Chapter 3
Automorphic Representations

Abstract In this chapter we give a definition of an automorphic representation in the category of unitary representations of the adelic points of an algebraic group. This definition will subsequently be refined, by enlarging the category of representations considered to the category of admissible representations, and then later by replacing representations of the archimedian component of the adelic group by \((g, K)-\text{modules}.

3.1 Representations of locally compact groups

Our goal in this section is to give a definition of an automorphic representation that is completely natural from the point of view of trace formulae.

We will start by introducing some basic abstract representation theory (i.e. representation theory that requires no more structure than a locally compact group). Our basic reference is [Fol95]. Throughout this chapter we let \(G\) be a locally compact second-countable (hausdorff) topological group (for example \(G(R)\) for a locally compact topological ring \(R\) such as the adele ring \(\mathbb{A}_F\)). Let \(V\) be a topological vector space over \(\mathbb{C}\) and let

\[
\text{End}(V)
\]
be the space of all continuous linear maps from \(V\) to itself. We let \(\text{GL}(V) \subset \text{End}(V)\) be the group of invertible continuous linear maps with continuous inverse.

Often we will be concerned with the case where \(V\) is a Hilbert space, that is, a complex inner product space complete with respect to the metric induced by the inner product that admits a countable orthonormal basis. In particular, Hilbert spaces will always be assumed to be separable. The inner product will be denoted \((\cdot, \cdot)\) and the norm by \(\|\cdot\|_2\). In this case we let \(U(V)\) be the subgroup of \(\text{GL}(V)\) preserving \((\cdot, \cdot)\).
Definition 3.1. A representation of $G$ is a continuous group homomorphism 

$$\pi : G \rightarrow \text{GL}(V).$$

If $V$ is a Hilbert space and $\pi$ maps $G$ to the subspace of unitary operators, that is

$$\langle \pi(g)\varphi_1, \pi(g)\varphi_2 \rangle = \langle \varphi_1, \varphi_2 \rangle$$

for all $\varphi_1, \varphi_2 \in V$ and $g \in G$, then $\pi$ is said to be unitary.

If we wish to specify that $\pi$ acts on a ?-space (e.g. Fréchet, Hilbert, locally convex) we often to $\pi$ as a ?-representation.

A morphism of representations (or a $G$-intertwining map) is a morphism of topological vector spaces that are $G$-equivariant; it is an isomorphism if the underlying morphism of topological vector spaces is an isomorphism. If two representations are isomorphic then they are also called equivalent. Two unitary representations $(\pi_1, V_1)$ and $(\pi_2, V_2)$ are unitarily equivalent if there is a $G$-equivariant isomorphism of Hilbert spaces $V_1 \rightarrow V_2$. Thus unitary equivalence implies equivalence, but not conversely.

A subrepresentation of $\pi$ is a closed subspace $W \leq V$ that is preserved by $\pi$, a quotient of $\pi$ is a topological vector space $W$ equipped with an action of $G$ and a $G$-equivariant continuous linear surjection $V \rightarrow W$, and a subquotient is a subrepresentation of a quotient.

We will mostly be interested in unitary representations, and we need language to describe when a representation that is not originally presented as a unitary representation actually “is” unitary. First, we say that a representation on a Hilbert space is unitarizable if it is equivalent to a unitary representation. Sometimes the natural representations to study are not on Hilbert spaces, or even complete spaces (see, for example, Chapter 5). However, one can still discuss unitarity in this situation as follows. Recall that a pre-Hilbert space is a complex inner product space that admits a countable orthonormal basis. Thus a Hilbert space is simply a pre-Hilbert space that is complete with respect to the metric induced by the inner product. A representation $\pi$ of $G$ on a pre-Hilbert space $V$ extends by continuity to the completion $\overline{V}$ of $V$ (which is a Hilbert space). With this in mind we say that a representation $\pi$ on a pre-Hilbert space $V$ is pre-unitary if $\pi(G) \leq \text{U}(V)$. Thus for every pre-unitary representation of $G$ we obtain, by continuity, a canonical unitary representation of $G$ on $\overline{V}$. Finally, we say that a representation is pre-unitarizable if it is equivalent to a pre-unitary representation. Of course, in practice, these distinctions are often ignored, and one sometimes calls a unitarizable representation, or even a pre-unitarizable representation, simply a unitary representation.

We now discuss the basic means of constructing unitary representations. Assume one has a continuous right action

$$X \times G \rightarrow X$$

$$(x, g) \mapsto xg$$

(3.1)
where $X$ is a locally compact Hausdorff space equipped with a $G$-invariant Radon measure $dx$. The Radon measure gives rise to an $L^2$-space, namely the space $L^2(X, dx)$ of square-integrable complex valued functions on $X$ endowed with the inner product

$$ (\varphi_1, \varphi_2) := \int_X \varphi_1(x)\overline{\varphi_2}(x)dx. \quad (3.2) $$

We obtain a left action of $G$ on $L^2(X, dx)$ via

$$ \pi(g)\varphi(x) := \varphi(xg). \quad (3.3) $$

**Theorem 3.1.1** The action of $G$ on $L^2(X, dx)$ defined as in (3.3) is continuous.

We refer to the action of $G$ on $L^2(X, dx)$ just constructed as the **natural action** or the **regular action**.

