Chapter 2
Adeles

Adeles make life possible.

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Abstract Automorphic representations are defined as certain representations of the adelic points of reductive algebraic groups. In this chapter, we introduce the adele ring and recall the basic properties of the adelic points of reductive groups. They play a central role in the arithmetic theory of algebraic groups and the class field theory.

2.1 Adeles

The arithmetic objects of interest in this book are constructed using global fields and their adele rings. We recall the construction of the adeles in this section; references include [CF86], [Neu99], [RV99].

Definition 2.1.1 A global field \( F \) is a field which is a finite extension of \( \mathbb{Q} \) or of \( \mathbb{F}_p(t) \) for some prime \( p \). Global fields over \( \mathbb{Q} \) are called number fields while global fields over \( \mathbb{F}_p(t) \) are called function fields.

To each global field \( F \) one can associate an adele ring \( \mathbb{A}_F \). Before defining this ring, we recall the related notions of a valuation (or a finite place) of a global field.

Definition 2.1.2 Let \( F \) be a field. A (non-archimedean) valuation on \( F \) is a map

\[ v : F \to \mathbb{R} \cup \{\infty\} \]

such that for all \( a, b \in F \), \( v \) satisfies the following:

(a) \( v(a) = \infty \) if and only if \( a = 0 \).
(b) \( v(a) + v(b) = v(ab) \).
These axioms are designed so that if one picks $0 < \alpha < 1$ then
\[ | \cdot |_{\alpha} : F \rightarrow \mathbb{R}_{\geq 0} \]
\[ x \mapsto \alpha^{v(x)} \]
is a nonarchimedean absolute value on $F$ in the sense of the following definition:

**Definition 2.1.3** Let $F$ be a field. An **absolute value**
\[ | \cdot | : F \rightarrow \mathbb{R}_{\geq 0} \]
is a function satisfying the following axioms:

(a) $|a|_v = 0$ if and only if $a = 0$.
(b) $|ab|_v = |a|_v |b|_v$.
(c) $|a + b|_v \leq |a|_v + |b|_v$.

It is **nonarchimedean** if it satisfies the following strengthening of (c):

(c') $|a + b|_v \leq \max(|a|_v, |b|_v)$.

If $| \cdot |_v$ satisfies (c) but not (c') we say $| \cdot |_v$ is **archimedean**.

We point out that (c') implies (c), but not conversely. We often implicitly exclude the **trivial absolute value** given by $|a|_v = 1$ for all $a \in F^\times$.

An absolute value $| \cdot |_v$ induces a metric on $F$ known as the $v$-adic metric. The completion of $F$ with respect to this metric is denoted $F_v$. This completion is a **local field**, that is, a field equipped with an absolute value that is locally compact with respect to the induced metric. All infinite locally compact fields arise as the completion of some global field [Lor08, §25, Theorem 2].

**Definition 2.1.4** A **place** of a global field $F$ is an equivalence class of absolute values, where two absolute values are said to be equivalent if they induce the same topology on $F$. A place is (non)archimedean if it consists of (non)archimedean absolute values.

We now describe these places and fix representative absolute values in each place. These representative absolute values are said to be the **normalized absolute values**. The places of a global field $F$ fall into two categories: the finite and infinite places.

The finite places are in bijection with the prime ideals of $\mathcal{O}_F$. The place $v$ associated to a prime $p$ of $\mathcal{O}_F$ is the equivalence class of an absolute value attached to the valuation
\[ v(x) := \max\{k \in \mathbb{Z} : x \in p^k \mathcal{O}_F\} . \]
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The normalized absolute value in this equivalence class is

$$|x|_v = q_v^{-v(x)}$$

where $q_v := |O_F/p|$. These valuations are all nonarchimedean.

The infinite places of a number field are all archimedean. They are indexed by embeddings $\tau : F \hookrightarrow \mathbb{C}$ up to conjugation; the associated normalized absolute value is

$$|x|_v := \begin{cases} |\tau(x)| & \text{if } \tau(F) \leq \mathbb{R} \\ \tau(x)\tau(x) & \text{if } \tau(F) \nleq \mathbb{R}. \end{cases}$$

Here on the right the $|\cdot|$ denotes the usual absolute value on $\mathbb{R}$ and the bar denotes complex conjugation. Notice that in the complex case (i.e. where $\tau(F) \nleq \mathbb{R}$) this is the square of the usual absolute value.

The infinite places of a function field are those attached to extensions of the absolute value $\|\cdot\|_\infty$ on $F_p(t)$. If $F_v$ is the completion of $F$ with respect to such a valuation, the associated normalized absolute value is

$$|x|_v := |N_{F_v/F_p(t)}((x))|_\infty^{1/[F_v:F_p(t)]}.$$ 

These valuations are archimedean, in contrast to the archimedean case. Henceforth we will always take $|\cdot|_v$ to be the unique normalized absolute value attached to $v$.

We note that the set of infinite places of $F$ is often denoted by $\infty$, and one often writes $v|\infty$ or $v \nmid \infty$ as shorthand for “$v$ is an infinite place of $F$” and “$v$ is a finite place of $F$,” respectively.

For any place $v$, write $F_v$ for the completion of $F$ with respect to some choice of absolute value associated to $v$. Since the absolute values corresponding to $v$ all induce the same topology on $F$, the field $F_v$ is independent of this choice. If $v$ is finite, then the ring of integers of $F_v$ is

$$O_{F_v} = \{ x \in F_v : |x|_v \leq 1 \};$$

it is a local ring with a unique maximal ideal

$$\varpi_v O_{F_v} := \{ x \in F_v : |x|_v < 1 \}.$$ 

Here $\varpi_v$ is a uniformizer for $F_v$, that is, a generator for the maximal ideal of $O_{F_v}$.

Let us make these constructions explicit when $F = \mathbb{Q}$. If $p \in \mathbb{Z}$ is a prime, then completing $\mathbb{Q}$ at the $p$-adic absolute value gives rise to the local field $\mathbb{Q}_p$. Its ring of integers is $\mathbb{Z}_p$ and the maximal ideal is $p\mathbb{Z}_p$. The residue field is $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z} \cong \mathbb{F}_p$, so the normalized absolute is just the usual $p$-adic
norm. There is only one infinite place of \( \mathbb{Q} \), denoted \( \infty \), and \( \mathbb{Q}_\infty \cong \mathbb{R} \). The normalized archimedean norm \( |\cdot|_\infty \) is the usual Euclidean norm on \( \mathbb{R} \).

The normalization allows one to prove that the following **product formula** holds:

**Proposition 2.1.1** For \( x \in F^\times \) one has
\[
\prod_v |x|_v = 1
\]
where the product is over all places \( v \) of \( F \).

**Definition 2.1.5** Let \( F \) be a global field. The ring of **adeles** of \( F \), denoted by \( \mathbb{A}_F \), is the restricted direct product of the completions \( F_v \) with respect to the rings of integers \( \mathcal{O}_{F_v} \). In other words,
\[
\mathbb{A}_F = \left\{ (x_v) \in \prod_v F_v : x_v \in \mathcal{O}_{F_v} \text{ for all but finitely many places } v \right\}, \quad (2.2)
\]
The restricted product is usually denoted by a prime:
\[
\mathbb{A}_F = \prod_v^\prime F_v.
\]
Note that \( \mathbb{A}_F \) is a subring of the full product \( \prod_v F_v \). If \( S \) is a finite set of places of \( F \) then we write
\[
\mathbb{A}_F^S = \prod_{v \notin S}^\prime F_v
\]
\[
:= \left\{ (x_v) \in \prod_{v \notin S} F_v : x_v \in \mathcal{O}_{F_v} \text{ for all but finitely many places } v \right\},
\]
\[
F_S := \mathbb{A}_{F,S} = \prod_{v \in S} F_v.
\]
Thus we may identify \( F_S \times \mathbb{A}_F^S = \mathbb{A}_F \).

