Chapter 1
Algebraic Groups

Before Genesis, God gave the devil free reign to construct much of creation, but dealt with certain matters himself, including semi-simple groups.

attributed to Chevalley by Harish-Chandra

Abstract In this chapter we briefly recall some of the basic notions related to affine algebraic groups. If the reader knows the definition of a reductive algebraic group (which can be found in §1.5) then the rest of the chapter can probably be skipped and then referred to later as needed.

1.1 Introduction

Someone beginning to learn the theory of automorphic representations does not need to internalize the content of this chapter. For the most part, one can get by with much less. For example, one can think of $GL_n$, which is really an affine group scheme over the integers, as a formal notational device already familiar from an elementary course in algebra: for commutative rings $R$ the notation $GL_n(R)$ refers to the invertible matrices with coefficients in $R$. Likewise, one knows that, e.g., $Sp_{2n}(R)$ is the group of matrices with coefficients in $R$ that fix the standard symplectic form on $R^{2n}$. One can think of an reductive group $G$ as some gadget of this type that assigns a group to every ring of some type (say $\mathbb{Q}$-algebras) and behaves like $GL_n$. In fact, for a first reading of this book, whenever one sees the assumption that $G$ is a reductive group one could assume that $G = GL_n$ for some integer $n$ and not lose a lot of the flavor.
That said, at some point it is not a bad idea for any student of automorphic representation theory (and really, any mathematician) to come to grips with the modern terminology of algebraic geometry and algebraic groups. It is useful, beautiful, and despite its reputation not really that counterintuitive or difficult. Thus we will use it in this chapter, mostly as a means of fixing notation and conventions, and make minor use of it later in the book.

Much of the material in this chapter can be found in one form or another in the references in algebraic groups [Bor91] [Hum75] [Spr09]. Unfortunately they were all written in the archaic language of Weil used before Grothendieck’s profound reimagining of the foundations of algebraic geometry. The most complete reference is [DG74], but it is hard to penetrate if one doesn’t have solid preparation in algebraic geometry. The reference [Wat79] is more accessible, but covers less. We also mention that J. Milne is currently reworking the three standard references in modern language. Hopefully his online notes (which were instrumental in the preparation of this chapter) will be formally published soon.

1.2 Affine schemes

Throughout this chapter we let $k$ be a commutative ring with identity. Let $\text{Alg}_k$ denote the category of commutative $k$-algebras with identity. For a $k$ algebra $A$, one obtains a functor

$$\text{Spec}(A) : \text{Alg}_k \to \text{Set}$$

$$R \mapsto \text{Hom}_{\text{Alg}_k}(A, R).$$

Moreover, if $\phi : A \to B$ is a morphism of $k$-algebras, then we obtain for each $k$-algebra $R$ a map

$$\text{Spec}(B)(R) \to \text{Spec}(A)(R)$$

$$\phi \mapsto \phi \circ \phi.$$

This collection of maps is an example of what is known as a natural transformation of functors. More precisely, to give a natural transformation $X \to Y$ of set-valued functors on $\text{Alg}_k$ is the same as giving for each $k$-algebra $R$ a map

$$X(R) \to Y(R)$$

such that for all morphisms of $k$-algebras $R \to R'$ the following diagram commutes:
One can check that (1.2) is a natural transformation of functors.

**Definition 1.1.** A **affine scheme over** \( k \) is a functor on the category of \( k \)-algebras of the form \( \text{Spec}(A) \). A **morphism of affine schemes** is a natural transformation of functors.

This definition gives us a category \( \text{AffSch}_k \) of affine schemes over \( k \), equipped with a contravariant anti-equivalence of categories

\[
\text{Spec} : \text{Alg}_k \longrightarrow \text{AffSch}_k.
\]

In other words, the category of affine schemes over \( k \) is the category of \( k \)-algebras “with the arrows reversed.” If \( k \) is understood we often omit explicit mention of it. By way of terminology, if \( X \) is an affine scheme then

\[
X(R)
\]

is called its \( R \)-valued points.