For the proof we follow [Fol95, Proposition 2.41]:

**Proof.** Let

$$ C_c(X) \leq L^2(X, dx) $$

be the subspace of compactly supported continuous functions. Let $U \subset G$ be an open neighborhood of the identity $1 \in G$ with compact closure, let $\varphi \in C_c(X)$, and let $W = \text{supp}(\varphi)U \subset X$. Then $W$ is relatively compact, and $\pi(g)\varphi$ is supported in $W$ if $g \in U$. Thus

$$ \|\pi(g)\varphi - \varphi\|_2 \leq \text{vol}(W)^{1/2}\|\pi(g)\varphi - \varphi\|_\infty. $$

This goes to zero as $g \to 1$ (see Exercise 3.1).

Since $X$ is locally compact and $dx$ is a Radon measure $C_c(X)$ is dense in $L^2(X, dx)$, given $\varphi_1 \in L^2(X, dx)$ we can choose $\varphi_2 \in C_c(X)$ such that $\|\varphi_1 - \varphi_2\|_2 < \varepsilon$ for any $\varepsilon > 0$. Then

$$ \|\pi(g)\varphi_1 - \varphi_1\|_2 \leq \|\pi(g)(\varphi_1 - \varphi_2)\|_2 + \|\pi(g)\varphi_2 - \varphi_2\|_2 + \|\varphi_2 - \varphi_1\|_2 $$

$$ \leq 2\varepsilon + \|\pi(g)\varphi_2 - \varphi_2\|_2 $$

which, by what we’ve already shown, goes to zero as $g \to 1$. □

### 3.2 Haar measures on locally compact groups

To give more concrete examples of representations it is useful to recall the notion of Haar measure. If $G$ is a locally compact group then there exists a positive Radon measure $d_xg$ on $G$ that is right invariant under the action of $G$: 
\[
\int_G f(gx)d_r g = \int_G f(g)d_r g \quad \text{for all } x \in G. \tag{3.4}
\]

[\text{Fol95, Theorem 2.10}].

Moreover, this measure is unique up to multiplication by an element of \(\mathbb{R}_{>0}\). A \textbf{right Haar measure} is a choice of such a measure. There is also a left invariant positive Borel measure \(d_l g = d_r (g^{-1})\), again unique up to scalars. Such a measure is known as a \textbf{left Haar measure}.

The existence of the right Haar measure gives via the recipe of (3.1) a representation of \(G\) on \(L^2(G) = L^2(G, dg)\) called the \textbf{(right) regular representation}.

\textbf{Definition 3.2.} A locally compact group \(G\) is \textbf{unimodular} if there is a (nonzero) constant \(C\) such that \(d_r g = Cd_l g\).

We note that an abelian group is trivially unimodular. Moreover a compact (Hausdorff) group is unimodular (see Exercise 3.2).

\textbf{Example 3.1.} The points of Borel subgroups are, in general, not unimodular. For example, if \(B \leq \text{GL}_2\) is the Borel subgroup of upper triangular matrices then for any local field \(F\) with normalized valuation \(|\cdot|\) we can write

\[
B(F) = \left\{ \begin{pmatrix} a & t \\ b & 1 \end{pmatrix} \in \text{GL}_2(F) \right\}.
\]

With respect to this decomposition one can take

\[
d_l g := \frac{dadbdt}{|a||b|} \quad \text{and} \quad d_r g := \frac{dadbdt}{|b|^2}.
\]

The failure of a group to be unimodular is measured by an abelian quasi-character called the \textbf{modular character}. If \(G\) is a locally compact group with right haar measure \(d_r g\), then the modular character is the homomorphism \(\delta_G : G \to \mathbb{R}_{>0}\) defined by:

\[
d_r(hg) = \delta_G(h)d_r g. \tag{3.5}
\]

We note that one can alternately define \(\delta_G(g)\) via the relation

\[
d_r(g^{-1}) = \delta_G(g)d_l g \tag{3.6}
\]

(compare Exercise 3.3).

\textbf{Remark 3.1.} Some references define the modular character to be the multiplicative inverse of \(\delta_G\). In particular, in [Fol95], \(\Delta(g) := \delta_G(g)^{-1}\). The choice of normalization we use in this book seems to be consistent with most of
the literature in automorphic representation theory; one motivation to define
things this way is provided by Proposition 3.4.1.

