Chapter 11
Rankin-Selberg $L$-functions

...I had not recognized in 1966, when I discovered after many months of unsuccessful search a promising definition of automorphic $L$-function, what a fortunate, although, and this needs to be stressed, unforeseen by me, or for that matter anyone else, blessing it was that it lay in the theory of Eisenstein series.

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Abstract In this chapter we sketch the theory of generic representations and Rankin-Selberg $L$-functions.

11.1 Paths to the construction of automorphic $L$-functions

Let $F$ be a global field and let $\pi$ and $\pi'$ be a pair of cuspidal automorphic representations of $A_{\text{GL}_n} \backslash \text{GL}_n(\mathbb{A}_F)$ and $A_{\text{GL}_m} \backslash \text{GL}_m(\mathbb{A}_F)$, respectively. An important analytic invariant of this pair is the Rankin-Selberg $L$-function

$$L(s, \pi \times \pi').$$

This is a meromorphic function on the complex plane satisfying a functional equation whose poles can be explicitly described in terms of $\pi$ and $\pi'$. These $L$-functions play a crucial role in automorphic representation theory. On the one hand they are used in the statement of the local Langlands correspondence for $\text{GL}_n$ (and hence, implicitly, for the global Langlands correspondence). This will be discussed in §12.5.
In this chapter we describe one method by which they can be defined, namely the Rankin-Selberg method. This requires the introduction of the notion of a generic representation. It turns out that all cuspidal representations of $\text{GL}_n(A_F)$ are generic, but cuspidal representations of other groups need not be generic. A much more detailed treatment of the Rankin-Selberg method is contained in [Cog07], which was our primary reference in the preparation of this chapter.

There are other approaches to defining these $L$-functions. One is via the so-called Langlands-Shahidi method. This method also uses the notion of a generic representation. The idea is that Rankin-Selberg $L$-functions (and some more general $L$-functions) occur in the constant term of certain Eisenstein series. This approach was historically important because it suggested both the general definition of a Langlands $L$-function, discussed in §12.7, and the general formulation of Langlands functoriality. We do not discuss this construction. It is discussed at length in the book [Sha10].

In the special case where $\pi'$ is the trivial representation there is an alternate approach to defining these $L$-functions due to Godement and Jacquet [GJ72]. It is a direct generalization of Tate’s construction of the $L$-functions of Hecke characters in his famous thesis [Tat67], which in turn is an adelic formulation of the classical work of Hecke.

11.2 Generic characters

Let $G$ be a quasi-split group over a global or local field $F$, let $B \leq G$ be a Borel subgroup, let $T \leq B$ a maximal torus and let $N \leq B$ the unipotent radical of $B$. We choose a pinning

$$(B, T, \{X_\alpha\}_{\alpha \in \Delta})$$

as in Definition 7.2. Let $E/F$ be a Galois extension over which $B$ and $T$ split. Then since $T(E)$ is preserved by $\text{Gal}(E/F)$ the root spaces $X_\alpha$ are permuted by $\text{Gal}(E/F)$ and we can choose a set of representatives $\{\beta\}$ for the $\text{Gal}(E/F)$ orbits; let $O(\beta)$ denote the orbit of $\beta$. For $E$-algebras $R$ the exponential map yields a group homomorphism

$$\exp : \prod_{\alpha \in O(\beta)} G_\alpha(R) \rightarrow G_E(R)$$

$$(x_\alpha) \mapsto \prod_{\alpha \in O(\beta)} \exp(x_\alpha X_\alpha).$$

Its image is a unipotent subgroup whose Lie algebra is spanned by $\{X_\alpha\}_{\alpha \in O(\beta)}$. This unipotent subgroup is in fact stable under $\text{Gal}(E/F)$, and hence is the $E$ points of a unipotent subgroup $N(\beta) \leq N$ that is isomorphic to $\text{Res}_{E/F} G_\alpha$. 

Varying $\beta$, we see that we have a subgroup
\[ \prod_\beta N(\beta) \leq N, \]
where the product is over all $\text{Gal}(E/F)$-orbits of simple roots $\alpha$. We note that inclusion induces an isomorphism
\[ \prod_\beta N(\beta) \sim N/N^{\text{der}}. \]

**Definition 11.1.** If $F$ is a local field, a character $\psi : N(F) \rightarrow \mathbb{C}^\times$ is called **generic** if $\psi|_{N(\beta)(F)}$ is nontrivial for each $\text{Gal}(E/F)$-orbit $\beta$. If $F$ is global a character $\psi : N(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$ trivial on $N(F)$ is called **generic** if $\psi_v$ is generic for all places $v$.

**Example 11.1.** Take $G = \text{GL}_n$ and let $B_n \leq \text{GL}_n$ be the Borel subgroup of upper triangular matrices. Moreover let $N_n \leq B_n$ be the unipotent radical. If $F$ is local any character of $N_n(F)$ is of the form
\[
\begin{pmatrix}
1 & x_{12} & * & * \\
1 & x_{23} & * & * \\
\vdots & \vdots & \ddots & * \\
1 & x_{(n-1)n}
\end{pmatrix} \mapsto \psi(m_1 x_{12} + \cdots + m_{n-1} x_{(n-1)n}) \tag{11.1}
\]
for some $m_1, \ldots, m_{n-1} \in F$, where $\psi : F \rightarrow \mathbb{C}^\times$ is a nontrivial character. The character is generic if and only if all of the $m_i$ are nonzero. If $F$ is global any character of $N_n(\mathbb{A}_F)$ trivial on $N_n(F)$ is of the same form, but now the $m_i$ must be in $F$ and $\psi$ must be trivial on $F$. Again, the character is generic if and only if $\prod_{i=1}^{n-1} m_i \neq 0$. If all $m_i = 1$ we call this the **standard character** attached to $\psi$.

