Chapter 16
Spectral Sides of Trace Formulae

A felicitous but unproved conjecture may be of much more consequence for mathematics than the proof of many a respectable theorem.

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Abstract An important tool for the study of automorphic forms is the Arthur-Selberg trace formula, or more generally, the relative trace formula. We consider the spectral side of these identities in this chapter.

16.1 The automorphic kernel function

The remaining chapters of this book are devoted to stating and proving simple versions of trace formulae. These formulae all have a geometric side involving integrals of a test function along certain orbits, and a spectral side involving period integrals of automorphic forms.

In any of these formulae the first step in this is to employ a fundamental idea first applied to the study of automorphic forms by Selberg [Sel56] that we will now describe. Let $G$ be a connected reductive group over a global field $F$ and as in §2.6 let

$$[G] := G(F) A_G \backslash G(\mathbb{A}_F).$$

(16.1)

For $x \in A_G \backslash G(\mathbb{A}_F)$, one has the regular representation

$$R(x): L^2([G]) \rightarrow L^2([G])$$

$$\varphi \mapsto (g \mapsto \varphi(gx)).$$

(16.2)
For \( f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F)) \), consider the integral operator, the smooth version of the regular representation,

\[
R(f) : L^2([G]) \rightarrow L^2([G])
\]

\[
\varphi \mapsto \left( g \mapsto \int_{A_G \backslash G(\mathbb{A}_F)} f(x)\varphi(gx)dx \right).
\]

One has

\[
R(f)\varphi(x) = \int_{A_G \backslash G(\mathbb{A}_F)} f(y)R(y)\varphi(x)dy
\]

\[
= \int_{A_G \backslash G(\mathbb{A}_F)} f(x)\varphi(xy)dy
\]

\[
= \int_{A_G \backslash G(\mathbb{A}_F)} f(x^{-1}y)\varphi(y)dy
\]

\[
= \int_{[G]} \sum_{\gamma \in G(F)} f(x^{-1}\gamma y)\varphi(y)dy.
\]

Here to justify the last step one uses the unfolding lemma 9.2.3. In other words, \( R(f) \) is an integral operator with kernel

\[
K_f(x, y) := \sum_{\gamma \in G(F)} f(x^{-1}\gamma y).
\]

This is the \textbf{automorphic kernel function} attached to \( f \). It is a smooth function of \( x \) and \( y \). In fact, for \((x, y)\) in a fixed compact set, the sum is finite as we now check: Let

\[
\Omega_1 \times \Omega_2 \subset A_G \backslash G(\mathbb{A}_F) \times A_G \backslash G(\mathbb{A}_F)
\]

be compact subsets. Let

\[
(x, y) \in \Omega_1 \times \Omega_2
\]

then the only nonzero summands in \( K_f(x, y) \) correspond to \( \gamma \) satisfying \( \gamma \in \Omega_1\text{Supp}(f)\Omega_2 \).

The kernel \( K_f(x, y) \) can also be expanded spectrally. The easiest contribution to describe is that given by the cuspidal spectrum as we now explain. By Theorem 9.1.1 the restriction \( R_{\text{cusp}}(f) \) of \( R(f) \) to \( L^2_{\text{cusp}}([G]) \) has kernel

\[
K_{f, \text{cusp}}(x, y) := \sum_{\pi} K_{\pi(f)}(x, y),
\]
16.1 The automorphic kernel function

where the sum is over isomorphism classes of cuspidal automorphic representations $\pi$ of $A \backslash G(\mathbb{A}_F)$ and

$$K_{\pi(f)}(x,y) = \sum_{\varphi \in \mathcal{B}_\pi} \pi(f)\varphi(x)\overline{\varphi(y)} \quad (16.5)$$

for $\mathcal{B}_\pi$ an orthonormal basis of $L^2_{\text{cusp}}(\pi)$, the $\pi$-isotypical subspace of $L^2_{\text{cusp}}([G])$. Here $\pi(f)$ is just the restriction of $R(f)$ to the space of $\pi$. We note that $\pi(f)$ need not be finite rank in general, but it is if $f$ is $K_\infty$-finite by admissibility of $\pi$ (see Exercise 16.1).