The following proposition provides a useful means of decomposing Haar
measures:

**Proposition 3.2.1** Suppose that \( S \) and \( T \) are closed subgroups of \( G \) with
compact intersection and that the product map \( S \times T \to G \) is open with
image of full measure with respect to \( d_\ell \)g. Then one can normalize the left
and right Haar measures on \( S \) and \( T \) so that

\[
\int_{G} f(g) \, d_\ell \! g = \int_{S \times T} f(st) \frac{\delta_T(t)}{\delta_G(t)} \, d_\ell s \, d_\ell t
= \int_{S \times T} f(st) \frac{\delta_G(t)}{\delta_G(t)} \, d_\ell s \, d_\ell t.
\]

In particular, if \( G \) is unimodular, then

\[
\int_{G} f(g) \, dg = \int_{S \times T} f(st) \, d_\ell s \, d_\ell t.
\]

**Proof.** The first assertion is [Kna86, Theorem 8.32] in the case where \( G \) is
a Lie group. The proof is the same in general. We also note that in the
notation of loc. cit. one has \( \Delta_G := \delta_G, \Delta_S := \delta_S, \) and \( \Delta_T := \delta_T. \) For the
second equality we use (3.6). \( \square \)

Sometimes we will require a slight generalization of the notion of a Haar
measure. Let \( G \) be a locally compact group and let \( H \subseteq G \) be a closed
subgroup. We will sometimes wish to integrate over the quotient \( H \setminus G. \) The
following theorem tells us precisely when this is possible (see, e.g. [?, Theorem
2.49] for a proof):

**Theorem 3.2.1** There is a right \( G \)-invariant Radon measure \( \mu \) on \( H \setminus G \) if
and only if \( \delta_G|_H = \delta_H. \) In this case, \( \mu \) is unique up to a constant factor, and
for a suitable choice of right Haar measures \( dg \) (resp. \( dh \)) on \( G \) (resp. on \( H \))
one has

\[
\int_{G} f(g) \, dg = \int_{H \setminus G} \left( \int_{H} f(hg) \, dh \right) \, d\mu(Hg)
\]

for \( f \in C_c(G). \) \( \square \)

The measure \( d\mu \) in the theorem will be denoted by \( \frac{dH}{H}. \) We remark that in
loc. cit. the author considers right invariant measures on \( G/H \) instead, but
the result above follows in the same manner, or from the version with \( H \setminus G \)
replaced by \( G/H. \)
3.3 Convolution algebras of test functions

To any locally compact group \( G \) we can associate the space of integrable functions
\[
L^1(G) := \left\{ \text{\(d_r g\)-measurable } f : G \to \mathbb{C} : \int_G |f(g)| d_r g < \infty \right\}
\]
and the subspace of compactly supported continuous functions \( C_c(G) \). This space is endowed with the structure of a \( \mathbb{C} \)-algebra with product
\[
f_1 \ast f_2(g) := \int f_1(gh^{-1}) f_2(h) d_r h.
\] (3.7)

This product is associative (Exercise 3.14).

**Example 3.2.** When \( G = \mathbb{R} \), the convolution \( f_1 \ast f_2 \) is exactly the usual convolution from Fourier analysis.

Given a representation \( \pi : G \to \text{GL}(V) \) of \( G \), if \( f \in C_c(G) \) one obtains a linear map
\[
\pi(f) : V \to V
\]
\[
\varphi \mapsto \int_G \pi(g)f(g) \varphi d_r g.
\]

If \( \pi \) is unitary then this map is bounded, and in fact in the unitary case it is well-defined for all \( f \in L^1(G) \) (Exercise 3.15). The definition (??) is chosen so that
\[
\pi(f_1 \ast f_2) = \pi(f_1) \circ \pi(f_2)
\]
(see Exercise 3.16). Thus, in particular, representations \( \pi \) of \( G \) give rise to algebra representations of \( C_c(G) \), that is, \( \mathbb{C} \)-algebra homomorphisms
\[
C_c(G) \to \text{End}(V).
\]

3.4 Haar measures on the points of algebraic groups

Let \( G \) be an (affine) algebraic group over a local or global field \( F \). It is useful to more explicitly describe the Haar measures on \( G(R) \) for a locally compact topological \( F \)-algebras \( R \) using differential forms. We will do this in the current section.

Let \( n = \dim_F G \). Then there is a nonzero top-dimensional left-invariant differential form
\[
\omega_L \in \wedge^n(g)
\]
which is unique up to multiplication by an element of $F^\times$. For every place $v$ of $F$ we therefore obtain a differential form $\omega_v := \omega_v \ell_v$ upon localizing. This allows us to define for each $v$ a Radon measure

$$C_v(G(F_v)) \longrightarrow \mathbb{C}$$

$$f \longmapsto \int_{G(F_v)} f(g) d|\omega_v|_v(g).$$

Since $\omega_v$ is left $G(F_v)$-invariant, the measure $d|\omega_v|_v$ is left $G(F_v)$-invariant, and hence is a left Haar measure. This is explained in more detail in the archimedian case in [Kna86, Chapter VIII.2], and in the nonarchimedian case in [Oes84, §2].

**Proposition 3.4.1** One has

$$\delta_{G(F_v)}(g) = |\det (\Ad(g) : g(F_v) \to g(F_v))|_v.$$  \hspace{1cm} (3.8)

**Proof.** One has

$$\Ad(g) \omega_v = c(g) \omega_v$$

where

$$c(g) := \det (\Ad(g) : g(F_v) \to g(F_v)).$$

We write this in terms of Haar measures:

$$d|\omega_v|_v(hg^{-1}) = d|\omega_v|_v(g(hg^{-1}) = |c(g)|_v d|\omega_v|_v(h).$$

On the other hand

$$d(hg^{-1}) = \delta_{G(F_v)}(g) d(h)$$

for any left Haar measure $d(h)$ on $G(F_v)$ by Exercise 3.3. We deduce that $|c(g)| = \delta_{G(F_v)}(g)$. \hfill $\square$