**Definition 1.2.** A functor \( S : \text{Alg}_k \longrightarrow \text{Set} \) is **representable** by a ring \( A \) if \( S = \text{Spec}(A) \). In this case we write

\[
\mathcal{O}(S) := A
\]

and refer to it as the **coordinate ring of** \( S \).

To ease comparison with other references, we note that we are defining schemes using their functors of points, whereas the usual approach is to define schemes as a topological space with a sheaf of rings on it satisfying certain desiderata and then associate a functor of points to the scheme. The two approaches are equivalent. The usual approach is desirable for many purposes, but the approach via the functor of points is more suitable for the study of algebraic groups. For more details on the usual approach and on the scheme theoretic concepts mentioned below we point the reader to [Har77], [EH00], [Mum99], [GW10]. The functor of points approach is used in [DG74].

In order to work with affine schemes in practice one must usually impose additional restrictions on the representing ring \( A \).

**Definition 1.3.** An affine scheme \( \text{Spec}(A) \) is **of finite type** if \( A \) is a finitely generated \( k \)-algebra.

Therefore \( \text{Spec}(A) \) is of finite type if and only if

\[
A \cong k[t_1, \ldots, t_n]/(f_1, \ldots, f_m)
\]
for some (finite) set of indeterminates \( t_1, \ldots, t_n \) and finite set of polynomials \( f_1, \ldots, f_m \). An important example of a scheme of finite type is affine \( n \)-space:

\[
\mathbb{G}^n_a := \text{Spec}(k[t_1, \ldots, t_n]).
\]

This is also denoted by \( \mathbb{A}^n \), but we avoid this notation because we will use the symbol \( \mathbb{A} \) for the adeles (which will be defined in §??).

Here are two other nice properties that the schemes of interest to us will often enjoy:

**Definition 1.4.** An affine scheme \( X \) is **reduced** if \( \mathcal{O}(X) \) has no nilpotent elements and **irreducible** if \( \mathcal{O}(X) \) has a unique minimal prime ideal, or, equivalently, if its nilradical is prime.

It is important to note that an affine scheme can be both reduced and irreducible (Exercise 1.4).

Assume for the moment that \( k \) is a field. An affine scheme \( X \) is **smooth** of dimension \( d \) if it is isomorphic to an affine scheme

\[
\text{Spec}(k[t_1, \ldots, t_n]/(f_1, \ldots, f_{n-d}))
\]

where the ideal in \( k[t_1, \ldots, t_n] \) generated by the \( f_i \) and all the \((n-d) \times (n-d)\) minors of the matrix of derivatives \( \partial f_i/\partial x_i \) is the whole ring \( k[t_1, \ldots, t_n] \).

We close this section by discussing closed subschemes.

**Definition 1.5.** A morphism of schemes

\[
X \longrightarrow Y
\]

is a **closed immersion** if the associated map \( \mathcal{O}(Y) \longrightarrow \mathcal{O}(X) \) is surjective.

**Remark 1.1.** If \( X \rightarrow Y \) is a closed immersion then \( X(R) \rightarrow Y(R) \) is injective for all \( k \)-algebras \( R \) (see Exercise 1.5).

If \( X \rightarrow Y \) is a closed immersion then \( \mathcal{O}(X) \cong \mathcal{O}(Y)/I \) for some ideal \( I \) of \( \mathcal{O}(Y) \). Motivated by this observation, one defines a **closed subscheme** of the scheme \( Y \) to be a scheme of the form \( \text{Spec}(\mathcal{O}(Y)/I) \) for some ideal \( I \subseteq \mathcal{O}(Y) \). Thus schemes of finite type over \( k \) are all closed subschemes of \( \mathbb{G}^n_a \) for some \( n \). One thinks of the closed subscheme \( k[t_1, \ldots, t_n]/(f_1, \ldots, f_m) \) as being the zero locus of \( f_1, \ldots, f_m \), although some care is required when applying this intuition when \( k \) is not an algebraically closed field, or even a field for that matter.