We endow \( \mathbb{A}_F \) with the **restricted product topology**. This is defined by stipulating that open sets are sets of the form
\[
U_S \times \prod_{v \notin S} \mathcal{O}_{F_v}
\]
where \( S \) is a finite set of places of \( F \) including the infinite places and \( U_S \subseteq F_S \) is an open set. We give \( \mathbb{A}_F^S = 0 \times \mathbb{A}_F^S \subset \mathbb{A}_F \) the subspace topology and \( F_S = \prod_{v \in S} F_v \) the product topology. Then the identification \( F_S \times \mathbb{A}_F^S = \mathbb{A}_F \) becomes an isomorphism of topological rings.
The topology on $\mathbb{A}_F$ is not the same as the topology induced on $\mathbb{A}_F$ by regarding it as a subset of the direct product $\prod_v F_v$. While $\prod_v F_v$ is not locally compact, for $\mathbb{A}_F$ one has the following:

**Proposition 2.1.2** The adele ring $\mathbb{A}_F$ of a global field $F$ is a locally compact Hausdorff topological ring.

*Proof.* We prove that $\mathbb{A}_F$ is locally compact and leave the other details to the reader. For any finite set $S$ of places $F$ including the infinite places, the subset

$$\prod_{v \in S} F_v \times \prod_{v \not\in S} O_{F_v}$$

(2.3)

is an open subring of $\mathbb{A}_F$ for which the induced topology coincides with the product topology. In particular (2.3) is locally compact by Tychonoff’s theorem. Every $x \in \mathbb{A}_F$ is contained in a set of the form (2.3), which shows that $\mathbb{A}_F$ is locally compact. \[\Box\]

There is a natural diagonal embedding $F \hookrightarrow \mathbb{A}_F$ given by sending an element of $F$ to all of its completions.

**Lemma 2.1.1** The subspace topology on $F$ arising from the diagonal embedding $F \hookrightarrow \mathbb{A}_F$ is the discrete topology.

*Proof.* Choose $x \in F^\times$. We will construct a neighborhood of $x$ in $\mathbb{A}_F$ containing no other element of $F$. Since $\mathbb{A}_F$ is a topological group under addition this will suffice to complete the proof.

For each finite place $v$ of $F$ let $n_v = v(x)$, so that $x \in \mathbb{A}_v^n$ but $x \not\in \mathbb{A}_v^{n_v+1}$ for all $v$. Note that $n_v = 0$ for all but finitely many places. For each infinite place $v$ let

$$U_v := \left\{ y \in F_v : |y - x|_v < \prod_{v \not\in \infty} |x|_v^{-\infty} \right\}$$

(here $|\infty|$ is the number of infinite places of $S$). Consider the open subset of $\mathbb{A}_F$ defined by

$$U = \prod_{v \not\in \infty} U_v \times \prod_{v \in \infty} \mathbb{A}_v^{n_v}.$$  

By construction, $x \in U$. Suppose that $y \in U$. Then $|x - y|_v \leq |x|_v$ for all finite places $v$. Thus

$$\prod_v |x - y|_v \leq \prod_{v \not\in \infty} |x|_v \times \prod_{v \in \infty} |x - y|_v < 1.$$  

since $|x - y|_v \leq |x|_v$ for all finite places $v$. By the product formula we conclude that $x = y$. \[\Box\]
We often identify $F$ with its image under the diagonal embedding. Given Lemma 2.1.1 the following theorem can be surprising the first time one sees it:

**Theorem 2.1.1 (Strong Approximation)** If $S$ is any finite nonempty set of places of $F$ then $F$ is dense in $A_S^S$.

Thus omitting one place is enough to move $F$ from being discrete to being dense. The proof can be found in any standard reference, see [Cas67, §15] for example.

We close this section by remarking that one can construct analogues of the finite adeles in more general situations, e.g. schemes of finite type over the ring of integers of a global field [Hub91]. The construction is quite a bit more involved than that given above.

### 2.2 Adelic points of affine schemes

Weil and Grothendieck both gave approaches (under different hypotheses) to topologizing the points of schemes of finite type over a topological ring (for example, $A_F$). Conrad gave a beautiful exposition/elaboration in [Con12b]. We lift Theorem 2.2.1 below and its proof from loc. cit. with little modification.

**Theorem 2.2.1** Let $R$ be a topological ring. There exists a unique way to topologize $X(R)$ for all affine schemes $X$ of finite type over $R$ such that

(a) the topology is functorial in $X$; that is if $X \to Y$ is a morphism of affine schemes of finite type over $R$, then the induced map on points $X(R) \to Y(R)$ is continuous;

(b) the topology is compatible with fibre products; this means that if $X \to Y$ and $Z \to Y$ are morphisms of affine schemes, all of finite type over $R$, then the topology on $X \times_Y Z(R)$ is exactly the fibre product topology;

(c) closed immersions of schemes $X \hookrightarrow Y$ induce topological embeddings $X(R) \hookrightarrow Y(R)$;

(d) if $X = \text{Spec} R[t]$ then $X(R)$ is homeomorphic with $R$ under the natural identification $X(R) \cong R$.

If $R$ is Hausdorff or locally compact, then so is $X(R)$. Moreover, if $R$ is Hausdorff then closed immersions induce closed embeddings, not just topological embeddings.

In the proof and throughout this book we take the convention that locally compact spaces are Hausdorff.

If the reader is uncomfortable with fiber products then they can omit the assertions in the theorem involving them and their proof. The only consequence of these facts we really need in the sequel is that with the definition
of the topology on $G^n_a(R)$ given above the natural bijection

$$G^n_a(R) \xrightarrow{\sim} \mathbb{R}^n$$

is a homeomorphism if we give the right hand side the product topology.

**Proof.** Let $X$ be an affine $R$-scheme. Pick an $R$-algebra isomorphism

$$A := \mathcal{O}(X) \cong R[t_1, \ldots, t_n]/I$$

for an ideal $I$, and identify $X(R)$ with the subset of $\mathbb{R}^n$ on which the elements of $I$ (thought of as polynomials on $\mathbb{R}^n$) vanish.

We start with uniqueness. By our assumption on compatibility with fiber products the natural bijection

$$\text{Spec } R[t_1, \ldots, t_n] \xrightarrow{\sim} \mathbb{R}^n$$

is a homeomorphism provided that we give the right hand side the product topology. By assumption, this induces a topological embedding $X(R) \hookrightarrow \mathbb{R}^n$.

We start with uniqueness. By our assumption on compatibility with fiber products the natural bijection

$$\text{Spec } R[t_1, \ldots, t_n] \xrightarrow{\sim} \mathbb{R}^n$$

is a homeomorphism provided that we give the right hand side the product topology. By assumption, this induces a topological embedding $X(R) \hookrightarrow \mathbb{R}^n$.

This completes the proof of uniqueness and also shows that $X(R)$ is Hausdorff if $R$ is Hausdorff. If $R$ is Hausdorff, then $0 \in R$ is closed, so viewing $X(R)$ as the vanishing locus of $f \in I$ (viewed as polynomials on $\mathbb{R}^n$) we see that $X(R)$ is closed. Thus if $R$ is locally compact $X(R)$ then is as well.