**Corollary 3.4.1** If $G$ is reductive, then $G(F_v)$ is unimodular.

**Proof.** Let $Z_G$ be the center of $G$ and $G^{\text{der}}$ its derived group. We then have a commutative diagram

$$
\begin{array}{ccc}
Z_G \times G^{\text{der}} & \overset{a}{\longrightarrow} & \mathbb{G}_m \\
\downarrow b & & \downarrow 1 \\
G & \overset{c}{\longrightarrow} & \mathbb{G}_m
\end{array}
$$

where $a$ and $c$ are given on points by

$$g \mapsto \det(\Ad(g) : g^{\text{der}} \to g),$$

the left vertical map $b$ is the product map, and the right vertical map is the identity. The map $a$ is trivial on $Z_G$ and it is trivial on $G^{\text{der}}$ since $X^*(G^{\text{der}})$...
is trivial. Since $b$ is surjective, $c$ is trivial as well, and we deduce the corollary. \hfill \Box

This provides us with a description of the left Haar measures $dg_v$ on $G(F_v)$ for every $v$. We similarly obtain right Haar measures $d_r(g) := d_r(g^{-1})$. One now wishes to obtain a right Haar measure on $G(A_F)$. To do this let $S$ be a finite set of places of $F$ including the archimedean places and let

$$K^S = \prod_{v \notin S} K_v \leq G(A_F^S)$$

be a maximal compact open subgroup. We then define a right (resp. left) Haar measure on $G(A_F)$ via

$$d_l g = \prod_{v \in S} d_l g_v \quad \text{and} \quad d_r g = \prod_{v \in S} d_r g_v, \quad (3.9)$$

where we assume that $d_l g_v(K_v) = d_r g_v(K_v) = 1$ for all $v \notin S$. Then it is easy to see that the measures $d_l g, d_r g$ so defined are left and right Haar measures on $G(A_F)$, respectively (see Exercise 3.5). If we wish, we can apply an analogous procedure and obtain left and right Haar measures on $A_G \backslash G(A_F)$.

We can now prove the following lemma:

**Lemma 3.4.1** Any right Haar measure on $G(A_F)$ is left invariant under $G(F)$.

**Proof.** Since the right Haar measure is constructed as the product of local Haar measures one has

$$\delta_{G(A_F)}(g) = \prod_v \delta_{G(F_v)}(g_v) = |\det(\text{Ad}(g) : g(A_F) \rightarrow g(A_F))|_{A_F}. \quad (3.10)$$

By the product formula we deduce that $\delta_{G(A_F)}(\gamma) = 1$ for $\gamma \in G(F)$. \hfill \Box

The proof here amounted to observing that $\delta_{G(A_F)}$ is trivial on $G(F)$. In fact there is a much larger subgroup of $G(A_F)$ on which the modular quasi-character is trivial, namely $G(A_F)^1$, defined as in §2.7:

**Lemma 3.4.2** The group $G(A_F)^1$ is unimodular. If $G$ is reductive, then $G(A_F)$ is unimodular.

**Proof.** The mapping

$$g \mapsto \det(\text{Ad}(g) : g \rightarrow g)$$

defines an element of $X^*(G)$, the character group of $G$. Since the modular character of $G(A_F)$ is just this character followed by the adelic norm by (3.10) we deduce that $G(A_F)^1$ is unimodular.
3.5 Automorphic representations in the $L^2$-sense

Finally, if $G$ is reductive, then the modular character of $G(\mathbb{A}_F)$ is the product of the local modular characters by our construction of the Haar measure, so we deduce that $G(\mathbb{A}_F)$ is unimodular by Corollary 3.4.1.

Now $G(F)$ is a discrete subgroup of $G(\mathbb{A}_F)$ and $G(\mathbb{A}_F)^1$. It is in particular closed in either of these groups. Thus in view of Lemma 3.4.1 and Lemma 3.4.2 $d_r g$ induces a right $G(\mathbb{A}_F)$-invariant Radon measure on $G(F)\backslash G(\mathbb{A}_F)$.

(see [Fol95, Theorem 2.49]). This quotient only has finite volume if $G(\mathbb{A}_F) = G(\mathbb{A}_F)^1$. Similarly any Haar measure on $G(\mathbb{A}_F)^1$ induces a right $G(\mathbb{A}_F)$-invariant Radon measure on $[G]$. The latter quotient always has finite volume (see Theorem 2.6.2).

The way we obtained the Haar measure on $G(\mathbb{A}_F)$ is a little unnatural. It would be more natural to just take the product $\prod_v \omega_v$ directly. However, this product is often not convergent. One can introduce a canonical family of positive real numbers $\lambda_v$ such that

$$\prod_v \lambda_v |\omega_v|_v$$

is convergent, and in this way one obtains the so-called Tamagawa measure on $G(\mathbb{A}_F)$. Upon choosing a suitable measure on $A_G$ we obtain a measure on $G(\mathbb{A}_F)^1$ and can thereby obtain the so-called Tamagawa measure on $[G]$. The measure of $[G]$ with respect to the Tamagawa measure is called the Tamagawa number of $G$. This number is intimately connected with the arithmetic of $G$. We refer to the reader to [Kot88] and the references therein for more details.