We thus arrive at the identity that underlies all trace formulae:

$$\sum_{\gamma \in G(F)} f(x^{-1}\gamma y) = \sum_{\pi} K_{\pi(f)}(x,y) + * \quad (16.6)$$

where the sum is over isomorphism classes of cuspidal automorphic representations of $A \backslash G(\mathbb{A}_F)$, and the $*$ denotes the contribution of the orthogonal complement of $L^2_{\text{cusp}}([G])$ in $L^2([G])$. This contribution will be made precise in §16.3. The left hand side is the geometric expansion of the kernel. The right hand side is the spectral expansion of the kernel.

The key point here is that the right hand side manifestly contains all automorphic representations of $A \backslash G(\mathbb{A}_F)$ whereas the left hand side, at least a priori, does not involve automorphic representations at all. It packages automorphic information in an entirely different manner. The goal is to play these two different manners of encoding automorphic representations off of each other.

An evident method of extracting information from (16.6) is taking traces as we now explain. As defined, $K_{\pi(f)}(x,y)$ is only a function in the $L^2$ sense; a priori it is meaningless to evaluate it at a point. But

$$(\varphi_1, \varphi_2) \rightarrow \int_{[G]} \varphi_1(g)\overline{\varphi_2(g)}dg$$

is the pairing defining the metric on $L^2([G])$. It follows that integrating $K_{\pi(f)}(x,y)$ along $[G]$ embedded diagonally in $[G] \times [G]$ is always well-defined, and

$$\sum_{\pi} \int_{[G]} K_{\pi(f)}(x,y)dx = \sum_{\pi} m(\pi)\text{tr} \pi(f), \quad (16.7)$$

where $m(\pi)$ is the multiplicity of $\pi$ in $L^2_{\text{cusp}}([G])$. Thus (16.7), in principle, is equal to the integral of the left hand side of (16.6) over the diagonal (minus the contribution of $*$). We say in principle because this integral is rarely convergent. In any case, at the expense of a massive amount of work one can make sense of the integral over the diagonal of the left hand side of (16.6) and one obtains the Arthur-Selberg trace formula.
For the moment we leave the geometric expansion of the kernel and focus on the spectral side. We return to the geometric side in §18.2 when we discuss trace formulae in simple settings.

16.2 Relative traces

Let $\pi$ be a cuspidal automorphic representation of $A_G \backslash G(\mathbb{A}_F)$. As just mentioned, the integral of $K_\pi(f)(x, y)$ over the diagonal copy of $[G]$ gives the trace of the operator $\pi(f)$.

There is no need to just integrate over the diagonal copy of $[G]$ however. Jacquet was the first to systematically study the integrals of $K_\pi(f)(x, y)$ over subgroups other than the diagonal copy of $[G]$ (apart from twisted versions of the diagonal embedding that appears in the twisted trace formula) [Jac97, Jac05]. These new integrals are called relative traces. They are introduced in this section.

We start by clarifying the status of the function $K_\pi(f)(x, y)$. As it stands, it is only defined up to a measurable set. It is easy to remedy this. For any closed subspace $V \leq L^2_{\text{cusp}}([G])$ let $B_V$ be an orthonormal basis of this subspace. For $f \in C^\infty_c(A_G \backslash G(\mathbb{A}_F))$, $R(f)|_V$ is of trace class by Theorem 9.1.1 and hence its kernel admits an $L^2$ expansion

$$K_{f,V}(x, y) = \sum_{\varphi \in B_V} R(f)\varphi(x)\varphi(y)$$

(16.8)

(see (9.11)).

**Lemma 16.2.1** There is a unique function that is smooth as a function of $(x, y) \in (A_G \backslash G(\mathbb{A}_F))^2$ that is equal to the kernel $K_{f,V}(x, y)$ almost everywhere.