We now prove existence. Note that there is a canonical and tautological injection

$$X(R) = \text{Hom}(A, R) \rightarrow R^A.$$  \hfill (2.5)

We claim that the topology defined using (2.4) as above is the same as the subspace topology defined by the canonical injection $X(R) \hookrightarrow R^A$, so it is independent of the choice of (2.4). Let $a_1, \ldots, a_n \in A$ correspond to $t_1$ (mod $I$), $\ldots$, $t_n$ (mod $I$) via (2.4), so the injection $X(R) \hookrightarrow \mathbb{R}^n$ defined by (2.4) is the composition of the natural injection $X(R) \hookrightarrow R^A$ and the map $R^A \rightarrow \mathbb{R}^n$ given by projection to the factors indexed by $(a_1, \ldots, a_n)$. Therefore every open set in $X(R)$ is induced by an open set in $R^A$ because $R^A \rightarrow \mathbb{R}^n$ is continuous. Since every element of $A$ is an $R$-polynomial in $a_1, \ldots, a_n$ and $R$ is a topological ring, it follows that the map $X(R) \rightarrow R^A$ is also continuous. Thus $X(R)$ has been given the subspace topology from $R^A$. This completes the proof of the claim. It also implies that the formation of the topology on $X(R)$ is functorial (i.e. morphisms of affine schemes induce continuous maps on $R$-points).

Consider a closed immersion

$$i : X := \text{Spec } A \hookrightarrow X' := \text{Spec } A'$$

corresponding to a surjective $R$-algebra map $h : A' \rightarrow A$. The map
\( j : R^A \to R^{A'} \)
\[
(r_a) \mapsto (r_{h(a')})
\]
is visibly a topological embedding; it topologically identifies \( R^A \) with the subset of \( R^{A'} \) cut out by a collection of equalities among components. Moreover \( j \) is a closed embedding when \( R \) is Hausdorff. We have
\[
X'(R) \cap j(R^A) = j(X(R))
\]
because a set theoretic map \( A \to R \) is an \( R \)-algebra homomorphism if and only if its composition with \( h \) is an \( R \)-algebra map. Hence \( i : X(R) \to X'(R) \) is an embedding of topological spaces and it is a closed embedding when \( R \) is Hausdorff. By forming products of closed immersions into affine spaces \( \mathbb{G}_a^n \), we see that
\[
(X \times_{\text{Spec } R} X')(R) \to X(R) \times X'(R)
\]
is a topological isomorphism via reduction to the trivial special case when \( X \) and \( X' \) are affine spaces.

Finally, we prove that for given maps \( X \to Y \) and \( Z \to Y \) between affine \( R \)-schemes the bijection \( (X \times Y)Z(R) \to X(R) \times Y_{X(R)}Z(R) \) is a topological isomorphism. Consider the (tautological) isomorphism
\[
X \times Y \cong (X \times R Z) \times Y \times R Y
\]
and its topological counterpart. The product map
\[
\mathcal{O}(Y) \otimes_R \mathcal{O}(Y) \to \mathcal{O}(Y)
\]
is surjective and hence
\[
Y \to Y \times_R Y
\]
is a closed immersion (i.e. affine schemes are separated).

Since we have already checked compatibility with fiber products over \( \text{Spec } R \), we see that we are reduced to the case in which one of the map defining the fiber product is a closed immersion. We have already proven that closed immersions yield topological embeddings, so we deduce compatibility with fiber products. \( \square \)

### 2.3 Relationship with restricted direct products

In practice we will be interested in the topological space \( X(\mathbb{A}_F) \) when \( X \) is an algebraic group. To explain how this works, consider first the case where \( X = \text{GL}_n \). There is a closed immersion \( \text{GL}_n \hookrightarrow \mathfrak{g}l_n \times \mathbb{G}_a \) given on points in a \( \mathbb{Z} \)-algebra \( R \) by
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\[ \text{GL}_n(R) \rightarrow \mathfrak{gl}_n(R) \times \mathbb{G}_m(R) \quad (2.6) \]

\[ g \mapsto (g, \det g^{-1}). \]

Here \( \mathfrak{gl}_n(R) \) is the affine scheme whose points in a \( \mathbb{Z} \)-algebra are the \( R \)-linear endomorphisms of \( R^n \); it can be identified as a scheme with \( \mathbb{G}_m^n \). Thus for any topological ring \( R \) the group \( \text{GL}_n(R) \) is endowed with the subspace topology if we view it as a subspace of \( \mathfrak{gl}_n(R) \times \mathbb{G}_m(R) \) via (2.6). For example, if \( v \) is an archimedean place of \( F \), then by this recipe \( \text{GL}_n(F_v) \) acquires its usual topology.

To describe the topology for nonarchimedean \( v \), we recall that to describe a topology on a space it suffices to give a neighborhood base for every point \( x \), that is, a set of open neighborhoods \( x \in U_\alpha \) such that every neighborhood of \( x \) contains some \( U_\alpha \). In a topological group, it suffices to just give a neighborhood base of the identity, since we can take translates of this neighborhood base to be neighborhood bases at every other point.

Now if \( \varpi_v \) is a uniformizer for \( \mathcal{O}_{F_v} \) then

\[ \{(I_n + \varpi_v^k \mathfrak{gl}_n(\mathcal{O}_{F_v})) \times (1 + \varpi_v^k \mathcal{O}_{F_v}) : k \in \mathbb{Z}_{\geq 1}\} \]

forms a neighborhood base of the point \((I_n, 1) \in \mathfrak{gl}_n(F_v) \times F_v\). It follows that

\[ \{I_n + \varpi_v^k \mathfrak{gl}_n(\mathcal{O}_{F_v}) : k \in \mathbb{Z}_{\geq 1}\} \]

forms a neighborhood base of the identity in \( \text{GL}_n(F_v) \). Note that this is the same topology we would obtain if we just gave \( \text{GL}_n(F_v) \subset \mathfrak{gl}_n(F_v) \) the subspace topology.

On the other hand, if \( S \) is any finite set of places of \( F \) including the infinite places then

\[ \left\{(I_n + m \mathfrak{gl}_n(\mathcal{O}^S_F)) \times (1 + m \mathcal{O}^S_F) : m \subset \mathcal{O}^S_F \right\} \]

forms a neighborhood basis for \((I_n, 1) \in \mathfrak{gl}_n(K^S_F) \times K^S_F\). Here \( m \) runs over proper ideals of \( \mathcal{O}^S_F \), and

\[ m \mathfrak{gl}_n(\mathcal{O}^S_F) := \prod_{v \notin S} \varpi_v^{m(v)} \mathfrak{gl}_n(\mathcal{O}_{F_v}). \]

If we intersect one of these neighborhoods with the image of \( \text{GL}_n(K^S_F) \) under (2.6) then we obtain

\[ \prod_{v \notin S} (I_n + m \mathfrak{gl}_n(\mathcal{O}_{F_v})) \times \prod_{v \notin S} \text{GL}_n(\mathcal{O}_{F_v}). \]
Here $mgl_n(\mathcal{O}_F) = \omega_n^{v(m)} gl_n(\mathcal{O}_F)$. This is not the same as the topology obtained by giving $GL_n(\mathbb{A}_F^S) \subset gl_n(\mathbb{A}_F^S)$ the subset topology.