3.5 Automorphic representations in the $L^2$-sense

Now assume that $G$ is an affine algebraic group over a global field $F$. Consider the quotient

$$[G] := G(F)\backslash G(\mathbb{A}_F)^1$$

as in (2.9).

By Lemma 3.4.1 a right Haar measure on $G(\mathbb{A}_F)$ is left invariant under $G(F)$ and with respect to it the space $[G]$ has finite volume (compare Theorem 2.6.2). Thus we have a well-defined Hilbert space

$$L^2([G])$$

with inner product defined as in (3.2) (but with $X$ replaced by $[G]$ and $dx$ replaced by the right Haar measure $d_r g$).
Definition 3.3. An automorphic representation of $G(\mathbb{A}_F)^1$ is an irreducible unitary representation $\pi$ of $G(\mathbb{A}_F)^1$ that is equivalent to a subquotient of $L^2([G])$.

Here a subquotient is a subrepresentation of a quotient representation. The reason we must consider subquotients, and not just subrepresentations, is discussed in §3.6 below.

Note that contrary to the usual convention we have not assumed that $G$ is reductive. Even without this assumption the preceding definition makes sense. It is only when one starts talking about admissible representations that the assumption of reductivity is really necessary. A concrete example of this is given right after [BS98, Proposition 3.2.11]. The nonreductive case has not received as much attention as the reductive case, but see [Sli84].

We end this chapter with §3.6 through 3.9. Sections 3.6 through 3.8 discuss how one can decompose unitary representations of locally compact groups in general, and §3.9 discusses why we have restricted our attention to affine algebraic groups instead of certain natural larger classes of groups. All of these sections can be omitted on a first reading.

Before delving into these sections we note that our definition of an automorphic representation leaves much to be desired in that it is unclear how one would go about studying these objects and how they are fit into arithmetic and geometry. In the subsequent two chapters we will develop the basic algebra of convolution operators on automorphic representations that are used to study them. It is a refinement of the algebra $L^1(G(\mathbb{A}_F))$ discussed in §3.3 above. For $F$ a number field and the decomposition $G(F_\infty) \times G(\mathbb{A}_F^\infty)$ naturally gives rise to a decomposition of $L^1(G(\mathbb{A}_F))$; the first factor is the archimedean factor which will be discussed in Chapter 4 and the second factor is the nonarchimedean factor which will be discussed in chapters 5 and 8.

This gives us tools to study automorphic representations, but it still does not shed much light into how they appear in geometry and arithmetic. The link with geometry will be made precise in Chapter 6. The primary link with arithmetic is much more profound, subtle, and conjectural; it is the content of the Langlands functoriality conjectures, explained in Chapter 12.

3.6 Decomposition of representations

If $V$ is a finite dimensional complex representation of a finite group $G$, then it is completely reducible, i.e. admits a direct sum decomposition

$$V = \bigoplus_i V_i$$
where the $V_i$ are irreducible representations of $G$ (see Exercise 3.12). Thus a subquotient of $V$ in this case is just a subrepresentation. When $G$ is no longer a finite group, these sorts of decompositions may no longer hold (see Exercise 3.13).

However, there is a very general theory which guarantees some sort of decomposition exists if we restrict our attention to unitary representations. Let $G$ be a locally compact (Hausdorff) second-countable group (for example, $G(\mathbb{A}_F)$ or $A_G \backslash G(\mathbb{A}_F)$ for some affine algebraic group $G$ over a global field $F$). Recall that a Borel space (a.k.a. measurable space) is called standard if it is isomorphic in the category of measurable spaces to $\mathbb{R}$ (with the standard Lebesgue measure), $\mathbb{Z}$, or a finite set (with the counting measure). For any unitary representation $\pi : G \to \text{GL}(V)$ there is a standard Borel space $X$ with a measure $\mu$ and a measurable field of unitary representations $(\pi_x, V_x)_{x \in X}$ equipped with a $G$-equivariant Hilbert space isomorphism

$$V \to \int_X V_x d\mu(x).$$

(3.11)

This is called a direct integral decomposition. We will not make these notions precise here; a nice introduction to these concepts (but without proofs) is given in [Fol95, §7], for proofs one can consult [Dix77]. Two motivating examples to keep in mind are the right regular representation of $\mathbb{R}$ on $L^2(\mathbb{R})$ and the representation $L^2(\mathbb{Z} \backslash \mathbb{R})$ induced by it. Fourier analysis tells us that

$$L^2(\mathbb{Z} \backslash \mathbb{R}) = \bigoplus_{n \in \mathbb{Z}} V_n \quad \text{and} \quad L^2(\mathbb{R}) = \int_{t \in \mathbb{R}} V_t dt.$$  

(3.12)

Here $V_t$ is the one-dimensional $\mathbb{C}$-vector space on which $\mathbb{R}$ acts via $t \mapsto e^{2\pi i nt}$. The decomposition of $L^2(\mathbb{Z} \backslash \mathbb{R})$ is said to be discrete and the decomposition of $L^2(\mathbb{R})$ is said to be continuous; this will be discussed in greater generality in §3.8 below.