The definition of a closed subscheme can also be given from a topological perspective as we now explain. There is a topological space attached to \( \text{Spec}(A) \) which is also denoted by \( \text{Spec}(A) \). It is the collection of all primes \( p \subseteq A \) equipped with the **Zariski topology**. To define this topology, for every subset \( S \subseteq A \) we let
1.2 Affine schemes

\[ V(S) = \{ p \in \text{Spec}(A) : p \supseteq S \}. \]

One checks that the sets \( V(S) \) are closed under infinite intersections and finite unions. Thus we can define the Zariski topology on \( \text{Spec}(A) \) to be the topology with \( V(S) \) as its closed sets. A morphism of schemes induces a morphism of the associated topological spaces, and the image of an injective morphism

\[ \text{Spec}(A/I) \rightarrow \text{Spec}(A) \]

is \( V(I) \) (see Exercise 1.1).

We close this section with the notion of fiber products. This is a simple and useful method to create new schemes from existing ones. Let

\[ X = \text{Spec} A, \quad Y = \text{Spec} B, \quad Z = \text{Spec} C \]

be \( k \)-schemes equipped with morphisms \( f : X \rightarrow Y \) and \( g : Z \rightarrow Y \). We can then form the fiber product

\[ X \times_Y Z := \text{Spec}(A \otimes_B C) \]

where \( A \) and \( C \) are regarded as \( B \)-algebras via the maps \( B \rightarrow A \) and \( B \rightarrow C \) induced by \( f \) and \( g \), respectively. This scheme has the property that for \( k \)-algebras \( R \) one has

\[ (X \times_Y Z)(R) = X(R) \times_{Y(R)} Z(R) \]

where the fiber product on the right is that in the category of sets:

\[ X(R) \times_{Y(R)} Z(R) := \{ (x, z) \in X(R) \times Z(R) : f(x) = g(z) \}. \quad (1.5) \]

By far the case that will be used the most frequently in the sequel is the so-called absolute product in the category of \( k \)-schemes. To define it note that every \( k \)-scheme \( X = \text{Spec} A \) comes equipped with a morphism \( X \rightarrow \text{Spec} k \), namely the morphism induced by the ring homomorphism \( k \rightarrow A \) that gives \( A \) the structure of a \( k \)-algebra. This morphism \( X \rightarrow \text{Spec} k \) is called the structure morphism. Thus for any pair of \( k \)-schemes \( X, Z \) we can form the absolute product

\[ X \times Z := X \times_k Z := X \times_{\text{Spec} k} Z. \]

It follows from (1.5) that for \( k \)-algebras \( R \),

\[ (X \times Z)(R) = X(R) \times Z(R). \]
1.3 Group schemes

Definition 1.6. An **affine group scheme** over $k$ is a functor
\[ \text{Alg}_k \rightarrow \text{Group} \]
representable by a $k$-algebra. A **morphism** of affine group schemes $H \rightarrow G$ is a natural transformation of functors from $H$ to $G$.

Here a natural transformation of functors is defined as in the previous section, though this time we require the associated maps
\[ X(R) \rightarrow Y(R) \]
to be group homomorphisms, not just maps of sets. It is clear that an affine group scheme over $k$ is in particular an affine scheme over $k$. In this book we will only be interested in affine group schemes (as opposed to elliptic curves, for instance), so we will often omit the word “affine.”

The most basic examples of affine group schemes are the additive and multiplicative groups:

**Example 1.1.** The **additive group** $\mathbb{G}_a$ is the functor assigning to each $k$-algebra $R$ its additive group,
\[ \mathbb{G}_a(R) := (R, +). \]
It is represented by the polynomial algebra $k[x]$:
\[ \text{Hom}_k(k[x], R) = R. \]

**Example 1.2.** The **multiplicative group** $\mathbb{G}_m$ is the functor assigning to each each $k$-algebra $R$ its multiplicative group,
\[ \mathbb{G}_m(R) = R^\times. \]
It is represented by $k[x, y]/(xy - 1)$.