Finally, for any set of places $S$ of $F$ including the infinite places it is not hard to see by modifying this argument that one has a topological isomorphism of locally compact groups

$$GL_n(\mathbb{A}_F) = GL_n(F_S) \times GL_n(\mathbb{A}_F^S).$$

If $F$ is a local field and $G$ is an algebraic group over $F$ to define the topology on $G(F)$ we choose an embedding $G \hookrightarrow GL_n(F)$ and give $G(F)$ the subspace topology. Theorem 2.2.1 tells us that this topology is independent of the choice of embedding $G \hookrightarrow GL_n(F)$. If $F$ is global instead we similarly obtain the topology on $G(\mathbb{A}_F)$.

A helpful way to organize these comments is to introduce the notion of a restricted direct product of topological spaces (we already encountered this notion in a special case when we defined the topology on $\mathbb{A}_F$ in the previous section). Let $\{X_\alpha\}_{\alpha \in A}$ be a set of locally compact topological spaces indexed by a countable set $A$, and for all $\alpha$ outside a finite subset $S_0$ of $A$ let $K_\alpha \subseteq X_\alpha$ be a collection of compact open subsets of $X_\alpha$. Then the restricted direct product of the $X_\alpha$ with respect to the $K_\alpha$ is the set

$$\prod'_{\alpha \in A} X_\alpha := \left\{ (x_\alpha) \in \prod_{\alpha \in A} X_\alpha : x_\alpha \in K_\alpha \right\}$$

with topology given by declaring a subset of $X$ to be open if it is of the form

$$U \times \prod_{\alpha \in A \setminus S} K_\alpha$$

where $S$ is a finite subset of $A$ including $S_0$ and $U \subseteq X_S$ is an open subset. One verifies that this is indeed a topology and with respect to this topology $X$ is locally compact. Note that if the $X_\alpha$ are topological groups and the $K_\alpha$ are topological subgroups then $\prod'_{\alpha \in A} X_\alpha$ is again a topological group. We also note that the topology does not change if we replace $S_0$ by any larger finite subset of $A$.

Now assume that $G$ is an algebraic group over the global field $F$. Choose a faithful representation $G \hookrightarrow GL_n$.

Identify $G$ with its image in $GL_n$ and define, for all $v \uparrow \infty$,

$$K_v := G(F_v) \cap GL_n(\mathcal{O}_{F_v}).$$

Then $K_v$ is a compact open subgroup of $G(F_v)$. Our discussion of the adelic topology on $G(\mathbb{A}_F)$ above can be summarized as follows:

**Proposition 2.3.1** One has an isomorphism of topological groups

$$GL_n(\mathbb{A}_F) = GL_n(F_S) \times GL_n(\mathbb{A}_F^S).$$
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\[ G(\mathbb{A}_F) \cong \prod_v G(F_v) \]

where the restricted direct product is defined with respect to the subgroups \( K_v \).

In fact, in most references \( G(\mathbb{A}_F) \) is defined using Proposition 2.3.1. This has the advantage of being concrete, but it makes it awkward to rigorously prove that the topology satisfies good functorial properties.

2.4 Hyperspecial subgroups and models

Let \( G \) be an algebraic group over a nonarchimedean local field \( F \) and let \( G \to \text{GL}_n \) be a faithful representation. Identify \( G \) with its image in \( \text{GL}_n \). In the previous section we made use of the fact that \( K := G(F) \cap \text{GL}_n(\mathcal{O}_F) \) is a compact open subgroup of \( G(F) \). This construction is actually algebraic in nature in the sense that \( K \) is the \( \mathcal{O}_F \)-points of an affine group scheme over \( \mathcal{O}_F \). We explain this in more detail in this section and use it as motivation to introduce the notion of a hyperspecial subgroup. The reader should feel free to skip this section since the construction will only come up in passing later in the book. Our presentation in this section was influenced by [Yu].

We start with a Dedekind domain \( \mathcal{O} \) with fraction field \( F \). For example, \( \mathcal{O} \) could be the ring of \( S \)-integers \( \mathcal{O}^F \) of a number field \( F \) for a finite set of places \( S \) or the ring of integers of a local field. Here the ring of \( S \)-integers is simply the localization of the set of integers \( \mathcal{O}_F \) of \( F \) with respect to the set of primes in \( S \). Let \( Z \) be an affine scheme over \( \mathcal{O} \). The generic fiber of \( Z \) is \( Z_F \). If \( F \) is local, then \( \mathcal{O} \) has a unique prime ideal \( \wp \); in this case we let \( k = \mathcal{O}/\wp \) be the residue field. The scheme \( Z_k \) is known as the special fiber of \( Z \).

Suppose we are given an affine scheme \( Y \) over \( F \); often it is useful consider schemes \( Y \) that have \( Y \) as their generic fiber and satisfy certain desiderata:

**Definition 2.4.1** A model \( Y \) of \( Y \) over \( \mathcal{O} \) is an affine scheme of finite type over \( \mathcal{O} \) of the form \( \text{Spec} \ A \) where \( A \subseteq \mathcal{O}(Y) \) and \( A \otimes \mathcal{O} F = \mathcal{O}(Y) \).

**Remark 2.1.** The assertion that an affine \( \mathcal{O} \)-scheme \( \text{Spec} \ A \) is flat is (by definition) the assertion that \( A \) is flat as an \( \mathcal{O} \)-module, which is equivalent to the assertion that \( A \) is torsion-free since \( \mathcal{O} \) is a Dedekind ring. Thus we see that models are flat schemes. Suppose conversely we are given an affine scheme of finite type \( Y = \text{Spec} \ A \) over \( \mathcal{O} \) together with an isomorphism \( Y_F \cong Y \). Assume moreover that \( Y \) is flat, which is to say that \( A \) is flat as an \( \mathcal{O} \)-module. Thus the map \( A \to \mathcal{O}(Y) \) induced by the isomorphism \( Y_F \cong Y \) is injective and we obtain a model of \( Y \).
Sometimes there are obvious models for affine schemes. For example, we can view $\text{GL}_n$ as an affine group scheme over $O$, and it is clearly a model of its generic fiber $\text{GL}_n F$. However, there are many other models of $\text{GL}_n$ (see Exercise 2.8 for an example).

We now describe a particular type of model of a reductive group. Assume that $F$ is a local nonarchimedean field.

**Definition 2.4.2** A reductive group $G$ over $F$ is **unramified** if it is quasisplit and there is a finite degree unramified extension $E/F$ such that $G_E$ is split.

The following theorem is foundational for the theory of automorphic representations. For a proof see [Tit79] and the references therein.

**Theorem 2.4.1** Assume that $G$ is unramified. Then there exists a model $G$ of $G$ over $O_F$ such that the special fiber of $G$ is reductive. If $G$ is model of $G$ over $O_F$ such that the special fiber of $G$ is reductive then the subgroup $G(O_F) \leq G(F)$ is a maximal compact subgroup of $G(F)$. $\square$

**Definition 2.4.3** A subgroup of $G(F)$ of the form $G(O_F)$ for a model $G$ as in Theorem 2.4.1 is called a **hyperspecial subgroup**.

**Example 2.1.** When $G = \text{GL}_n$, it is clear that $\text{GL}_n(O_F)$ is a hyperspecial subgroup of $\text{GL}_n(F)$. It turns out that all maximal compact subgroups of $\text{GL}_n(F)$ are conjugate to $\text{GL}_n(O_F)$ [Ser06, Chapter IV, Appendix 1]. In loc. cit. one also finds a proof that $\text{GL}_n(O_F)$ is maximal in $\text{GL}_n(F)$.