Given the lemma, we can and do identify the kernel of $R(f)$ restricted to a closed subspace of $L^2_{\text{cusp}}([G])$ with a smooth function.

**Proof.** By the Dixmier-Malliavin lemma, Theorem 4.2.2, we can write $f$ as a finite sum of functions of the form

$$f_1 \ast f_2 \ast f_3$$

for $f_1, f_2, f_3 \in C^\infty_c(A_G \backslash G(\mathbb{A}_F))$. It clearly suffices to prove the lemma for $f$ of this special form, so we assume that $f = f_1 \ast f_2 \ast f_3$. For $f \in C^\infty_c(A_G \backslash G(\mathbb{A}_F))$ let

$$f^\vee(g) := f(g^{-1}) \quad \text{and} \quad f^*(g) := \overline{f(g^{-1})},$$

(16.9)

where the bar denotes the complex conjugate. If $\varphi_1, \varphi_2 \in B_V$ then
16.2 Relative traces

\[
\int_{[G] \times [G]} \sum_{\varphi \in B_V} R(f) \varphi(x) \overline{\varphi}(y) \overline{\varphi_2}(x) \varphi_1(y) dx dy = (R(f) \varphi_1, \varphi_2)_{L^2([G])} = (\varphi_1, R(f^*) \varphi_2)_{L^2([G])} = \int_{[G] \times [G]} \varphi(x) \overline{\varphi}(y) R(f^*) \overline{\varphi_2}(x) \varphi_1(y) dx dy.
\]

We deduce that

\[
\sum_{\varphi \in B_V} R(f) \varphi(x) \overline{\varphi}(y) = \sum_{\varphi \in B_V} \varphi(x) R(f^*) \overline{\varphi}(y). \tag{16.10}
\]

Thus

\[
K_{f,V}(x, y) = \sum_{\varphi \in B_V} R(f_1 \ast f_2 \ast f_3) \varphi(x) \overline{\varphi}(y) = \sum_{\varphi \in B_V} R(f_2 \ast f_3) \varphi(x) R(f_1^*) \overline{\varphi}(y) = (R(f_2) \times R(f_1^*)) \sum_{\varphi \in B_V} R(f_3) \varphi(x) \overline{\varphi}(y). \tag{16.11}
\]

The function (16.11) is smooth as a function of \((x, y)\) by Proposition 4.2.1. This is the unique smooth function representing the kernel in the lemma. \(\square\)

We now prepare to define relative traces. Let

\[
H \leq G \times G
\]

be a subgroup. We will assume that \(H\) is connected and that it is the direct product (not semidirect product) of a unipotent and a reductive group. We set

\[
A_{G,H} := A_H \cap (A_G \times A_G). \tag{16.12}
\]

Note that the containment \(A_{G,H} \leq A_H\) is often proper. For example, it is proper when \(G = GL_2\) and \(H\) is the maximal torus of diagonal matrices. Implicit in the definition of \(R(f)\) is a choice of measure on \(A_G \backslash G(\mathbb{A}_F)\). To define relative traces we additionally choose a Haar measure \(d(h_l, h_r)\) on \(A_{G,H} \backslash H(\mathbb{A}_F)\). It induces a measure on \(A_{G,H} H(F) \backslash H(\mathbb{A}_F)\).

Let

\[
\chi : H(\mathbb{A}_F) \rightarrow \mathbb{C}^\times
\]

be a quasi-character trivial on \(A_{G,H} H(F)\). We define the relative trace of \(\pi(f)\) with respect to \(H\) and \(\chi\) to be
\[
\operatorname{rtr}_{H, \chi} \pi(f) = \int_{A_G, H(F) \backslash H(\mathbb{A}_F)} K_{\pi(f)}(h_{\ell}, h_r) \chi(h_{\ell}, h_r) d(h_{\ell}, h_r). \quad (16.13)
\]

This is well-defined by the following lemma:

**Lemma 16.2.2** For any closed subspace \( V \leq L^2_{\text{cusp}}([G]) \) one has
\[
\int_{A_G, H(F) \backslash H(\mathbb{A}_F)} |K_{f,V}(h_{\ell}, h_r)\chi(h_{\ell}, h_r)| d(h_{\ell}, h_r) < \infty.
\]

**Proof.** As in the proof of Lemma 16.2.1 we may assume \( f = f_1 * f_2 * f_3 \). Then by (16.11)
\[
K_{f,V}(x, y) = (R(f_2) \times R(f'_1)) \sum_{\varphi \in B_V} R(f_3)\varphi(x)\overline{\varphi}(y). \quad (16.14)
\]

The right hand side is rapidly decreasing by Theorem 9.7.1 and hence the integral in the lemma converges by Proposition 14.3.1. \( \square \)

The key property of relative traces is contained in the following proposition:

**Proposition 16.2.1** Let \( \pi \) be a cuspidal automorphic representation of \( G(\mathbb{A}_F) \) that is trivial on \( A_G \). The representation \( \pi \otimes \pi' \) of \( (A_G, G(\mathbb{A}_F))^2 \) is \((H, \chi)\)-distinguished if and only if \( \operatorname{rtr}_{H, \chi} \pi(f) \neq 0 \) for some \( f \in C_\infty(A_G \backslash G(\mathbb{A}_F)) \).

In the proof of Proposition 16.2.1 we are about to give we will apply Lemma 16.2.2 in the special case where \( V \) is a cuspidal automorphic representation of \( A_G \backslash G(\mathbb{A}_F) \). In this case we have two natural notions of smoothness of an element of \( \varphi \in L^2_{\text{cusp}}([G]) \). The first is that \( \varphi \) is a smooth vector, that is, it is fixed by a compact open subgroup \( K_\infty \leq G(\mathbb{A}_F) \) and it is smooth under the action of \( G(F_\infty) \) in the sense of §4.2. The second is that \( \varphi \) is a smooth function, that is, it is (equal a.e., and hence can be identified with) an element of \( C^\infty(G(\mathbb{A}_F)) \) invariant on the left by \( A_G G(F) \). It is obvious that if \( \varphi \) is a smooth function then it is a smooth vector. At least when \( \varphi \) is \( K_\infty \)-finite we have a converse:

**Lemma 16.2.3** An element \( \varphi \in L^2_{\text{cusp}}(\pi) \) that is \( K_\infty \)-finite is equal to a smooth function almost everywhere.

**Proof.** Since \( \varphi \) is \( K_\infty \)-finite it is equal almost everywhere to an automorphic form by Theorem 6.5.2. \( \square \)

**Proof of Proposition 16.2.1:** Recall the definition of the period integral \( P_\chi \) from (14.3). One has
\[
P_\chi(K_{\pi(f)}) = \operatorname{rtr}_{H, \chi} \pi(f),
\]
where the convergence is justified by Lemma 16.2.2. Certainly $K_{\pi(f)}$ lies in the $\pi \otimes \pi^\vee$ isotypic subspace of $L^2([G]^2)$ so we deduce the “if” direction.

Now assume that $\pi \otimes \pi^\vee$ is $(H,\chi)$-distinguished. By Lemma 14.3.1 and Exercise 16.3 there are $K_\infty$-finite smooth functions $\varphi, \varphi' \in L^2_{\text{cusp}}(\pi)$ such that $\mathcal{P}_\chi(\varphi' \otimes \varphi) \neq 0$. Upon renormalizing we can and do assume that $\varphi$ has $L^2$-norm 1. Using Proposition 4.5.2 and Exercise 5.6 choose $f \in C^\infty_c(A_G \backslash G(A_F))$ such that $\pi(f)$ maps $\varphi$ to $\varphi'$ and sends any vector in $L^2(\pi)$ orthogonal to $\varphi$ to zero. Then

$$r_{trH,\chi} \pi(f) = \mathcal{P}_\chi(\varphi' \otimes \varphi).$$

Thus investigating which representations of $(A_G \backslash G(A_F))^2$ are $(H,\chi)$-distinguished is equivalent to studying the linear forms

$$C^\infty_c(A_G \backslash G(A_F)) \to \mathbb{C}$$

$$f \mapsto r_{trH,\chi} \pi(f).$$

The main result of this chapter relates the kernel function $K_{f,\text{cusp}}(x,y)$ to these distributions. It amounts to the computation of the cuspidal contribution to the spectral side of the relative trace formula:

