{v \in S' - S} \varphi_0_v).$$

(5.4)

Consider the vector space

$$W := \bigotimes' W_v := \lim_{\rightarrow S} W_S$$

where the transition maps are given by (5.4). This is is the \textbf{restricted tensor product} of the $W_v$ with respect to the $\varphi_{0v}$. Thus $W$ is the set of sequences

$$(\varphi_v)_{v \in \Xi} \in \bigotimes_v W_v$$
such that \( w_v = \varphi_{0v} \) for all but finitely many \( v \in \Xi \). We note that if we are given for each \( v \in \Xi \) a \( \mathbb{C} \)-linear map 

\[
B_v : W_v \rightarrow W_v
\]

such that \( B_v(w_{0v}) = w_{0v} \) for all but finitely many \( v \in \Xi \) then this gives a map 

\[
B = \otimes_v B_v : W \rightarrow W
\]

\[
\otimes \varphi_v \mapsto \otimes B_v(\varphi_v).
\]

We now define a restricted directed product of algebras. Suppose we are given \( \mathbb{C} \)-algebras (not necessarily with unit) \( \{ A_v : v \in \Xi \} \) and idempotents \( a_{0v} \in A_v \) for all \( v \in \Xi - \Xi_0 \). If \( S \subseteq S' \) there is a map 

\[
A_S \rightarrow A_{S'}
\]

\[
\otimes_{v \in S} a_v \mapsto \otimes_{v \in S} a_v \otimes (\otimes_{v \in S' - S} a_{0v})
\]

Consider the algebra 

\[
A := \otimes'_{v \in \Xi} A_v := \lim_{S \rightarrow S'} A_S
\]

where the transition maps are given by (5.5). This is the restricted tensor product of the \( A_v \) with respect to the \( a_{0v} \). Finally, if \( W_v \) is an \( A_v \)-module for all \( v \in \Xi \) such that \( a_{0v} \varphi_{0v} = \varphi_{0v} \) for almost all \( v \), then \( \otimes_v W_v \) is an \( A \)-module. The isomorphism class of \( W \) as an \( A \)-module in general depends on the choice of \( \{ \varphi_{0v} \} \). However, if we replace the \( \varphi_{0v} \) by nonzero scalar multiples we obtain isomorphic \( A \)-modules.

An easy example of this construction is the ring of polynomials in infinitely many variables. It can be given the structure of a restricted tensor product of algebras 

\[
\mathbb{C}[X_1, X_2, \ldots] = \otimes' \mathbb{C}[X_i]
\]

where we take \( a_{0i} \) to be the identity in \( \mathbb{C}[X_i] \).

The fundamental example of this construction for our purposes is the isomorphism 

\[
C_c^\infty(G(A_F^\infty)) \cong \otimes' C_c^\infty(G(F_v))
\]

where \( F \) is a global field and \( \infty \) is the set of infinite places. The idempotents are 

\[
e_{K_v} := \frac{1}{\text{meas}(K_v)} \mathbb{1}_{K_v}
\]

where \( K_v \leq G(F_v) \) is a (choice of) hyperspecial subgroup for all \( v \) outside a finite set of places of \( F \) including \( \infty \). Here the measure in question is the Haar measure used in the definition of the convolution product on \( C_c^\infty(G(F_v)) \).
We really can take any choice of compact open subgroup here (see Exercise 5.13), so the isomorphism above is not canonical. Of course, since all choices lead to isomorphic algebras, the representation theory will be essentially the same (see Exercise 5.14).

### 5.7 Flath’s theorem

Let \( \Xi_0 \) be a finite set of places of \( F \) including the infinite places. Enlarging \( \Xi_0 \) if necessary we assume that if \( v \not\in \Xi_0 \) then \( G_{F_v} \) is unramified. If \( v \not\in \Xi_0 \) we let \( K_v \leq G(F_v) \) be a choice of hyperspecial subgroup.

**Definition 5.8.** A \( C_c^\infty(G(\mathbb{A}_F^\infty)) \)-module \( W \) is factorizable if we can write

\[
W \cong \bigotimes_v W_v, \tag{5.6}
\]

where the restricted direct product is with respect to elements \( \phi_0_v \in W_v^K_v \), \( \dim(W_v^K_v) = 1 \), and the isomorphism (5.6) intertwines the action of \( C_c^\infty(G(\mathbb{A}_F^\infty)) \) with the action of \( \bigotimes_v C_c^\infty(G(F_v)) \), the restricted direct product being with respect to the idempotents \( e_{K_v} \).