We quickly note a few useful facts from [Tit79] about hyperspecial subgroups. First, a $G(F)$-conjugate of a hyperspecial subgroup is again hyperspecial. Second, though every compact subgroup of $G(F)$ is contained in a maximal one, the maximal compact subgroups are not $G(F)$ conjugate in general (in contrast to the case of $\text{GL}_n$).

Now suppose that we are given an unramified reductive group $G$. One might ask how one ought to go about finding a hyperspecial subgroup of $G$. One method proceeds via constructing schematic closures as we now explain. We again revert to the assumptions at the beginning of this section, so $O$ is a Dedekind domain with fraction field $F$. Let $Y$ be a scheme of finite type over $F$ and let $\mathcal{Y}$ be a model of $Y$. Suppose in addition we are given a closed immersion of $F$-schemes

$$X \longrightarrow Y; \quad (2.7)$$

we identify $X$ with a closed subscheme of $Y$. Let

$$A := \text{Im}(O(Y) \rightarrow O(Y) \rightarrow O(X)).$$

Since $O(Y) \rightarrow O(Y)$ is injective and (2.7) is a closed immersion we see that $X := \text{Spec} A$ comes equipped with a closed immersion $X \rightarrow \mathcal{Y}$ and $X_F = X$. We leave the proof of the following lemma as an exercise (see Exercise 2.4)
Lemma 2.4.1 The scheme $\mathcal{X}$ is a model of $X$ and

$$\mathcal{X}(\mathcal{O}) = X(F) \cap \mathcal{Y}(\mathcal{O}).$$

Remark 2.2. The scheme $\mathcal{X}$ earns its moniker as a schematic closure via a universal property, see Exercise 2.5.

If $\mathcal{Y} = \text{GL}_n$, viewed as an affine group scheme over $\mathcal{O}$, then the condition that $\mathcal{O}(\text{GL}_n) \to \mathcal{O}(\text{GL}_nF)$ is injective is clearly satisfied. Thus given any faithful representation $G \to \text{GL}_n$, we can form the schematic closure $\mathcal{G}$ of $G$; it is a scheme over $\mathcal{O}$ with generic fiber $G$. In the special case where $F$ is a local field and $\mathcal{O} = \mathcal{O}_F$ is its ring of integers we obtain

$$\mathcal{G}(\mathcal{O}_F) := G(F) \cap \text{GL}_n(\mathcal{O}_F),$$

which we denoted by $K$ earlier. Thus in our construction of the restricted direct topology from the previous section we implicitly used schematic closures.

Now suppose that $F$ is global and that we are given a faithful representation $G \to \text{GL}_n$ of a connected reductive group $G$. We can then take the schematic closure $\mathcal{G}$ of $G$ in $\text{GL}_n(\mathcal{O}_F)$. One has the following proposition:

Proposition 2.4.1 For almost all nonarchimedean places $v$ of $F$ the subgroup $G(\mathcal{O}_F)$ is hyperspecial.

Thus taking a “global” schematic closure gives us hyperspecial subgroups at almost all places. This proposition and variants of it are extremely useful in applications, so we indicate the proof after giving some preparation.

Recall that an $\mathcal{O}$-scheme $\mathcal{Y}$ is smooth if it is flat and of finite type and for all algebraically closed fields $\overline{k}$ that admit a nonzero $\mathcal{O}$-algebra morphism $\mathcal{O} \to \overline{k}$ the fiber $\mathcal{Y}(\overline{k})$ is smooth. The following lemma is a very special case of [GW10, Proposition 6.18]:

Lemma 2.4.2 Let $S_0$ be a finite set of finite places of $F$ (possibly empty), let $\mathcal{Y}$ be a smooth $\mathcal{O}_F^{S_0}$-scheme and let $X \leq \mathcal{Y}_F$ be a smooth subscheme of its generic fiber. Let $\mathcal{X}$ be the schematic closure of $X$ in $\mathcal{Y}$. Then there is a finite set $S$ of finite places containing $S_0$ such that $\mathcal{X}_{\mathcal{O}_F^S}$ is smooth over $\mathcal{O}_F^S$.  

We now indicate the proof of Proposition 2.4.1:

Proof of Proposition 2.4.1: The schematic closure $\mathcal{G}$ is again a group scheme over $\mathcal{O}_F$ (compare Exercise 2.6 below). It is also smooth over $\mathcal{O}_F^S$ for a sufficiently large finite set $S$ of finite places of $F$. The result now follows from [Con14, Proposition 3.1.9].
2.5 Approximation in algebraic groups

For a global field $F$ and a finite set $S$ of places of $F$, the image of $F$ under the diagonal embedding $F \to F_S$ is dense. This is fairly easy to prove and can be viewed as a kind of Chinese remainder theorem (compare Exercise 2.7). If $X$ is an affine scheme of finite type over $F$ and $S$ is a finite set of places of $F$ then one can ask if a similar phenomenon holds:

**Definition 2.5.1** Let $S$ be a finite set of places of $F$. The affine scheme $X$ satisfies **weak approximation with respect to $S$** if the image of the diagonal embedding $X(F) \to X(F_S)$ is dense. It satisfies **strong approximation with respect to $S$** if $X(F)$ is dense in $X(\AA_F^S)$.

Here when we speak of density we are of course using the canonical topologies on $X(F_S)$ and $X(\AA_F^S)$ afforded by Theorem 2.2.1.

Despite the relative ease of proving weak approximation when $X = \mathbb{G}_m$, establishing whether or not weak approximation holds for a general affine scheme is very difficult. In general it is false. We refer to [Har04] for a more detailed discussion.

Our goal in this section is to describe when weak and strong approximation hold in settings related to algebraic groups. To simplify the discussion we often assume that $F$ is a number field; additional complications come up in the general case. Our primary reference is [PR94, Chapter 7]. We start with the following proposition, the proof of which we leave as an exercise:

**Proposition 2.5.1** Let $G$ be a connected algebraic group over a number field $F$ with Levi decomposition $G = MN$. Then $G$ admits weak (resp. strong) approximation with respect to a finite set $S$ of places of $F$ if and only if $M$ does. \hfill $\Box$

Thus in the number field case studying strong and weak approximation of algebraic groups is equivalent to studying it for the smaller set of reductive groups. It turns out that under suitable restrictions on the set of places $S$ weak approximation always holds:

**Theorem 2.5.1** Let $G$ be a connected algebraic group over a number field $F$. There is a finite set $S_0$ of finite places of $F$ such that $G$ has weak approximation for any finite set of places of $F$ not containing $S_0$. \hfill $\Box$

We refer to [PR94, §7.3] for the proof.

If we wish to guarantee that we can take $S_0$ to be empty we need to assume additional arithmetic conditions on the group $G$. To state them, recall that a **central isogeny** of connected algebraic groups $G \to G'$ over a field $k$ is a surjective homomorphism whose kernel is finite and contained in the center of $G$. A semisimple group $G$ is **simply connected** if any central isogeny $G' \to G$ is an isomorphism, and **adjoint** if any central isogeny $G \to G'$ is an isomorphism.
If \( G \to G' \) is a central isogeny between two semisimple \( k \) groups then \( G \) and \( G' \) have the same root system, but not necessarily the same root datum. In general, simply connected groups have the largest center out of the set of semisimple groups with the same root system and adjoint groups have trivial center. For example, \( \text{SL}_n \) is simply connected and \( \text{PGL}_n \) is adjoint. The quotient map \( \text{SL}_n \to \text{PGL}_n \) is a central isogeny. See [Bor91], [Hum75], [Spr09].