**Theorem 16.2.1** Let $f \in C^\infty_c(A_G \backslash G(A_F))$. One has

$$\int_{A_G, H(F) \backslash H(A_F)} K_{f,\text{cusp}}(h_\ell, h_r) \chi(h_\ell, h_r) d(h_\ell, h_r) = \sum_\pi r_{trH,\chi} \pi(f).$$

Moreover, the integral on the left and the sum on the right are absolutely convergent. In particular, if $R(f)$ has cuspidal image then

$$\int_{A_G, H(F) \backslash H(A_F)} K_f(h_\ell, h_r) \chi(h_\ell, h_r) d(h_\ell, h_r) = \sum_\pi r_{trH,\chi} \pi(f).$$

After the proof of Theorem 16.2.1 we describe the general spectral expansion of $K_f(x,y)$ and then, in §16.4, describe how to construct function $f$ such that $R(f)$ has cuspidal image.

**Proof.** By the Dixmier-Malliavin lemma, Theorem 4.2.2, we can and do assume $f = f_1 * f_2 * f_3$. Using (16.10) we have

$$\sum_\pi K_{\pi(f)}(x,y) = \sum_\pi \pi(f_2) \otimes \pi(f_3^{\vee}) K_{\pi(f_3)}(x,y) \leq \sum_\pi |\pi(f_2) \otimes \pi(f_3^{\vee}) K_{\pi(f_3)}(x,y)|.$$

By Proposition 9.6.1 and with the notation of that proposition this is bounded by a constant depending on $B_1, B_2 \in \mathbb{R}_{>0}$ and $f_1, f_2$ times
\[ \sum_{\pi} \max_{\alpha_1, \alpha_2 \in \Delta} (\alpha(s_x))^{-B_1} (\alpha(s_y))^{-B_2} \|K_\pi(f_3)(x, y)\|_2 \]
\[ = \sum_{\pi} \max_{\alpha_1, \alpha_2 \in \Delta} (\alpha(s_x))^{-B_1} (\alpha(s_y))^{-B_2} \text{tr} \pi(f_3^* f_3). \quad (16.15) \]

Since the operator $R_{\text{cusp}}(f_3^* f_3)$ is trace class by Theorem 9.7.1 this converges absolutely. We conclude that the sum
\[ \sum_{\pi} K_\pi(x, y) \quad (16.16) \]
converges absolutely and uniformly on compact subsets of $[G] \times [G]$ and hence is a continuous function. On the other hand the operator $R_{\text{cusp}}(f)$ is trace class by Theorem 9.1.1 therefore $K_{f, \text{cusp}}(x, y)$ is equal to (16.16) in the $L^2$ sense. Since $K_{f, \text{cusp}}(x, y)$ is smooth by Lemma 16.2.1 we conclude that it is equal to (16.16) pointwise. We can now use the estimate (16.15) together with Proposition 14.3.1 to deduce the theorem. \qed

### 16.3 The full expansion of the automorphic kernel

Using Langlands’ decomposition of the entire spectrum $L^2([G])$ we now give the full spectral expansion of $K_f(x, y)$, not just its restriction to the cuspidal subspace.

We will use the notation and terminology developed in Chapter 10. Let $f \in C^\infty_c(A_G \backslash G(\mathbb{A}_F))$. The automorphic kernel function $K_f(x, y)$ has an expansion
\[ K_f(x, y) = \sum_P \sum_{\sigma} \sum_{\varphi, \lambda} E(x, I(\sigma, \lambda)(f) \varphi, \lambda) \overline{E(y, \varphi, \lambda)} d\lambda \quad (16.17) \]
where the sum on $P$ is over all standard parabolic subgroups of $G$, the sum on $\sigma$ is over all irreducible representations of $G(\mathbb{A}_F)$ occurring in the discrete spectrum of $H_P$, and $\mathcal{B}_\sigma$ is over an orthonormal basis of the $\sigma$-isotypic subspace of $\sigma$ in $H_P$. This expression, a priori, only converges in $L^2$. However, it actually converges pointwise by an analogue of the argument proving Theorem 16.2.1 (see [Art78, §4]).