In view of the assumption that \( \dim(W_v^K_v) = 1 \) the module \( \bigotimes_v W_v \) only depends on the choice of the \( \phi_0_v \) up to isomorphism. Note that in this case \( W \) is admissible and irreducible if and only if the \( W_v \) are for all \( v \).

**Theorem 5.7.1 (Flath)** Every admissible irreducible representation \( W \) of \( C_c^\infty(G(\mathbb{A}_F^\infty)) \) is factorizable.

For the moment let \( G \) be a group of td-type (see §5.1 for the definition). The following is the easy case of Theorem 5.7.1:

**Theorem 5.7.2** Let \( G_1, G_2 \) be td groups and let \( G = G_1 \times G_2 \).

(a) If \( V_i \) is an admissible irreducible representation of \( G_i \) for \( i = 1, 2 \) then \( V_1 \otimes V_2 \) is an admissible irreducible representation of \( G \).

(b) If \( V \) is an admissible irreducible representation of \( G \) then there exists admissible irreducible representations \( V_i \) of \( G_i \) for \( i = 1, 2 \) such that \( V \cong V_1 \otimes V_2 \). Moreover the isomorphism classes of the \( V_i \) are uniquely determined by \( V \).

**Proof.** We first prove (a). By the irreducibility criterion Proposition 5.3.1 for every compact open subgroup \( K_1 \times K_2 \leq G_1 \times G_2 \) the representation \( V_1^{K_1} \otimes V_2^{K_2} \) of \( C_c^\infty(G_1//K_1) \times C_c^\infty(G_2//K_2) \) is irreducible. But one knows that \( C_c^\infty(G_1 \times G_2) = C_c^\infty(G_1) \times C_c^\infty(G_2), \) \( C_c^\infty(G_1 \times G_2//K_1 \times K_2) = C_c^\infty(G_1//K_1) \times C_c^\infty(G_2//K_2) \), and \( (V_1 \otimes V_2)^{K_1 \times K_2} = V_1^{K_1} \otimes V_2^{K_2} \), so this implies that \( V_1 \otimes V_2 \) is admissible and irreducible.
Conversely, let $W$ be an admissible $G$-module. Choose $K = K_1 \times K_2$ such that $W^K \neq 0$ (this is possible by smoothness). Then since $W^K$ is finite dimensional there exists finite-dimensional $C^\infty_c(G_i // K_i)$-modules $W_i(K_i)$ and an isomorphism of $C^\infty_c(G // K)$ modules $W^K \to W_1(K_1) \otimes W_2(K_2)$. Varying $K$, we obtain a decomposition

$$W \cong W_1 \otimes W_2$$

as $C^\infty_c(G) \cong C^\infty_c(G_1 \times G_2)$-modules, where

$$W_1 := \lim_{\rightarrow} W_1(K_1) \quad \text{and} \quad W_2 := \lim_{\rightarrow} W_2(K_2).$$

□

We now prove Flath’s theorem:

**Proof of Theorem 5.7.1:** For each finite set of finite places $S$ containing the set of places $v$ where $G_{F_v}$ is ramified let $K^S := \prod_v K_v \leq G(\mathbb{A}_F^S)$ be a fixed compact open subgroup with $K_v$ hyperspecial for every $v$.

Choose an isomorphism

$$C^\infty_c(G(\mathbb{A}_F^S)) \cong \bigotimes_v C^\infty_c(G(F_v)) \quad (5.7)$$

where the restricted direct product is constructed using the idempotents $e_{K_v}$ for hyperspecial $K_v \leq G(F_v)$. Use (5.7) to identify these two algebras for the remainder of the proof. We then have a well-defined subalgebra

$$A_S := C^\infty_c(G(F_S^\infty)) \otimes e_{K_S} \leq C^\infty_c(G(\mathbb{A}_F^\infty)),$$

where $e_{K_S} = \bigotimes_{v \in S} e_{K_v}$ is $\frac{1}{\meas(K_S)} 1_{K_S}$. By Corollary 5.5.1 and Theorem 5.7.2, as a representation of $A_S$ we have an isomorphism

$$W^K_S \cong \bigotimes_{v \in S} W_v \otimes W^S$$

where $W^S$ is a one-dimensional $\mathbb{C}$-vector space on which $e_{K_S}$ acts trivially. Hence, by admissibility,

$$W = \bigcup_S W^K_S \cong \lim_{\rightarrow} \bigotimes_{v \in S} W_v \otimes W^S \quad (5.8)$$

with respect to the obvious transition maps (compare §5.6). On the other hand, (5.7) induces an identification

$$C^\infty_c(G(\mathbb{A}_F^\infty)) = \bigcup_S A_S = \lim_{\rightarrow} A_S \quad (5.9)$$
where the direct limit is taken with respect to the obvious transition maps (again, compare §5.6), and it is clear that (5.8) is equivariant with respect to (5.9).