Decompositions like the decomposition of $L^2(\mathbb{R})$ abound in the theory of automorphic forms (compare §??). This allows us to explain why one takes an automorphic representation of $G(\mathbb{A}_F)^1$ to be a subquotient, not just a subrepresentation, of $L^2(\mathbb{G})$. Indeed, one would like to consider all of the $V_t$ occurring in (3.12) as pieces of the representation on $L^2(\mathbb{R})$, but they are not subrepresentations. They are only subquotients:

**Lemma 3.6.1** For every $t \in \mathbb{R}$ there is a continuous linear $\mathbb{R}$-intertwining map

$$L^2(\mathbb{R}) \to V_t.$$

There is no $\mathbb{R}$-intertwining continuous linear maps

$$V_t \to L^2(\mathbb{R}).$$

**Proof.** For each $t$ one has a continuous $\mathbb{R}$-intertwining linear map
\[ L^2(\mathbb{R}) \rightarrow V_t \]
\[ f \mapsto \int_{\mathbb{R}} f(x) e^{-2\pi itx} \, dx \]

by Fourier theory.

For the converse statement, note that a linear \( \mathbb{R} \)-intertwining map

\[ V_t \rightarrow L^2(\mathbb{R}) \]

is either zero or has image contained in the \( \mathbb{C} \)-span of \( x \mapsto e^{2\pi itx} \). On the other hand \( x \mapsto e^{2\pi itx} \) is not \( L^2(\mathbb{R}) \), so we deduce that any linear \( \mathbb{R} \)-intertwining map \( V_t \rightarrow L^2(\mathbb{R}) \) is zero.

\[ \square \]

Unfortunately the direct integral decomposition (3.11) is not unique and is fairly badly behaved for general locally compact topological groups. If one restricts to the class of groups that are of \textbf{type I} then the situation is much more pleasant. We will discuss this in \S3.8.

### 3.7 The Fell topology

To describe the refined decomposition that exists for type I groups \( G \) it is useful to define the \textbf{unitary dual} \( \hat{G} \) of \( G \). Our basic reference is [Fol95, \S7.2]. As a set, \( \hat{G} \) is the set of unitary equivalence classes of unitary representations. The unitary dual is \( \hat{G} \) equipped with the \textbf{Fell topology}. The procedure for defining this topology is very similar to the procedure for defining the Zariski topology as in \S1.2. To define this topology, we recall from \S3.3 every representation \((\pi, V)\) of \( G \) gives rise to an algebra homomorphism \( C_*(G) \rightarrow \text{End}(V) \) which extends to \( L^1(G) \) if \( \pi \) is unitary. We can extend this homomorphism still further by completing. If we define

\[ \|f\|_* = \sup_{[\pi] \in \hat{G}} \|\pi(f)\|_1 \]

then \( \|f\|_* \leq \|f\|_1 \) and \( \|\cdot\|_* \) is a seminorm (here \([\pi]\) is the equivalence class of \( \pi \)). We define \( C^*(G) \) to be the completion of \( L^1(G) \) with respect to \( \|\cdot\|_* \). The convolution operation extends by continuity to \( C^*(G) \), and endows it with the structure of an algebra, called the \textbf{\( C^* \)-algebra of the group} \( G \). Thus for each unitary representation \((\pi, V)\) of \( G \) we obtain a representation of algebras

\[ C^*(G) \rightarrow \text{End}(V). \]

This is the motivating example of a \( C^* \)-algebra, but we will not give the (elementary) definition of a \( C^* \)-algebra, referring instead to [Fol95, \S1.1]. Now given a unitary representation \( \pi \) of \( G \), we obtain a closed two-sided ideal

\[ \ker(\pi) = \{ f \in C^*(G) : \pi(f) = 0 \}. \]
If $\pi$ is irreducible, this ideal is called a primitive ideal of $C^*(G)$. In fact it depends only on the equivalence class of $\pi$ in the category of unitary representations. The set of all primitive ideals in $C^*(G)$ is called $\text{Prim}(G)$:

$$\text{Prim}(G) := \{ \ker(\pi) : [\pi] \in \hat{G} \}.$$ 

If we think of this as the analogue of the prime spectrum of a ring, then proceeding as follows is reasonable: For any subset $S \subset \text{Prim}(G)$ we let

$$V(S) := \left\{ p \in \text{Prim}(G) : p \supset \bigcap_{q \in S} q \right\}.$$ 

By convention $V(\emptyset) = \emptyset$. Then one checks that the $V(S)$ just defined form the closed sets of a topology on $\text{Prim}(G)$, called the hull-kernel or Jacobson topology. Now we have a natural map

$$\hat{G} \rightarrow \text{Prim}(G)$$

$$[\pi] \mapsto \ker(\pi)$$

and we can pull back the topology on $\text{Prim}(G)$ to obtain a topology on $\hat{G}$. In other words the open sets in $\hat{G}$ are just the inverse images of open sets in $\text{Prim}(G)$. This is the Fell topology. Since the Fell topology is a topology the set of Borel sets on $\hat{G}$ is defined; this give $\hat{G}$ the structure of a Borel space. When speaking of measures and measurable functions on $\hat{G}$ we will always mean measures and measurable functions with respect to this Borel structure (i.e. set of measurable sets). It is useful to point out that if $G$ is of type I then as a Borel space it is standard [Fol95, Theorem 7.6].