### 11.3 Generic representations

Let $G$ be a quasi-split connected reductive group over a local field $F$ and let $\psi : N(F) \rightarrow \mathbb{C}^\times$ be a character. Let $(\pi, V)$ be an admissible representation of $G(F)$.

**Definition 11.2.** Assume $F$ is nonarchimedean. A $\psi$-**Whittaker functional** on $V$ is a continuous linear functional $\lambda : V \rightarrow \mathbb{C}$ such that
\[
\lambda(\pi(n) \varphi) = \psi(n) \lambda(\varphi)
\]
for \( \varphi \in V \) and \( n \in N(F) \).

In this nonarchimedean setting, continuous means locally constant.

If \( F \) is archimedean, we assume that \( V \) is a Hilbert space and even that \( \pi \) is unitary. Then for every \( X \in U(g) \) we can define a seminorm \( \| \cdot \|_X \) on \( V_{\text{sm}} \) via

\[
\| \varphi \|_X := \| \pi(X) \varphi \|_2.
\]

We can give \( V_{\text{sm}} \) a locally convex topology via these seminorms, and it then makes sense to speak of continuous linear functionals. It is this notion of continuity we use in the following definition:

**Definition 11.3.** Assume \( F \) is archimedean. A \( \psi \)-Whittaker functional on \( V \) is a continuous linear functional \( \lambda : V \rightarrow \mathbb{C} \) such that

\[
\lambda(\pi(n)\varphi) = \psi(n)\lambda(\varphi)
\]

for \( \varphi \in V_{\text{sm}} \) and \( n \in N(F) \).

Assume, as above, that in the archimedean setting \((\pi, V)\) is unitary. Assume in addition that \( \psi \) is generic.

**Definition 11.4.** An irreducible admissible representation \((\pi, V)\) of \( G(F) \) is \( \psi \)-generic if it has a nonzero \( \psi \)-Whittaker functional on \( V \).

The following theorem of Shalika [Sha74] has turned out to be extremely important:

**Theorem 11.3.1 (Shalika)** The space of \( \psi \)-Whittaker functionals on an irreducible admissible representation of \( G(F) \) is at most 1-dimensional. \( \square \)

We now define the related notion of a Whittaker function. The space of Whittaker functions on \( G(F) \) is the space of smooth functions

\[
W(\psi) := \{ \text{smooth } W : G(F) \rightarrow \mathbb{C} \text{ satisfying } W(ng) = \psi(n)W(g) \}. \quad (11.2)
\]

This space admits an action of \( G(F) \):

\[
G(F) \times W(\psi) \rightarrow W(\psi)
\]

\[
(g, W) \mapsto (x \mapsto W(xg)).
\]

Let \((\pi, V)\) be an irreducible admissible representation \( \pi \) of \( G(F) \). A Whittaker model of \( \pi \) is a nontrivial \( G(F) \)-intertwining map

\[
V_{\text{sm}} \rightarrow W(\psi)
\]

(in the nonarchimedean case, \( V = V_{\text{sm}} \) by assumption). As a corollary of Theorem 11.3.1 we have the following (see Exercise 11.2):
11.3 Generic representations

**Corollary 11.3.1** A Whittaker model of \( \pi \) exists if and only if \( \pi \) is generic. If \( \pi \) is generic, then it admits a unique Whittaker model. \( \square \)

Motivated by Corollary 11.3.1 if \((\pi, V)\) is generic we let

\[
W(\pi, \psi) \tag{11.3}
\]

be the image of any intertwining map \( V_{\text{sm}} \to W(\psi) \). We write the map explicitly as

\[
V_{\text{sm}} \to W(\pi, \psi) \\
\varphi \mapsto W_{\psi}^{\varphi}.
\]

By Schur's lemma this map is well-defined up to multiplication by a nonzero complex number.

Now assume that \( F \) is global, let \( \pi \) be a cuspidal automorphic representation of \( G(\mathbb{A}_F) \) realized in a subspace \( V \leq L^2_{\text{cusp}}([G]) \), and let \( \varphi \in V_{\text{sm}} \). We then define the **global \( \psi \)-Whittaker function**

\[
W_{\psi}^{\varphi}(g) := \int_{[N]} \varphi(ng) \overline{\psi}(n) dn. \tag{11.4}
\]

We say that \((\pi, V)\) is globally **\( \psi \)-generic** if

\[
W_{\psi}^{\varphi}(g) \neq 0
\]

for some \( \varphi \in V \) and \( g \in G(\mathbb{A}_F)^1 \). We remark that a priori this notion depends on the realization of \( \pi \) in \( L^2_{\text{cusp}}([G]) \) if it does not occur with multiplicity 1. It is not hard to check that if \( \pi \) is globally \( \psi \)-generic and \( \pi \cong \otimes'_v \pi_v \) then each \( \pi_v \) is \( \psi_v \)-generic (Exercise 11.3).

The notion of a generic representation is only useful if we have an interesting supply of generic representations. The following theorem implies that all cuspidal automorphic representations of \( \text{GL}_n(\mathbb{A}_F) \) are generic, and moreover they admit an expansion in terms of Whittaker functionals:

**Theorem 11.3.2** Let \( \varphi \in L^2_{\text{cusp}}([\text{GL}_n]) \) be a smooth vector in the space of a cuspidal automorphic representation \( \pi \) of \( \text{GL}_n \backslash \text{GL}_n(\mathbb{A}_F) \). If \( \psi : N_n(\mathbb{A}_F) \to \mathbb{C}^\times \) is a generic character then one has

\[
\varphi(g) = \sum_{\gamma \in N_{n-1}(F) \backslash \text{GL}_{n-1}(F)} W_{\psi}^{\varphi}((\gamma \begin{pmatrix} 1 & \; \\ \; & 1 \end{pmatrix}) g). \tag{11.5}
\]

The expression for \( \varphi \) in (11.5) is called its **Whittaker expansion**. It is a generalization (but not the only generalization) of the well-known Fourier expansion of a modular form.