### 16.4 Functions with cuspidal image

The general spectral expansion of $K_f(x, y)$ is formidable, and it becomes more serious when one tries to integrate the kernel. In particular it is not integrable over the diagonal copy of $[G]$. Arthur has spent much of his career investigating truncated versions of the kernel. In particular these truncated
versions are integrable over the diagonal and can be given spectral interpretations ([Art05] is a nice introduction).

If we place assumptions on the test function \( f \) then the expansion can be made much simpler. One such simplifying assumption is that the operator \( R(f) \) has cuspidal image. In this case \( K_f,\text{cusp}(x, y) = K_f(x, y) \). Lindenstrauss and Venkatesh [LV07] have defined a large class of functions with purely cuspidal image that essentially have no kernel when restricted to the cuspidal spectrum, provided one is only interested in functions spherical at infinity. It would be interesting to remove this assumption.

Before their work, there was a standard example of such functions that can be used to good effect in studying local factors of automorphic representations. We recall it now.

**Definition 16.1.** Let \( v \) be a finite place of a global field \( F \). A function \( f_v \in C_\infty^c(G(F_v)) \) is said to be \( F \)-supercuspidal or simply supercuspidal if
\[
\int_{N(F_v)} f_v(gnh) dn = 0
\]
for all proper parabolic subgroups \( P \leq G \) (defined over \( F \)) with unipotent radical \( N \) and all \( g, h \in G(F_v) \).

**Lemma 16.4.1** If \( f \in C_\infty^c(A_G \backslash G(\mathbb{A}_F)) \) is supercuspidal at some place \( v \) then \( R(f) \) has cuspidal image.

**Proof.** Let \( P \leq G \) be a proper \( F \)-rational parabolic subgroup with unipotent radical \( N \). As usual let \( [N] := N(F) \backslash N(\mathbb{A}_F) \). For \( \varphi \in L^2([G]) \), \( R(f)\varphi \) is smooth and hence can be integrated over any compact subset. For all \( x \in A_G \backslash G(\mathbb{A}_F) \) we have
\[
\int_{[N]} R(f)\varphi(nx) dn = \int_{[N]} \int_{A_G \backslash G(\mathbb{A}_F)} f(g)\varphi(nxg) dg dn \\
= \int_{[N]} \int_{A_G \backslash G(\mathbb{A}_F)} f(x^{-1}n^{-1}g)\varphi(g) dg dn \\
= \int_{[N]} \int_{A_G \backslash G(\mathbb{A}_F)} \sum_{\delta \in N(F)} f(x^{-1}n^{-1}\delta g)\varphi(g) dg dn.
\]
Taking a change of variables \( g \rightarrow \delta^{-1}g \) this becomes
\[
\int_{A_G \backslash N(F) \backslash G(\mathbb{A}_F)} \int_{[N]} \sum_{g \in N(F)} f(x^{-1}n^{-1}g)\varphi(g) dn dg \\
= \int_{A_G \backslash N(F) \backslash G(\mathbb{A}_F)} \int_{N(\mathbb{A}_F)} f(x^{-1}n^{-1}g)\varphi(g) dn dg \\
= 0,
\]
where the last equality follows from the fact that the inner integral
\[
\int_{N(\mathbb{A}_F)} f(x^{-1} n^{-1} g) \varphi(g) dn
\]
vanishes since \( f \) is supercuspidal at some place. \( \square \)

Essentially all examples of supercuspidal functions are obtained using the following lemma (see Exercise 16.4):

**Lemma 16.4.2** Assume that \( Z_G(F_v) \) is compact for some \( v \) and that \((\pi_v, V)\) is an irreducible supercuspidal representation of \( G(F_v) \). If \( f_v \) is a matrix coefficient of \( \pi_v \) then \( f_v \) is \( F \)-supercuspidal.