For the following theorem we refer again to [PR94, §7.3]:

**Theorem 2.5.2** A simply connected or adjoint semisimple group \( G \) over a number field \( F \) admits weak approximation with respect to any set finite \( S \) of places \( F \).

We finish our discussion of weak approximation by considering the case where \( X \) is not an algebraic group, but a particular type of homogeneous space. Let \( G \) be a reductive group over a number field \( F \) and let \( H \leq G \) be a reductive subgroup. Then there is a natural action of \( G \) on

\[ X := G/H := \text{Spec}(O(G)^H). \]

If \( \mathcal{F} \) is an algebraic closure of \( F \) and \( F \leq L \leq \mathcal{F} \) is a subfield then

\[ X(L) = (G(\mathcal{F})/H(\mathcal{F}))^{\text{Gal}(\mathcal{F}/L)} \]

with the natural Galois action. For more details on algebraic group actions we refer to [MFK94] or ?? below.

To state weak approximation theorems in this context, we recall that an algebraic torus \( T \) is **quasi-trivial** if \( X^*(T) \) is a permutation \( \text{Gal}(\bar{F}/F) \)-module. A connected reductive group \( G \) is **quasi-trivial** if the torus \( G/G^\text{der} \) is quasi-trivial and \( G^\text{der} \) is simply connected. We record the following theorem of Borovoi [Bor09, 3.12], which generalizes Theorem 2.5.1:

**Theorem 2.5.3** Let \( G \) be a connected reductive quasi-trivial group and assume that \( H \) is connected. There is a finite set \( S_0 \) of finite places of the number field \( F \) such that \( G/H \) has weak approximation for any finite set of places of \( F \) not containing \( S_0 \).

In particular, \( G/H \) always has strong approximation with respect to \( \infty \).

Similarly we have a generalization of Theorem 2.5.2:

**Theorem 2.5.4** Assume that \( G \) is connected, reductive, and quasi-trivial and that \( H/H^\text{der} \) splits over a cyclic extension of \( F \). Then \( G/H \) satisfies weak approximation for any finite set \( S \) of places of \( F \).

This is [Bor09, Corollary 3.14]. We note that Borovoi actually works in a more general context where \( G \) is not necessarily reductive, but we have restricted to the situation above for simplicity.
We now turn to a discussion of strong approximation. We recall that a connected algebraic group over a global field $F$ is almost simple if Lie $G$ is a simple Lie algebra, that is, a Lie algebra with no proper ideals. The basic result on strong approximation is the following. For the proof see [Pra77]:

**Theorem 2.5.5** Let $G$ be a connected absolutely almost simple algebraic group over a global field $F$ and let $S$ be a finite set of places of $F$. If $G$ is simply connected and $G(F_S)$ is noncompact then $G$ satisfies strong approximation relative to $S$. \hfill $\square$

**Remark 2.3.** At least in the number field case there is a converse to this theorem, see [PR94, Theorem 7.12].

Here is an analogue for certain homogeneous spaces [Rap14, Proposition 2.4]:

**Theorem 2.5.6** Assume that $H \leq G$ are almost simple algebraic groups that are simply connected and semisimple. Then $G/H$ satisfies strong approximation relative to $S$ if and only if $G/H(F_S)$ is noncompact. \hfill $\square$

### 2.6 The adelic quotient

Let $G$ be an affine algebraic group over a global field $F$ (so we do not assume $G$ to be reductive or connected). Then the subgroup $G(F) \leq G(\mathbb{A}_F)$ is discrete (compare Exercise 2.2) and we can consider the quotient $G(F) \backslash G(\mathbb{A}_F)$. In this section we recall basic facts on the topology of $G(F) \backslash G(\mathbb{A}_F)$. More precise results will be recalled in the following section.

The first result we recall is the following (see [Con12a] for the proof):

**Theorem 2.6.1** (Borel, Conrad, Oesterlé, Prasad) For any finite set $S$ of places of $F$ containing the infinite places and for any compact open subgroup $K^S \leq G(\mathbb{A}_F^S)$ the quotient

$$G(F) \backslash G(\mathbb{A}_F^S)/K^S$$

is finite. \hfill $\square$

We note that one does not even have to assume that $G$ is smooth (although this is automatic in the characteristic zero case, see Theorem 1.5.2). Theorem 2.6.1 is known as **finiteness of class numbers** (for affine algebraic groups). Indeed, in the special case $G = \text{GL}_1$, $K^\infty := \hat{O}_F^\times$ the set above can be identified with the class group of $F$, and hence the theorem implies the finiteness of the class group.

It is not hard to see that in general the quotient $G(F) \backslash G(\mathbb{A}_F)$ itself is infinite. However, we could ask when the quotient is compact or finite volume with respect to a suitable measure (see §3.5), and there is a complete answer
to this question which we recall in Theorem 2.6.2 below. However, it is useful to first eliminate a trivial obstruction to the quotient $G(F) \backslash G(A_F)$ being finite volume or compact.

The units $F^\times$ of $F$ embed into $\mathbb{A}_F^\times$ diagonally as a discrete subspace. Since there the idelic norm $|\cdot|_F$ provides a continuous nontrivial homomorphism

$$|\cdot|_F := \prod_v |\cdot|_v : F^\times \backslash \mathbb{A}_F^\times \longrightarrow \mathbb{R}_{>0},$$

we conclude that $F^\times \backslash \mathbb{A}_F^\times$ is noncompact and actually it has infinite volume with respect to the Haar measure (for more on Haar measures see §3.2 below).

An analogous phenomenon occurs for other groups, and this motivates the following definition:

$$G(\mathbb{A}_F)^1 := \bigcap_{\chi \in X^*(G)} \ker(\cdot | \circ \chi : G(\mathbb{A}_F) \to \mathbb{R}_{>0}).$$

Here $|x| := |x|_F = \prod_v |x|_v$. Note that $G(F)$ is contained $G(\mathbb{A}_F)^1$ in virtue of the product formula Proposition 2.1.1. Moreover, $G(F)$ is discrete in $G(\mathbb{A}_F)^1$ (see Exercise 2.2).

In the special case where $G$ is reductive we now define a subgroup

$$A_G \leq G(F_\infty) \leq G(\mathbb{A}_F)$$

such that

$$A_G G(\mathbb{A}_F)^1 = G(\mathbb{A}_F),$$

the product being direct.

When $F$ is a number field we let $A_G$ be the identity component of the $\mathbb{R}$-points of the greatest $\mathbb{Q}$-split torus in $\text{Res}_{F/\mathbb{Q}} Z_G$. When $F$ is a function field of characteristic $p$ temporarily write $Z$ for the largest $\mathbb{F}_p((t))$-split torus in the center of $\text{Res}_{F/\mathbb{F}_p((t))} Z_G$.

Then if $Z$ has rank $d$ there is an isomorphism $Z(\mathbb{F}_p((t)) \cong (\mathbb{F}_p((t)))^d$. We let $A_G$ be the subgroup $(\mathbb{Z})^d$. We note that in either case the inclusion $G(\mathbb{A}_F)^1 \to G(\mathbb{A}_F)$ induces an isomorphism $G(\mathbb{A}_F)^1 \cong A_G \backslash G(\mathbb{A}_F)$. Thus in the reductive case it is largely a manner of taste as to whether one works with $G(\mathbb{A}_F)^1$ or $A_G G(\mathbb{A}_F)^1$.