We now prove this theorem in the special case \( n = 2 \). Consider the function
This is a continuous function on the compact abelian group \( F \setminus \mathbb{A}_F \), and hence admits a Fourier expansion. If we fix a nontrivial character \( \psi : F \setminus \mathbb{A}_F \to \mathbb{C}^\times \), then all other characters are of the form \( \psi_\alpha(x) := \psi(\alpha x) \) for \( \alpha \in F \) (see Lemma B.1.2). Thus the Fourier expansion of the function (11.6) is

\[
\sum_{\alpha \in F^\times} \int_{F \setminus \mathbb{A}_F} \varphi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \overline{\psi(\alpha y)} dy.
\]

(11.7)

Since \( \varphi \) is cuspidal, the \( \alpha = 0 \) term vanishes identically. Taking a change of variables \( x \mapsto \alpha^{-1} x \) for each \( \alpha \in F^\times \) and using the left \( \text{GL}_2(F) \)-invariance of \( \varphi \) we see that (11.6) is equal to

\[
\sum_{\alpha \in F^\times} W^\varphi_\psi \left( \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \right) g.
\]

(11.8)

Setting \( x = 0 \) we obtain Theorem 11.3.2 when \( n = 2 \). We will not prove Theorem 11.3.2 because we cannot improve on the exposition in [Cog07, §1.1]. The basic idea is to proceed as in the \( n = 2 \) case, using abelian Fourier analysis on certain abelian subgroup of \( N_n(\mathbb{A}_F) \). However, since \( N_n(\mathbb{A}_F) \) is no longer abelian for \( n > 3 \) one has to combine this with a clever inductive argument using the fact that \( \varphi \) is cuspidal.

Using Theorem 11.3.2, we obtain the following related theorem of Shalika [Sha74, Theorem 5.5]:

**Theorem 11.3.3 (Multiplicity one for \( \text{GL}_n \))** An irreducible admissible representation of \( \text{GL}_n(\mathbb{A}_F) \) occurs with multiplicity at most one in \( L^2_{\text{cusp}}([\text{GL}_n]) \).

We note that Theorem 11.3.3 is false for essentially every reductive group that is not a general linear group.

**Proof.** Let \((\pi_1, V_1), (\pi_2, V_2)\) be two realizations of a given cuspidal automorphic representation \((\pi, V)\). Choose equivariant maps \( L_i : V \to V_i \). We then obtain Whittaker functionals

\[
\lambda_i : V_{sm} \to \mathbb{C},
\]

\[
\varphi \mapsto W^{L_i(\varphi)} \psi(I_n),
\]

where \( I_n \) is the identity element. They are nonzero by Theorem 11.3.2. By Theorem 11.3.1 one therefore has \( \lambda_1 = c \lambda_2 \) for some \( c \in \mathbb{C}^\times \). Thus

\[
L_1(\varphi)(g) = \sum_{\gamma \in N_{n-1}(F) \setminus \text{GL}_{n-1}(F)} W^{L_1(\varphi)} \psi \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \right) g
\]

\[
= c \sum_{\gamma \in N_{n-1}(F) \setminus \text{GL}_{n-1}(F)} W^{L_2(\varphi)} \psi \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \right) g.
\]
This implies in particular that $V_1$ and $V_2$ have nonzero intersection, and hence are equal. 

11.4 Formulae for Whittaker functions

Let $(\pi, V)$ be a globally generic cuspidal representation of the quasi-split group $G$, realized on a closed subspace $V \leq L^2((G))$.

Using Flath’s theorem we can choose an isomorphism

$$V_{\text{sm}} \xrightarrow{\sim} \otimes_v V_v \rightarrow \otimes_v \mathcal{W}(\pi_v, \psi_v).$$

(11.9)

Let $\varphi \in V_{\text{sm}}$ be a pure tensor, i.e. a vector that maps to a vector of the form $\otimes_v \varphi_v$ under the first isomorphism. Then, by local uniqueness of Whittaker models upon normalizing the composite isomorphism $V_{\text{sm}}$ by multiplying by a suitable nonzero complex number we have

$$W_{\varphi}(g) = \prod_v W_{\varphi_v}(g).$$

(11.10)

Thus to compute the global Whittaker function $W_{\varphi}(g)$ it suffices to compute the local Whittaker functions $W_{\varphi_v}$. When $\pi_v$ and $\psi_v$ are unramified this can be accomplished using famous Casselman-Shalika formula [CS80].

For the remainder of this section let $F$ be a nonarchimedean local field and let $\psi : F \to \mathbb{C}^\times$ be a nontrivial unramified character. The Casselman-Shalika formula is valid for any unramified reductive group over $F$, but to simplify our discussion we will assume that $G$ is a split group. Let $B \leq G$ be the Borel subgroup with split maximal torus $T \leq B$. We let $K \leq G(F)$ be a hyperspecial subgroup in good position with respect to $(B, T)$. We note that by the Iwasawa decomposition one has

$$G(F) = \prod_{\mu \in X^*(T)} N(F)\mu(\varpi)K$$

(11.11)

where $N \leq B$ is the unipotent radical of $B$.