### 3.8 Type I groups

We have not (and will not) given the definition of a type I group. However, we can give a characterization using the following theorem [Fol95, Theorem 7.6]:

**Theorem 3.8.1** If $G$ is a second countable locally compact group then the following are equivalent:

(a) The group $G$ is of type I.

(b) The Fell topology on $\hat{G}$ is $T_0$.

(c) The map $\hat{G} \rightarrow \text{Prim}(G)$ is injective.

We point out that the groups of primary concern to us in this book are fortunately of type I.
**Theorem 3.8.2** If $G$ is an algebraic group over a local field $F$ then $G(F)$ is of type I.

*Proof.* For the archimedean case see [Dix77, §13.11.12]. For the nonarchimedean case see [Sli84]. □

One can find a proof of the following in [Clo07, Appendix]:

**Theorem 3.8.3** If $G$ is a reductive group over a global field $F$ then $G(\mathbb{A}_F)$ is of type I.

It should be pointed out that if $G$ is not reductive then $G(\mathbb{A}_F)$ need not be of type I [Moo65].

For $\pi \in \hat{G}$ let $V_\pi$ be the space of (a particular realization of) $\pi$. The basic refinement of (3.11) is the following (see [Clo07, Theorem 3.3]):

**Theorem 3.8.4** Let $(\rho, V)$ be a unitary representation of $G$. Then there exists a Borel measurable multiplicity function

$$\hat{G} \rightarrow \{1, 2, \ldots, \infty\}$$

$$\pi \mapsto m(\pi)$$

and a positive measure $\mu$ on $\hat{G}$ such that

$$(\rho, V) = \int_{\hat{G}} \left( \bigoplus_{\pi} m(\pi) V_\pi \right) d\mu(\pi).$$

The decomposition is unique up to changing the measures and multiplicity functions on sets of measure zero.

One application of this theorem is that it allows us to define the discrete spectrum of a representation $V$. Since $\mu$ is a measure on a standard Borel space Lebesgue’s decomposition theorem implies that there is a unique decomposition

$$\mu = \mu_{\text{pt}} + \mu_{\text{cont}}$$

where $\mu_{\text{pt}}$ is zero outside a countable set and $\mu_{\text{cont}}$ is zero on every countable set. The **discrete spectrum** of $V$ is then

$$V_{\text{disc}} := \int_{\hat{G}} \left( \bigoplus_{\pi} m(\pi) V_\pi \right) \mu_{\text{pt}}(\pi)$$

(3.13)

and the **continuous spectrum** is

$$V_{\text{cont}} := \int_{\hat{G}} \left( \bigoplus_{\pi} m(\pi) V_\pi \right) \mu_{\text{cont}}(\pi).$$

(3.14)
We note that since the measure in (3.13) is supported in a countable set this direct integral is really a direct sum.

Of course in practice it would be better to have an explicit description of the decomposition of Theorem 3.8.4 for naturally occurring representations \( V \). Langlands’ profound and foundational work on Eisenstein series provide such a decomposition in our primary case of interest, namely when \( V = L^2([G]) \). We will return to this point in §10.2 below.

### 3.9 Why affine groups?

In the preceding discussion we have always assumed that \( G \) is an affine algebraic group. There are two possible avenues for generalization. First, we could look at covering groups of \( G(A_F) \) that are not algebraic. There is a great deal of beautiful theory that has developed around this topic and it is starting to become systematic. However, due to the authors’ lack of experience with the theory and additional complications that come up we have decided not to treat this topic.

If we forgo covering groups we could still generalize from affine groups to algebraic groups that are not necessarily affine. To get some idea of how this enlarges the set of groups in question we recall that a general algebraic group \( G \) over a number field \( F \) (not necessarily affine) has a unique linear algebraic subgroup \( G^\text{aff} \subseteq G \) such that \( G^A := G/G^\text{aff} \) is an abelian variety by a famous theorem of Chevalley [Con02].

Now consider an abelian variety \( A \). In [Con12] one finds a proof of a result of Weil stating that \( A(A_F) \) can still be defined as a topological space. In fact it is compact [Con12, Theorem 4.4]. Moreover \( A(F) \leq A(A_F) \) is still a discrete subgroup. However, it fails to be closed as soon as it is infinite [Con12, Theorem 4.4] because \( A \) is proper over \( F \). Thus the quotient \( A(F) \backslash A(A_F) \) is not even Hausdorff. Thus \( A(A_F) \) and the adelic quotient \( A(F) \backslash A(A_F) \) are very different objects than in the affine case, and we do not know whether or not it makes sense to pursue automorphic forms in this setting.

### Exercises

3.1. In the situation of the proof of Theorem 3.1.1 prove that if \( \varphi \in C_c(X) \) then \( \|\pi(g)\varphi - \varphi\|_\infty \to 0 \) as \( g \to 1 \).