The adelic quotient of $G$ is the quotient

$$[G] := G(F) \backslash G(\mathbb{A}_F)^1.$$

In the reductive case the isomorphism $G(\mathbb{A}_F)^1 \longrightarrow A_G \backslash G(\mathbb{A}_F)$ induces a canonical isomorphism

$$[G] \longrightarrow G(F) A_G \backslash G(\mathbb{A}_F)$$

(2.10)
that intertwines the action of $G(\mathbb{A}_F)^1$ on the left hand side and the action of $A_G \setminus G(\mathbb{A}_F)$ on the left. In the reductive case we will therefore allow ourselves to let the symbol $[G]$ denote either $A_G G(F) \setminus G(\mathbb{A}_F)$ or $G(F) \setminus G(\mathbb{A}_F)^1$. The basic topological or measure theoretic properties of the quotient are summarized in the following theorem:

**Theorem 2.6.2 (Borel, Conrad, Harder, Oesterlé)** The group $G(\mathbb{A}_F)^1$ is unimodular. The quotient $[G]$ defined in (2.9) has finite volume with respect to the measure induced by a Haar measure on $G(\mathbb{A}_F)$, and $[G]$ is compact if and only if for every $F$-split torus $T \leq G$ one has $T_{G} \leq R_u(G_{\mathbb{A}})$. □

See [Con12a] for a proof. The notion of a Haar measure and a unimodular group will be recalled in §3.2.

For comparison with the classical theory of automorphic forms it is convenient to relate the adelic quotient to locally symmetric spaces (compare Chapter 6). For this purpose, if $K \leq G(\mathbb{A}_F)$ is a compact open subgroup let

$$h := h(K^\infty) = |G(F) \setminus G(\mathbb{A}_F^\infty)/K^\infty|.$$  

By Theorem 2.6.1, $h < \infty$. Let $t_1, \ldots, t_h$ denote a set of representatives for $G(F) \setminus G(\mathbb{A}_F^\infty)/K^\infty$. We then have a homeomorphism

$$\prod_{i=1}^{h} \Gamma_i(K^\infty) A_G \setminus G(F_\infty) \longrightarrow G(F) A_G \setminus G(\mathbb{A}_F)/K^\infty \quad (2.11)$$

given on the $i$th component by

$$\Gamma_i(K^\infty) A_G g_\infty \longmapsto G(F) A_G g_\infty t_i \Gamma_i(K^\infty),$$

where

$$\Gamma_i(K^\infty) := G(F) \cap t_i \cdot A_G \setminus G(F_\infty) K^\infty \cdot t_i^{-1}.$$ 

Notice that the $\Gamma_i(K^\infty)$ are discrete subgroups of $G(F)$ and they are moreover arithmetic in the following sense:

**Definition 2.6.1** Let $G \leq GL_n$ be a linear algebraic group. A subgroup $\Gamma \leq G(F)$ is arithmetic if it is commensurable with $G(\mathcal{O}_F)$, where $\mathcal{G}$ is the schematic closure of $G$ in $GL_n/\mathcal{O}_F$.

The notion of arithmeticity does not depend on the choice of representation $G \leq GL_n$ (see Exercise 2.13). The subgroups of $G(F)$ that are intersections of a compact open subgroup of $G(\mathbb{A}_F^\infty)$ and $G(F)$ are known as congruence subgroups. Not every arithmetic group is a congruence subgroup in general, although for some groups $G$ this is the case. For more information the reader can consult the literature on the so-called congruence subgroup problem.

We end this section by considering the special case where $G = GL_2/\mathbb{Q}$. Then $K^\infty = GL_2(\hat{\mathbb{Z}})$ is a maximal compact open subgroup, where $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ is the profinite completion of $\mathbb{Z}$. Moreover, if we denote by
2.7 Reduction theory

\[ K_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}) : N|c \right\} \quad \text{and} \]

\[ \Gamma_0(N) := K_0(N) \cap \text{GL}_2(\mathbb{Z}) \]

then

\[ \Gamma_0(N)\backslash \text{GL}_2(\mathbb{R}) = \text{GL}_2(\mathbb{Q})\backslash \text{GL}_2(\mathbb{A}_\mathbb{Q})/K_0(N). \]

If we let \( K_\infty = \text{SO}_2(\mathbb{R}) \) then :

\[ \Gamma_0(N)\backslash (\mathbb{C} - \mathbb{R}) \rightarrow \text{GL}_2(\mathbb{Q})\backslash \text{GL}_2(\mathbb{A}_\mathbb{Q})/K_\infty K_0(N) = \text{GL}_2(\mathbb{Q})\backslash (\mathbb{C} - \mathbb{R}) \times \text{GL}_2(\mathbb{A}_\mathbb{Q}^\infty)/K_0(N). \]

where on the left \( \Gamma_0(N) \) acts via Möbius transformations:

\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az + b}{cz + d}. \]

2.7 Reduction theory

As discussed in the previous section, the adelic quotient \([G]\) is not compact as soon as the maximal reductive quotient of \( G \) has a parabolic subgroup (over the base field \( F \)). We now explain how to control this noncompact space.

To be more precise about this assume that \( G \) is a connected reductive group over a global field \( F \). In the number field case this assumption can be weakened to allow for arbitrary connected affine groups of finite type over \( F \) at the cost of introducing more notation [PR94, §4.3].

The following theorem provides what is known as the Iwasawa decomposition:

Theorem 2.7.1 Let \( G \) be a reductive group over a global field \( F \) and let \( P \leq G \) be a parabolic subgroup. There exists a maximal compact subgroup \( K \leq G(\mathbb{A}_F) \) such that

\[ G(\mathbb{A}_F) = P(\mathbb{A}_F)K. \quad (2.12) \]

We sketch the proof of this theorem in Appendix A.

We now assume that \( P \) is a minimal parabolic subgroup of \( G \) with unipotent radical \( N \). Fix a Levi subgroup \( M \leq P \). Let \( H \) be a maximal split torus of \( G^{\text{der}} \) contained in \( M \), then \( T := HZ_G \) is a maximal split torus of \( G \). As recalled in §1.9 the set of nonzero weights of \( T \) acting on \( \text{Lie } P \) is a base \( \Delta \subset \Phi(G, T) \) for the set of roots of \( T \) in \( G \).

A Siegel set in \( G(\mathbb{A}_F)^1 \) is a set of the form

\[ \mathfrak{S}(t) := \mathfrak{S}(t) = \Omega A_H(t)K, \quad (2.13) \]

where \( \Omega \subset N(\mathbb{A}_F) M(\mathbb{A}_F)^1 \) is a relatively compact subset, \( t \in \mathbb{R}_{>0} \), and
\[ A_H(t) := \{ x \in A_H : |\alpha(x)| \geq t \text{ for all } \alpha \in \Delta \}. \]

Here \(| \cdot |\) denotes the usual norm on \(\mathbb{R}\) when \(F\) is a number field and the canonical norm on \(\mathbb{F}_p((t))\) when \(F\) is a function field of characteristic \(p\).