Using the notation of §7.6, for $\lambda \in a_T^\vee_F$ we can then form the induced representation $I(\lambda)$, where the induction is with respect to the Borel subgroup $B$. Its unique unramified subquotient is denoted by $J(\lambda)$. Any unramified representation is equivalent to such a $J(\lambda)$. The choice of $T$ and $B$, by duality, give rise to a maximal torus and Borel subgroup $\hat{T} \leq \hat{B} \leq \hat{G}$ as explained in §7.3. This gives rise to a notion of dominant weight of $\hat{T}$ in $\hat{B}$, say $\mu \in X^*(\hat{T}) = X_*(T)$. By Cartan-Weyl theory, there is a unique isomorphism class $V(\mu)$ of irreducible representation of $\hat{G}$ attached to $\mu$. Let $\chi_\mu$ be its character.
Theorem 11.4.1 (Casselman and Shalika) Any function $W$ in the one-dimensional space $W(J(\lambda), \psi)^K$ satisfies $W(I) \in \mathbb{C}^\times$, $I$ is the identity element. If $\mu$ is a dominant weight then

$$\frac{W(\mu(\varpi))}{W(I)} = \delta_{B_n}^{1/2}(\mu(\varpi)) \chi_{\mu}(e^{-\lambda}).$$

If $\mu$ is not a dominant weight then $W(\mu(\varpi)) = 0$.

Here one recalls that $e^\lambda$ is discussed in Lemma 7.6.1. This is proven in [CS80]; to translate the formula in loc. cit. to the formula above one uses the Weyl character formula.

It is instructive to write this formula down more explicitly when $G = \text{GL}_n$. In this case it is due to Shintani [Shi76]. We assume that $B = B_n$ is the Borel subgroup of upper triangular matrices and $T = T_n \leq B_n$ is the maximal torus of diagonal matrices. In this case weights can be identified with tuples $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ as in (1.9). The cocharacter $\mu$ attached to such a tuple is recorded in (1.9). The dominant weights are those with $k_1 \geq \cdots \geq k_n$.

The associated irreducible representation of $\text{GL}_n$ is $S_{k_1, \ldots, k_n}(V_{st})$, where $V_{st}$ is the standard representation and $S_{k_1, \ldots, k_n}$ is the Schur functor attached to the partition $k_1, \ldots, k_n$ of $\sum_{i=1}^n k_i$. Let $\chi_{k_1, \ldots, k_n}$ be its character.

Corollary 11.4.1 (Shintani) If $G = \text{GL}_n$ and $W \in W(J(\lambda), \psi)^{\text{GL}_n(O_F)}$ is the unique vector satisfying $W(I_n) = 1$ then

$$W \left( \varpi^{k_1} \ldots \varpi^{k_n} \right) = \begin{cases} \delta_{B_n}^{1/2}(\mu(\varpi)) \chi_{k_1, \ldots, k_n}(e^{-\lambda}) & \text{if } k_1 \geq \cdots \geq k_n, \\ 0 & \text{otherwise.} \end{cases}$$

11.5 Local Rankin-Selberg $L$-functions

Let $F$ be a local field and let $\psi : F \to \mathbb{C}^\times$ be a nontrivial character. Our goal in this section is to define local Rankin-Selberg $L$-functions for generic representations, at least when $F$ is nonarchimedean. The fundamental idea here is that these $L$-functions are the smallest rational functions in $q^{-s}$ that cancel the poles of a family of zeta functions. The definition of this family was originally given in [JPSS83], and the archimedean case was treated definitively in [Jac09]. We refer to these papers for proofs of the unproved statements we make below.

Let $\pi, \pi'$ be irreducible admissible generic representations of $\text{GL}_n(F)$ and $\text{GL}_m(F)$, respectively. Let $\psi : F \to \mathbb{C}^\times$ be an additive character. Use it to
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define a standard generic character of $N_n(F)$ and $N_m(F)$ as in Example 11.1. We denote these characters again by $\psi$. Let

$$W \in W(\pi, \psi), \quad W' \in W(\pi', \psi).$$

If $m < n$ let

$$\Psi(s; W, W') := \int W(h) W'(h) \left| \det(h) \right|^{s-(n-m)/2} dh,$$

$$\tilde{\Psi}(s; W, W') := \int \int W(h x I_{n-m-1} 1) dx W'(h) \left| \det h \right|^{s-(n-m)/2} dh,$$

where the top integral is over $N_m(F) \setminus \text{GL}_m(F)$, the outer integral on the bottom is over $N_m(F) \setminus \text{GL}_m(F)$, and the inner integral on the bottom is over $(n-m-1) \times m$ matrices with entries in $F$.

If $m = n$, then for each $\Phi \in S(F^n)$ (the Schwartz space) let

$$\Psi(s; W, W', \Phi) := \int_{N_n(F) \setminus \text{GL}_n(F)} W(g) W'(g) \Phi(e_n g) \left| \det g \right|^s dg.$$  (11.13)

Here $e_n \in F^n$ is the elementary vector with 0’s in the first $n-1$ entries and 1 in the last entry.

**Proposition 11.5.1** Assume that $F$ is nonarchimedean.

(a) The integrals (11.12) and (11.13) converge for $\text{Re}(s)$ sufficiently large. For $\pi$ and $\pi'$ unitary they converge absolutely for $\text{Re}(s) \geq 1$. For $\pi$ and $\pi'$ tempered, they converge absolutely for $\text{Re}(s) > 0$.

(b) Each integral is a rational function of $q^{-s}$.

(c) The $\mathbb{C}$-linear span of the integrals in $\mathbb{C}[q^s, q^{-s}]$ as $W, W'$ (and when $m = n$ the Schwartz function $\Phi$) is a principal ideal $\mathcal{I}(\pi, \pi')$.