Exercises

5.1. Prove that a group of td-type is Hausdorff, locally compact, and totally disconnected.

5.2. Let $V$ be a representation of the td group $G$. Define

$$V_{sm} := \bigcup_K V^K$$

where the union is over all compact open subgroups $K \leq G$. Prove that $V_{sm}$ is preserved by $G$ and is a smooth subrepresentation of $G$.

5.3. Suppose that $V$ is a Hilbert space representation of the td group $G$. Define $V_{sm}$ as in the previous exercise. Show that $V_{sm}$ admits a countable algebraic (also known as Hamel) basis. Show that if $V$ is infinite dimensional then $V$ does not admit a countable algebraic basis. Conclude that if $V$ is infinite-dimensional then $V_{sm} \nsubseteq V$.

5.4. Suppose that $V$ is a smooth admissible representation of the td group $G$ that is preunitary, that is, there is a nondegenerate inner product on $V$ that is preserved by $G$. Prove that every $G$-invariant subspace $V_1 \leq V$ admits a complement, that is, a $G$-invariant subspace $V_2 \leq V$ such that $V_1 \oplus V_2 = V$.

5.5. Let $p$ be a prime number in $\mathbb{Z}$. Prove that there are no constants $c_1, c_2 \in \mathbb{R}_{>0}$ such that

$$c_1 |·|_p \leq |·|_{\infty} \leq c_2 |·|_p,$$

where $|·|_{\infty}$ denotes the usual archimedian norm on $\mathbb{Q}$ and $|·|_p$ denotes the $p$-adic norm.

5.6. Prove the equality (5.1).

5.7. Prove Lemma 5.3.1.

5.8. Let $G$ be a td group and $V$ an admissible representation of $G$. Prove Schur’s lemma: any intertwining map $V \rightarrow V$ is a scalar multiple of the identity.

5.9. Prove Lemma 5.3.2.
5.10. Show that for any admissible representation \( V \) of a \( \text{td} \) group \( G \)
\[
\text{Hom}(V^K, \mathbb{C}) = (V^\vee)^K
\]
for all compact open subgroups \( K \leq G \). Conclude that the contragredient and Hermitian contragredient of an admissible representation of a \( \text{td} \) group is again admissible.

5.11. Prove that if \( V \) is an admissible representation of a \( \text{td} \) group then there are canonical isomorphisms \( V \to (V^\vee)^\vee \) and \( V \to (V^*)^* \).

5.12. Let \( F \) be a nonarchimedian local field. Consider the spherical Hecke algebra \( C_c^\infty(\text{GL}_n(F) \sslash \text{GL}_n(O_F)) \). For \( f \in C_c^\infty(\text{GL}_n(F)) \) let \( f^\dagger(g) := f(g^t) \).
Show that for every \( f_1, f_2 \in C_c^\infty((\text{GL}_n(F)) \)
\[
(f_1 * f_2)^\dagger(g) = (f_2^\dagger * f_1^\dagger)(g).
\]
Show, on the other hand, that \( f^\dagger = f \) for \( f \in C_c^\infty(\text{GL}_n(F) \sslash \text{GL}_n(O_F)) \).
Conclude that \( C_c^\infty(\text{GL}_n(F) \sslash \text{GL}_n(O_F)) \) is commutative.

5.13. Let \( F \) be a global field. For all \( v \) outside a finite set of places including the infinite places choose hyperspecial subgroups \( K_v \leq G(F_v) \). Show that
\[
G(A_\infty) \cong \prod_{v \neq \infty} G(F_v)
\]
where the restricted tensor product is taken with respect to the \( K_v \). Note that we allow for infinitely many of the \( K_v \)'s (perhaps all of them) to be different from those obtained by choosing a model of \( G \) as in Proposition 2.3.1.

5.14. Let \( F \) be a global field. For all \( v \) outside a finite set of places including the infinite places choose hyperspecial subgroups \( K_{1v}, K_{2v} \leq G(F_v) \). For \( i = 1, 2 \), let \( A_i \) be \( \otimes_{v \neq \infty} C_c^\infty(G(F_v)) \) where the restricted direct product is taken with respect to \( \text{meas}(K_v)^{-1} 1_{K_v} \). Construct a bijection between irreducible admissible \( A_1 \)-modules and irreducible admissible \( A_2 \)-modules.
References