The basic result of reduction theory is the following:

**Theorem 2.7.2 (Reduction theory)** There exists a Siegel set such that

\[ G(F)S(t) = G(\mathbb{A}_F). \]

**Proof.** See [Spr94]. To aid the reader we note that in loc. cit. the group \(G(F)\) acts on the right. Thus one has to take an inverse to move \(G(F)\) to the left, and this changes the inequality in the analogue of \(A_H(t)\) in loc. cit. to that we used in defining \(A_H(t)\). \(\square\)

To understand what is going on in Theorem 2.7.2 it is useful to consider what is arguably the simplest nontrivial case, that when \(G = \text{GL}_2\). In this case we can take \(K = O_2(\mathbb{R})\text{GL}_2(\hat{\mathbb{Z}})\) and let \(B = P\) be the Borel subgroup of upper triangular matrices (since it is a Borel subgroup it is a minimal parabolic subgroup). Then \(\Delta\) consists of the single root

\[ \alpha : T(\mathbb{R}) \rightarrow \mathbb{R}_+^\times \]

\[ (t_1 \ t_2) \mapsto t_1 t_2^{-1}. \]

In this setting reduction theory tells us that there is a compact subgroup \(\Omega \leq N(\mathbb{A}_F)\) such that

\[ \text{GL}_2(\mathbb{A}_F) = \Omega A_H(t)K. \]

for small enough \(t \in \mathbb{R}_{>0}\). This is the content of Exercise 2.15. We remark that it is this example that gives reduction theory its name. Indeed, a slight strengthening of Theorem 2.7.2 in this case yields the familiar fact from the reduction theory of positive definite binary quadratic forms over \(\mathbb{Q}\), namely that every form of a given discriminant is representable by a reduced form.

**Exercises**

2.1. Prove Proposition 2.1.1.

2.2. Let \(R \rightarrow R'\) be a continuous map of topological rings and let \(X\) be an affine scheme of finite type over \(R\). Show that \(X(R) \rightarrow X(R')\) is continuous. Show moreover that if \(R \rightarrow R'\) is a

(a) topological embedding
(b) open topological embedding
(c) closed topological embedding
(d) topological embedding onto a discrete subset
2.7 Reduction theory

then so is \( X(R) \rightarrow X(R') \).

2.3. Let \( G \) be an algebraic group over a local field \( F \), let \( \rho : G \rightarrow \text{GL}_n \) be a faithful representation, and let \( K \leq \text{GL}_n(F) \) be a compact open subgroup. Prove that \( \rho(G(F)) \cap K \) is a compact open subgroup of \( \rho(G(F)) \).

2.4. Prove Lemma 2.4.1.

2.5. Let \( \mathcal{O} \) be a Dedekind domain with fraction field \( F \) and let \( \mathcal{Y} \) be a flat \( \mathcal{O} \)-scheme. Suppose that \( X \rightarrow Y := Y_F \) is a closed immersion. Let \( \mathcal{X} \) be the schematic closure of \( X \) in \( \mathcal{Y} \). Then for any closed immersion \( Z \rightarrow \mathcal{Y} \) whose generic fiber is an isomorphism onto \( X \) there is a unique closed immersion \( \mathcal{X} \rightarrow Z \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{X} & \rightarrow & \mathcal{Y} \\
\downarrow & & \downarrow \\
Z & \rightarrow & \mathcal{Y}
\end{array}
\]

2.6. Let \( \mathcal{O} \) be a Dedekind domain with fraction field \( F \). Let \( \mathcal{G}/\mathcal{O} \) be a flat group scheme of finite type with generic fiber \( G := \mathcal{G}_F \). Let \( H \) be a group scheme over \( F \) equipped with a morphism of group schemes \( H \rightarrow G \) that is a closed immersion. Show that the schematic closure \( \mathcal{H} \) of \( H \) in \( \mathcal{G} \) is a group scheme.

2.7. Let \( S \) be a finite set of places of a global field \( F \). Prove that \( F \) is dense in \( F_S \).

2.8. Let \( m > 1 \) be an integer. Let

\[
A := \mathbb{Z}[x_{ij}, t_{ij}, y]_{1 \leq i,j \leq n}/\left( \det(x_{ij})y - 1, (x_{ij}) - 1 + m(t_{ij}) \right).
\]

Here we view \((x_{ij}), (t_{ij})\) as matrices in order to make sense out of the polynomial relations. Show that \( \mathcal{G} := \text{Spec}(A) \) is a model of \( \text{GL}_n \) over \( \mathbb{Z} \) and that

\[
\mathcal{G}(\mathbb{Z}) := \{ g \in \text{GL}_n(\mathbb{Z}) : g \equiv 1 \pmod{m\text{gl}_n(\mathbb{Z})} \}.
\]

2.9. Let \( X_1, X_2 \) be affine schemes of finite type over a global field \( F \). Show that \( X_1 \) and \( X_2 \) admit weak (resp. strong) approximation with respect to a finite set \( S \) of places \( F \) if and only if \( X_1 \times X_2 \) does. Deduce that if \( G \) is an algebraic group over a number field \( F \) then \( G \) is satisfies weak (resp. strong) approximation with respect to a finite set \( S \) of places \( F \) if and only if a Levi subgroup of \( G \) does.

2.10. Let \( F \) be a global field. Show that \( \text{GL}_n \) admits weak approximation with respect to any proper subset of the places of \( F \).
2.11. Let $F$ be a number field. Show that $\text{GL}_n$ does not admit strong approximation with respect to the set of infinite places of $F$ if the class number of $F$ is not 1.

2.12. Assume that the affine algebraic group $G$ over the global field $F$ satisfies strong approximation with respect to a finite set $S$ of places $F$. Show that

$$|G(F) \backslash G(\mathbb{A}_F^S)/K^S| = 1$$

for any compact open subgroup $K^S \leq G(\mathbb{A}_F^S)$.

2.13. Let $\rho_i : G \to \text{GL}_{n_i}, i \in \{1, 2\}$ be a pair of faithful representations of the affine algebraic group $G$ over the number field $F$. Let $\mathcal{G}_i$ be the schematic closure of $\rho_i(G)$ in $\text{GL}_{n_i}(\mathcal{O}_F)$. Show that a subgroup $\Gamma \leq G(F)$ is commensurable with $\mathcal{G}_1(\mathcal{O}_F)$ if and only if it is commensurable with $\mathcal{G}_2(\mathcal{O}_F)$.

2.14. Let $B_n \leq \text{GL}_n$ be the Borel subgroup of upper triangular matrices. Show that

$$\text{GL}_n(\mathbb{R}) = B_n(\mathbb{R})O_n(\mathbb{R})\text{GL}_n(\mathbb{C}) = B_n(\mathbb{C})U_n(\mathbb{R})$$

and that for a nonarchimedean local field $F$

$$\text{GL}_n(F) = B_n(F)\text{GL}_n(\mathcal{O}_F).$$

Here for $\mathbb{R}$-algebras $R$, $O_n(R) := \{ g \in \text{GL}_n(R) : gg^t = I_n \}$ and $U_n(R) := \{ g \in \text{GL}_n(\mathbb{C} \otimes \mathbb{R}) : g\bar{g}^t = I_n \}$. Deduce the adelic Iwasawa decomposition for $\text{GL}_n(\mathbb{A}_F)$.

2.15. Let

$$\Omega = \{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in [0, 1] \}.$$

Show that $\text{GL}_2(\mathbb{A}_F) = \Omega A_H(t)K$ for $t$ sufficiently small.

2.16. For each integer $N \geq 1$ let $\Gamma(N) \leq \text{GL}_n(\mathbb{Z})$ be the kernel of the reduction map

$$\text{GL}_n(\mathbb{Z}) \to \text{GL}_n(\mathbb{Z}/N).$$

Prove that a subgroup of $\text{GL}_n(\mathbb{Q})$ is a congruence subgroup if and only if it contains $\Gamma(N)$ for some $N$ as a subgroup of finite index.
References