In this nonarchimedean case, one proves that there is a unique polynomial $P_{\pi, \pi'} \in \mathbb{C}[x]$ satisfying $P_{\pi, \pi'}(0) = 1$ such that $P_{\pi, \pi'}(q^{-s})^{-1}$ is a generator of $\mathcal{I}(\pi, \pi')$. One sets

$$L(s, \pi \times \pi') := P_{\pi, \pi'}(q^{-s})^{-1}.$$  (11.14)

In the archimedean case one actually defines $L(s, \pi \times \pi')$ using the local Langlands correspondence, which was established very early on in the theory by Langlands himself. We will explain this in more detail in §12.3 below. Right now we note the following important bound:

**Theorem 11.5.1** If $\pi$ and $\pi'$ are unitary then $L(s, \pi \times \pi')$ is holomorphic and nonzero for $\text{Re}(s) \geq 1$.  \(\square\)

See [BR94, §2] for a proof. Technically speaking they only state that the $L$-function is holomorphic at this point, but the argument actually proves that it is nonzero as well.
To discuss the archimedean case we say that a holomorphic function \( f(s) \) of \( s \in \mathbb{C} \) is **bounded in vertical strips** if it is bounded in

\[
V_{\sigma_1, \sigma_2} := \{ s : \sigma_1 \leq \text{Re}(s) \leq \sigma_2 \}
\]

for all \( \sigma_1 < \sigma_2 \in \mathbb{R} \). A meromorphic function \( f(s) \) is bounded in vertical strips if for each \( \sigma_1 < \sigma_2 \) there is a polynomial \( P_{\sigma_1, \sigma_2} \in \mathbb{C}[x] \) such that \( P_{\sigma_1, \sigma_2} f(s) \) is holomorphic and bounded in vertical strips in the original sense.

**Theorem 11.5.2** Assume \( F \) is archimedean. The local Rankin-Selberg integrals (11.12) and (11.13) converge absolutely for \( \text{Re}(s) \) sufficiently large and admit meromorphic continuations to functions of \( s \) that are bounded in vertical strips. \( \Box \)

Let

\[
w_{m,n} := \begin{pmatrix} I_m \\ w_{n-m} \end{pmatrix}, \quad w_n := \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} \in \text{GL}_n(\mathbb{Z})
\]

and for functions \( f \) on \( \text{GL}_n(F) \) or \( \text{GL}_n(\mathbb{A}_F) \) let \( \rho(w_{m,n}) f(x) := f(xw_{m,n}) \) and \( \tilde{f}(g) := f(w_n g^{-t}) \).

**Theorem 11.5.3** There is a meromorphic function \( \gamma(s, \pi \times \pi', \psi) \in \mathbb{C}(q^{-s}) \) in the nonarchimedean case such that if \( m < n \) we have

\[
\tilde{\Psi}(1-s, \rho(w_{n,m}) \tilde{W}, \tilde{W}') = \omega'(-1)^{n-1} \gamma(s, \pi \times \pi', \psi) \Psi(s, W, W')
\]

and if \( m = n \) we have

\[
\Psi(1-s, \tilde{W}, \tilde{W}', \Phi) = \omega'(-1)^{n-1} \gamma(s, \pi \times \pi', \psi) \Psi(s, W, W', \Phi)
\]

for all \( W \in W(\pi, \psi), \ W' \in W(\pi', \psi) \). Here \( \omega' \) is the central quasi-character of \( \pi' \).

We warn the reader that the functional equations in [Jac09] are different. The root of this is that the definition of the contragredient \( \tilde{W} \) of \( W \) in loc. cit. is different.

We need one other factor, the local \( \varepsilon \)-factor:

\[
\varepsilon(s, \pi \times \pi', \psi) := \frac{\gamma(s, \pi \times \pi', \psi) L(s, \pi \times \pi')}{L(1-s, \pi' \times \pi'^\vee)}.
\]  

(11.15)

**Proposition 11.5.2** In the nonarchimedean case \( \varepsilon(s, \pi \times \pi') \) is a function of the form \( cq^{-fs} \) for some real number \( f \) and \( c \in \mathbb{C}^\times \).

The \( \varepsilon \)-factor is a useful arithmetic invariant of the pair \( (\pi, \pi') \). Assume that \( \pi' \) is trivial. In this case the ideal \( \varpi^f \) is called the **conductor** of \( \pi \). One
writes \( f := f(\pi) \), and calls it the exponent of the conductor, or sometimes just the conductor. Let

\[
K_1(\varpi f(\pi)) := \left\{ g \in \text{GL}_n(\mathcal{O}_F) : g \equiv \begin{pmatrix} \ast \\ \vdots \\ \ast \\ 0 \ldots 0 1 \end{pmatrix} \pmod{\varpi f(\pi)} \right\}.
\]

Then one has the following (see [Mat13]):

**Theorem 11.5.4** The number \( f(\pi) \) is the smallest nonnegative integer such that the space \( V \) of \( \pi \) satisfies

\[ V^{K_1(\varpi f(\pi))} \neq 0. \]

Moreover, \( \dim_\mathbb{C} V^{K_1(\varpi f(\pi))} = 1 \).

11.6 The unramified Rankin-Selberg \( L \)-functions

In this section we assume that \( F \) is nonarchimedean and that \( \pi \) and \( \pi' \) are irreducible unramified representations of \( \text{GL}_n(F) \) and \( \text{GL}_m(F) \), respectively. Assume that \( \psi : F \to \mathbb{C}^\times \) is unramified and let \( W \in \mathcal{W}(\pi, \psi) \) and \( W' \in \mathcal{W}(\pi', \overline{\psi}) \) be the unique spherical vectors satisfying \( W(I_n) = W'(I_m) = 1 \).