3.2. Prove that compact (Hausdorff) topological groups are unimodular (hint: average a left Haar measure).

3.3. Show that for a locally compact topological group \( G \) the following are equivalent:
\[ d_r(hg) = \delta(h) d_r g, \]
\[ d_r(gh) = \delta_G(h)^{-1} d_r g, \]
\[ d_r(g^{-1}) = \delta_G(g)^{-1} d_r g, \]
\[ d_r(g^{-1}) = \delta_G(g) d_r g. \]

Show in addition that
\[ d_r g = \delta(g) d_r g. \]

**3.4.** Let \( G \) be a reductive algebraic group over a global field \( F \), let \( P \leq G \) be a parabolic subgroup, and let \( K \leq G(\mathbb{A}_F) \) be a maximal compact subgroup such that the Iwasawa decomposition
\[ G(\mathbb{A}_F) = P(\mathbb{A}_F) K \]
holds (see Theorem A.1.1). Show that the Haar measures on \( G(\mathbb{A}_F) \), \( P(\mathbb{A}_F) \), and \( K \) can be normalized so that
\[ dg = d\ell p dk. \]

**3.5.** Show that if \( \Omega \subset G(\mathbb{A}_F) \) is a compact set, then there is a finite set of places \( S \) of \( F \) depending on \( \Omega \) such that the projection of \( \Omega \) to \( G(\mathbb{A}_S) \) is contained in a maximal compact open subgroup \( K^S \leq G(\mathbb{A}_S) \). Deduce that the products (3.9) are left and right Haar measures on \( G(\mathbb{A}_F) \), respectively.

**3.6.** Let \( F \) be a nonarchimedian local field. Let \( dx \) be a Haar measure on \( F \), normalized so that \( dx(O_F) = 1 \), and let \( dx^\times \) be a Haar measure on \( F^\times \), normalized so \( dx^\times(O_F^\times) = 1 \). Show that
\[ \frac{1}{1 - q} dx(O_F^\times) = dx^\times(O_F^\times) \]
where \( q \) is the order of the residue field of \( O_F \).

**3.7.** Let \( F \) be a nonarchimedian local field and let \( dg \) be the Haar measure on \( GL_2(F) \) such that \( dg(GL_2(O_F)) = 1 \). For \( n \geq 0 \) compute
\[ dg(\{ g \in gl_2(O_F) \cap GL_2(F) : |\det g| = q^{-n} \}). \]

**3.8.** Suppose that \( G \) is a connected reductive group over a global field \( F \) and that \( A_G \neq 1 \). Prove that \( G(F)\backslash G(\mathbb{A}_F) \) has infinite volume with respect to the measure induced by a Haar measure on \( G(\mathbb{A}_F) \).

**3.9.** Suppose that \( G \) is a connected reductive group over a global field \( F \). Show that \([ G \] has finite volume using reduction theory (Theorem 2.7.2).

**3.10.** Prove that
3.9 Why affine groups?

\[ d(x_{ij}) = \frac{\wedge_{i,j} dx_{ij}}{|\det x_{ij}|} \]

is a (right and left) Haar measure on

\[ \text{GL}_n(R) := \{(x_{ij}) \in \text{gl}_n(R) : \det(x_{ij}) \in R^\times \} \]

if \( R \) is either a local field or the adeles of a global field and \(|\cdot|\) is the norm on the local field or the adelic norm, respectively.

3.11. Let \( F \) be a global field and let \( R \) be an \( F \)-algebra. For \((x_{ij}) \in \text{GL}_2(R)\) let

\[ \omega = \frac{dx_{11} \wedge dx_{21} \wedge dx_{12} \wedge dx_{22}}{x_{11}x_{22} - x_{21}x_{12}} \]

viewed as a left-invariant top-dimensional differential form on \( G \). For a nonarchimedean place \( v \) of \( F \), compute

\[ d|\omega|_v(\text{GL}_2(O_{F_v})) \]

Prove that

\[ \prod_v d|\omega|_v(\text{GL}_2(O_{F_v})) \]

does not converge, but that

\[ \prod_v \frac{d|\omega|_v(\text{GL}_2(O_{F_v}))}{\zeta_F(1)} \]

does converge.

3.12. Prove that any finite dimensional representation \( V \) of a finite group \( G \) is completely reducible, i.e. it decomposes into a finite direct sum of irreducible representations (Hint: prove first that \( V \) admits a \( G \)-invariant inner product).

3.13. Let \( B \leq \text{GL}_2 \) be the Borel subgroup of upper triangular matrices. Prove that the natural representation

\[ B(\mathbb{C}) \rightarrow \text{GL}_2(\mathbb{C}) \]

is not completely reducible; that is, it does not decompose into a direct sum of irreducible subrepresentations.

3.14. Let \( G \) be a locally compact group. Prove that the convolution product on \( C_c(G) \) is associative.

3.15. If \( \pi : G \rightarrow \text{GL}(V) \) is a unitary representation of \( G \) then show that \( \pi(f) \in \text{End}(V) \) for all \( f \in L^1(G) \).

3.16. Let \( G \) be a locally compact group and let \( f, g \in C_c(G) \). Prove that for any representation \( \pi \) of \( G \) one has \( \pi(f * g) = \pi(f) * \pi(g) \).
References


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