The assumption that \( Z_G(F_v) \) is compact is not essential; see the discussion of truncated matrix coefficients in [HL04]. This lemma indicates the intrinsic limitation of supercuspidal functions: supercuspidal functions can only be used to study representations that are supercuspidal at some place. For a precise statement see Exercise 16.5.

**Proof.** Let \( P \) be a proper parabolic subgroup of \( G \) with unipotent radical \( N \). Let \( V/V(N) \) be the space of coinvariants defined as (8.11) in §8.3. If
\[
\int_{N(F_v)} f_v(gnh) dn \neq 0
\]
for some \( g, h \in G(F_v) \) then upon realizing \( V \) as a subspace of \( C_\infty(G(F_v)) \) as in the proof of Proposition 8.5.2 we obtain a nonzero map
\[
V \longrightarrow V/V(N)
\]
\[
f_v \longmapsto \left( g \mapsto \int_{N(F_v)} f_v(gnh) dn \right)
\]
contradicting the supercuspidality of \( \pi_v \). \( \square \)

**Exercises**

16.1. Let \( K_\infty \) be a maximal compact subgroup of \( G(F_\infty) \). If \( f \) is \( K_\infty \)-finite, then prove that \( \pi(f) \) has finite image.

16.2. Prove that \( K_{f,V}(x, y) \) is independent of the choice of orthonormal basis.

16.3. For \( 1 \leq i \leq 2 \) let \( G_i \) be a reductive group over a number field \( F_i \), let \( K_{i,\infty} \leq A_{G_i}(F_\infty) \) be maximal compact subgroup, and let \( \pi_i \) be a cuspidal automorphic representation of \( A_{G_i}(F_\infty) \). Then a smooth function in
functions with cuspidal image

\[ L^2_{\text{cusp}}([G_1 \times G_2])(\pi_1 \otimes \pi_2) \]  

(16.18)

need not be in the algebraic tensor product

\[ L^2_{\text{cusp}}([G_1])(\pi_1) \otimes L^2_{\text{cusp}}([G_2])(\pi_2). \]

Despite this, show that a \( K_1 \infty \times K_2 \infty \)-finite smooth function in (16.18) is a finite sum of functions of the form \( \varphi_1 \otimes \varphi_2 \) where \( \varphi_i \) is a \( K_i \infty \)-finite smooth function in \( L^2_{\text{cusp}}([G_i])(\pi_i) \) for \( 1 \leq i \leq 2 \).

16.4. Let \( v \) be a finite place of the number field \( F \) and let \( G \) be a reductive group over \( F \). Assume that \( Z_{G}(F_v) \) is compact. Prove that any supercuspidal function is a finite sum of matrix coefficients of supercuspidal representations of \( G(F_v) \).

16.5. Under the assumptions of Exercise 16.4 let \( f = f_v f_v^r \in C^\infty_c(G(\mathbb{A}_F)) \) where \( f_v \) is a matrix coefficient of an irreducible supercuspidal representation \( \pi'_v \). Prove that if \( \pi \) is a cuspidal automorphic representation of \( G(\mathbb{A}_F) \) such that \( K_\pi(f)(x,y) \neq 0 \) then \( \pi_v \cong \pi'_v \).

16.6. Let \( K_\infty \) be a maximal compact subgroup of \( G(F_\infty) \). If \( f \) is \( K_\infty \)-finite prove that

\[ \text{rtr}_{H, \chi} \pi(f) = \sum_{\varphi \in \mathcal{B}_\pi} \mathcal{P}_\chi(\pi(f)\varphi \times \overline{\varphi}) \]

where we take the basis \( \mathcal{B}_\pi \) to consist of \( K_\infty \)-finite forms (and hence smooth by Proposition 4.4.1).
References