We assume that \( \pi \cong J(\lambda) \) and \( \pi' \cong J(\lambda') \), where \( \lambda \in \mathfrak{a}_{T_n}^* \mathbb{C} \) and \( \lambda' \in \mathfrak{a}_{T_m}^* \mathbb{C} \).

Our aim in this section is to prove the following identity:

**Theorem 11.6.1** For \( \text{Re}(s) \) sufficiently large if \( n > m \) one has

\[
\Psi(s, W, W', \psi) = \det \left( I_{mn} - q^{-s} e^{-\lambda} \otimes e^{-\lambda'} \right)^{-1}.
\]

If \( m = n \) one has

\[
\Psi(s, W, W', 1_{\mathcal{O}_F^n}) = \det \left( I_{n^2} - q^{-s} e^{-\lambda} \otimes e^{-\lambda'} \right)^{-1}.
\]

Implicit in the definition of the local Rankin-Selberg integrals is a choice of measure on \( \text{GL}_m(F) \) and \( N_m(F) \) so that we can form a quotient measure on \( N_m(F) \backslash \text{GL}_m(F) \). In order for the equalities above to be valid we must choose the unique Haar measures that assign volume 1 to \( \text{GL}_m(\mathcal{O}_F) \) and \( N_m(\mathcal{O}_F) \), respectively.

**Proof.** For each \( m \in \mathbb{Z}_{\geq 1} \) let \( T^+(m) \) be the set of the tuples \( \mu = (k_1, \ldots, k_m) \in \mathbb{Z}^m \) with \( k_1 \geq \cdots \geq k_m \geq 0 \). This can be identified with a subset of the dominant weights of \( T_m^+ \) or a cocharacter of \( T_m \). If \( n > m \) let
be the injection given by extending the tuple $k_1, \ldots, k_m$ by zeros.

Assume for the moment that $n > m$. Then by Corollary 11.4.1 we have

$$
\Psi(s; W, W') = \sum_{\mu \in T^+(m)} W(\mu(\varpi)) W'(\mu(\varpi)) \left| \det(\mu(\varpi)) \right|^{s-(n-m)/2} \delta^{-1}_{B_m} (\mu(\varpi))
$$

$$
= \sum_{\mu \in T^+(m)} \chi_{k_1, \ldots, k_m} (e^{-\lambda}) \chi_{k_1, \ldots, k_m, 0, \ldots, 0} (e^{-\lambda}) q^{-|\mu| s} \left| \det(\mu(\varpi)) \right|^{s-(n-m)/2} \delta^{-1}_{B_m} (\mu(\varpi))
$$

$$
= \det(I_{mn} - q^{-s} e^{-\lambda} \otimes e^{-\lambda'})^{-1},
$$

where we let $|\mu| = k_1 + \cdots + k_m$ and, as before, $\chi_{k_1, \ldots, k_m, 0, \ldots, 0}$ denotes the character associated to $S_{k_1, \ldots, k_m, 0, \ldots, 0}$. This last identity is known as the Cauchy identity [Bum13, Chapter 38]. Similarly, when $n = m$ we have

$$
\Psi(s; W, W', \mathbb{1}_{\mathcal{O}_F})
$$

$$
= \sum_{\mu \in T^+(n)} W(\mu(\varpi)) W'(\mu(\varpi)) \mathbb{1}_{\mathcal{O}_F} (e_n \mu(\varpi)) \left| \det(\mu(\varpi)) \right|^{s-1} \delta^{-1}_{B_1} (\mu(\varpi))
$$

$$
= \sum_{\mu \in T^+(n)} \chi_{k_1, \ldots, k_n} (e^{-\lambda}) \chi_{k_1, \ldots, k_n} (e^{-\lambda'}) q^{-|\mu| s} \left| \det(\mu(\varpi)) \right|^{s-1} \delta^{-1}_{B_1} (\mu(\varpi))
$$

$$
= \det(I_{n^2} - q^{-s} e^{-\lambda} \otimes e^{-\lambda'})^{-1}.
$$

The calculation above implies that $\det(I_{mn} - q^{-s} e^{-\lambda} \otimes e^{-\lambda'})$ divides $L(s, \pi \times \pi')^{-1}$ (as a polynomial in $q^{-s}$). More is true (see [JPSS83]):

**Theorem 11.6.2** One has $L(s, \pi \times \pi') = \det(I_{mn} - q^{-s} e^{-\lambda} \otimes e^{-\lambda'})^{-1}$. □

### 11.7 Global Rankin-Selberg $L$-functions

Let $\pi$ and $\pi'$ be cuspidal automorphic representations of $A_{GL_n} \setminus GL_n(\mathbb{A}_F)$ and $A_{GL_m} \setminus GL_m(\mathbb{A}_F)$, respectively.

We have previously defined the local Rankin-Selberg $L$-functions $L(s, \pi_v \times \pi'_v)$. The global Rankin-Selberg $L$-function is the product

$$
L(s, \pi \times \pi') := \prod_v L(s, \pi_v \times \pi'_v).
$$

(11.16)

If $\psi : F \setminus \mathbb{A}_F \to \mathbb{C}^\times$ is a nontrivial character we also define
\[ \varepsilon(s, \pi \times \pi') := \prod_v \varepsilon(s, \pi_v \times \pi'_v, \psi_v). \]

The reason that \( \psi \) is not encoded in to the left hand side of this equation is that the right hand side is in fact independent of the choice of \( \psi \).

The following theorem collects the basic facts about Rankin-Selberg \( L \)-functions:

**Theorem 11.7.1** The Rankin-Selberg \( L \)-function admits a meromorphic continuation to the plane, holomorphic except for possible simple poles at \( s = 0, 1 \). There are poles at \( s = 0, 1 \) if and only if \( m = n \) and \( \pi \cong \pi'^\vee \). One has a functional equation

\[ L(s, \pi \times \pi') = \varepsilon(s, \pi, \pi')L(1 - s, \pi'^\vee \times \pi'^\vee). \]  

(11.17)

One extremely important consequence of this theorem is the strong multiplicity one theorem:

**Theorem 11.7.2 (Strong multiplicity one)** Let \( \pi \) and \( \pi' \) be cuspidal automorphic representations of \( \text{GL}_n(\mathbb{A}_F) \) and let \( S \) be a finite set of places of \( F \). If \( \pi^S \cong \pi'^S \) then \( \pi \cong \pi' \).

We leave the proof as Exercise 11.6. It is a consequence of Proposition 11.5.1, Theorem 11.5.1 and Theorem 11.7.1. Despite its name Theorem 11.7.2 does not imply the Theorem 11.3.3 above. For a beautiful generalization of the strong multiplicity one theorem we refer the reader to the work of Dinakar Ramakrishnan on automorphic analogues of the Chebatarev density theorem.

Let \( \varphi \) be in the space of \( \pi \) and \( \varphi' \) in the space in \( \pi' \). We outline the basic idea behind this theorem in the case \( m = n - 1 \). We will not justify any of the convergence statements made in our outline. For full details we refer the reader to [JPSS83].

In this case we take

\[ I(s; \varphi, \varphi') := \int_{\text{GL}_{n-1}(F) \backslash \text{GL}_{n-1}(\mathbb{A}_F)} \varphi(h) \varphi'(h) |\det h|^{s-1/2} dh. \]  

(11.18)

Let \( \overline{\varphi}(g) := \varphi(g^{-t}) \) and \( \overline{\varphi}'(g) := \varphi(g^{-t}) \). Taking a change of variables \( h \mapsto h^{-t} \) we see that

\[ I(s; \varphi, \varphi') = I(1 - s; \overline{\varphi}, \overline{\varphi'}). \]  

(11.19)

This simple change of variables is what ultimately powers the proof of theorem 11.7.1.

Define

\[ \Psi(s; W^\varphi \psi, W^{\varphi'} \psi') = \int_{\text{GL}_{n-1}(\mathbb{A}_F) \backslash \text{GL}_{n-1}(\mathbb{A}_F)} \psi^\varphi \left( \begin{pmatrix} h & \ 0 \\ 0 & 1 \end{pmatrix} \right) W^{\varphi'} \psi'(h) |\det h|^{s-1/2} dh. \]

This is a **global Rankin-Selberg integral**.
Theorem 11.7.3 For \( \text{Re}(s) \) sufficiently large \( I(s; \varphi, \varphi') = \Psi(s, W^{\varphi}_{\psi}, W^{\varphi'}_{\psi}) \).

In particular, the functional equation (11.19) immediately implies an analogous one for the global Rankin-Selberg integral \( \Psi(s, W^{\varphi}_{\psi}, W^{\varphi'}_{\psi}) \).

Proof. Replacing \( \varphi \) by its Whittaker expansion we have

\[
I(s; \varphi, \varphi') = \int_{\text{GL}_{n-1}(F) \setminus \text{GL}_{n-1}(\mathbb{A}_F)} \varphi(h) \varphi'(h) | \det h |^{s-1/2} dh
\]

where the inner sum is over \( \gamma \in \text{GL}_{n-1}(F) \setminus \text{GL}_{n-1}(\mathbb{A}_F) \). Since \( \varphi'(h) \) is automorphic on \( \text{GL}_{n-1}(\mathbb{A}_F) \) and \( | \det(\gamma) | = 1 \) for \( \gamma \in \text{GL}_{n-1}(F) \), we may interchange the order of summation and integration and obtain

\[
I(s; \varphi, \varphi') = \int_{\text{GL}_{n-1}(F) \setminus \text{GL}_{n-1}(\mathbb{A}_F)} W^{\varphi}_{\psi}(u) \varphi'(h) | \det h |^{s-1/2} dh.
\]

This integral is absolutely convergent for \( \text{Re}(s) \gg 0 \) which justifies the interchange.

Let us first integrate over \([N_{n-1}]\). View \( N_{n-1} \hookrightarrow N_n \) as matrices of the form \( (u \ 1) \). Then for \( u \in N_{n-1}(\mathbb{A}_F) \) one has \( W^{\varphi}_{\psi}(u(g) = \psi(u) W^{\varphi}_{\psi}(g) \).

\[
I(s; \varphi, \varphi') = \int_{N_{n-1}(\mathbb{A}_F) \setminus N_{n-1}(\mathbb{A})} \int_{[N_{n-1}]} W^{\varphi}_{\psi}(u) \varphi'(uh) du | \det h |^{s-1/2} dh
\]

\[
= \int_{N_{n-1}(\mathbb{A}_F) \setminus N_{n-1}(\mathbb{A})} W^{\varphi}_{\psi}(h) \int_{[N_{n-1}]} \psi(u) \varphi'(uh) du | \det h |^{s-1/2} dh
\]

\[
= \int_{N_{n-1}(\mathbb{A}_F) \setminus N_{n-1}(\mathbb{A})} W^{\varphi}_{\psi}(h) W^{\varphi'}_{\psi}(h) | \det h |^{s-1/2} dh
\]

\[
= \Psi(s; W^{\varphi}_{\psi}, W^{\varphi'}_{\psi}).
\]

\( \square \)

In the case \( m = n - 1 \) above Theorem 11.7.1 can be deduced from theorems 11.5.3 and 11.7.3; we leave this as Exercise 11.7.

11.8 The nongeneric case

In the theory above crucial use was made of the fact that cuspidal automorphic representations are globally generic. Indeed, our definition of local \( L \)-functions was given in terms of the Whittaker model.
We now explain how to handle the general case. Assume first that $F$ is a local field. From §10.6 we know that every pair of admissible irreducible representation $\pi$ of $\text{GL}_n(F)$ and $\pi'$ of $\text{GL}_m(F)$ can be written as an isobaric sum

$$\pi \cong \bigoplus_{i=1}^{k} \pi_i$$

and

$$\pi' \cong \bigoplus_{j=1}^{k'} \pi'_i,$$

where the $\pi_i$ and $\pi'_i$ are essentially square integrable and hence generic [JS83, §1.2].

We then define, for nontrivial characters $\psi : F \to \mathbb{C}^\times$, that

$$L(s, \pi \times \pi') = \prod_{i=1}^{k} \prod_{j=1}^{k'} L(s, \pi_i \times \pi'_j),$$

$$\gamma(s, \pi \times \pi', \psi) = \prod_{i=1}^{k} \prod_{j=1}^{k'} \gamma(s, \pi_i \times \pi'_j, \psi),$$

$$\varepsilon(s, \pi \times \pi', \psi) = \prod_{i=1}^{k} \prod_{j=1}^{k'} \varepsilon(s, \pi_i \times \pi'_j, \psi).$$

Of course, one has to check that this is consistent with our earlier definitions. In other words, we must know that if $\pi$ and $\pi'$ are generic, then this procedure gives the same result as if we used our original definition of the local factors. This can be done.

We then adopt the analogous convention in the global case and obtain Rankin-Selberg integrals for isobaric automorphic representations. The analytic properties of these $L$-functions can be read off from the corresponding ones for cuspidal representations (see Theorem 11.7.1).

### 11.9 Converse theory

Let $\pi$ be an admissible irreducible representation of $\text{GL}_n(\mathbb{A}_F)$. If $\pi$ is a cuspidal automorphic representation then for all cuspidal automorphic representations $\pi'$ of $\text{GL}_m(\mathbb{A}_F)$ we have seen in Theorem 11.7.1 that the $L$-function $L(s, \pi \times \pi')$ admits a functional equation and a meromorphic continuation to the plane that is bounded in vertical strips. Converse theory asks if automorphic representations can be characterized by the analytic properties of their $L$-functions. It has been used to establish cases of Langlands functoriality [CKPSS04]. In addition, it is philosophically important because it says that an $L$-function that satisfies reasonable assumptions has to come from an automorphic representation, solidifying the tight connection between these two objects.
Theorem 11.9.1 Let $\pi$ be an irreducible admissible representation of $\text{GL}_n(\mathbb{A}_F)$. Assume that for all cuspidal automorphic representations $\pi'$ of $\text{GL}_m$ with $1 \leq m \leq \max(n-2,1)$ the $L$-functions $L(s, \pi \times \pi')$ and $L(s, \pi' \times \pi'')$ have analytic continuations to the plane that are holomorphic and bounded in vertical strips of finite width. Assume moreover that for all $\pi'$ as above

$$L(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi') L(1-s, \pi'' \times \pi'').$$

Then $\pi$ is a cuspidal automorphic representation. \qed

Here to define $\varepsilon(s, \pi \times \pi')$ we fix a nontrivial character $\psi : F \backslash \mathbb{A}_F \to \mathbb{C}^\times$ and formally define

$$\varepsilon(s, \pi \times \pi') := \prod_v \varepsilon(s, \pi_v \times \pi'_v, \psi_v). \quad (11.20)$$

We refer to [CPS99] for a proof of Theorem 11.9.1 and for variants.

**Exercises**

11.1. Let $B_1, B_2 \leq \text{GL}_n$ be Borel subgroups defined over a nonarchimedean local field $F$. For $1 \leq i \leq 2$ let $N_i$ be the unipotent radical of $B_i$ and let $\psi_i$ be a generic character of $N_i(F)$. Prove that an irreducible admissible representation $\pi$ of $\text{GL}_n(F)$ is $\psi_1$-generic if and only if it is $\psi_2$-generic.

11.2. Prove Corollary 11.3.1.

11.3. Let $G$ be a split reductive group over a global field $F$ and assume that $\pi$ is a cuspidal globally $\psi$-generic representation of $V$. If $\pi \cong \otimes_v \pi_v$ prove that each $\pi_v$ is $\psi_v$-generic.

11.4. Prove Theorem 11.4.1 when $n = 2$.

11.5. Assume that $J(\lambda)$ is irreducible unitary unramified generic representation of $\text{GL}_n(F)$ where $\lambda \in \mathfrak{a}_T^\circ \mathbb{C}$. Let $\psi : F \to \mathbb{C}^\times$ be unramified, let $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\pi', \psi')$ be the unique spherical Whittaker functionals satisfying $W(I_n) = W'(I_n) = 1$. Using the identity

$$\det(I - q^{-s} e^\lambda \otimes e^{-\lambda}) \Psi(s, W, W', \mathbb{I}_{\mathcal{O}_F}) = 1$$

and Proposition 11.5.1 prove that every eigenvalue of $e^\lambda$ lies in $(q^{-1/2}, q^{1/2})$.

11.6. If $\pi$ and $\pi'$ are cuspidal automorphic representations of $\text{GL}_n(\mathbb{A}_F)$ such that $\pi^S \cong \pi'^S$ for some finite set of places $S$ of $F$ then $\pi \cong \pi'$.

11.7. Prove Theorem 11.7.1 in the case $m = n - 1$ using Theorem 11.7.3 and the local functional equations of Theorem 11.5.3.
References