These affine group schemes are both abelian in the sense that their $R$-valued points are always abelian. The most basic example of a nonabelian group scheme is the general linear group $\text{GL}_n$. It is the functor taking a $k$-algebra $R$ to the group of $n \times n$ invertible matrices $(x_{ij})$ with coefficients $x_{ij}$ in $R$. The representing ring is
\[ k[x_{ij}][y]/(\det(x_{ij}) \cdot y - 1). \]
Note that $\text{GL}_1 = \mathbb{G}_m$. If one wishes to be coordinate free, then for any finite rank free $k$-module $V$ one can define
GL_V(R) := \{R-module automorphisms V \rightarrow V\}.

A choice of isomorphism V \cong k^n induces an isomorphism GL_V \cong GL_n.

We isolate a particularly important class of morphisms with the following definition:

**Definition 1.7.** A representation of an affine group scheme G is a morphism r : G \rightarrow GL_V. It is faithful if it is injective.

**Definition 1.8.** A group scheme G is said to be linear if it admits a faithful representation G \rightarrow GL_V for some V.

We will usually be concerned with linear group schemes. We shall see below in Theorem 1.5.1 that this is not much loss of generality if k is a field.

1.4 Extension and restriction of scalars

Let k \rightarrow k' be a homomorphism of rings. Given a k-algebra R, one obtains a k'-algebra R \otimes_k k'. Moreover, given a k' algebra R', one can view it as a k-algebra in the tautological manner. This gives rise to a pair of functors

\[ \text{Alg}_k \rightarrow \text{Alg}_{k'} \quad \text{and} \quad \text{Alg}_{k'} \rightarrow \text{Alg}_k \]

known as base change and restriction of scalars, respectively. These functors are useful in that they allow us to change the base ring k.

For affine schemes we similarly have a base change functor

\[ k' : \text{AffSch}_k \rightarrow \text{AffSch}_{k'} \]

given by \( X_{k'}(R') = X(R') \); the ring representing \( X_k \) is simply \( O(X) \otimes_k k' \). In fact this is a special case of the fiber product construction of \( \S 1.2 \); in the notation of that section one takes \( Y = \text{Spec} k \) and \( Z = \text{Spec} k' \) and let us equipped with the map \( Z \rightarrow Y \) induced by \( k \rightarrow k' \).

The functor in the opposite direction is a little more subtle. One can always define a set valued functor

\[ \text{Res}_{k' / k} X'(R) := X'(k' \otimes_k R) \]

called the (Weil) restriction of scalars. A priori this is just a set valued functor on \( \text{Alg}_k \), but if \( k'/k \) satisfies certain conditions then it is representable, and hence we obtain a functor

\[ \text{Res}_{k' / k} : \text{AffSch}_{k'} \rightarrow \text{AffSch}_k. \]

For example, it is enough to assume that \( k'/k \) is a field extension of finite degree, or more generally that \( k'/k \) is finite and locally free [BLR90, Theorem 4, \( \S 7.6 \)].
Example 1.3. The **Deligne torus** is

\[ S := \text{Res}_{\mathbb{C}/\mathbb{R}} \text{GL}_1. \]

One has \( S(\mathbb{R}) = \mathbb{C}^\times \) and \( S(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times \). If \( V \) is a real vector space, then to give a representation \( S \to \text{GL}(V) \) is equivalent to giving a Hodge structure on \( V \) [Mil05].

Example 1.4. Let \( d \) be a square free integer and \( L = \mathbb{Q}(\sqrt{d}) \). If we view \( L \) as a two-dimensional vector-space over \( \mathbb{Q} \) one obtains an embedding of \( L^\times \) into \( \text{GL}_2(\mathbb{Q}) \). This can be refined into an embedding

\[ \text{Res}_{L/\mathbb{Q}}(\mathbb{G}_m) \hookrightarrow \text{GL}_2 \]

of reductive groups over \( \mathbb{Q} \). One can choose the embedding so the \( R \)-valued points of its image is

\[ \{ \left( \begin{array}{cc} a & db \\ b & a \end{array} \right) \in \text{GL}_2(R) \} \]

for \( \mathbb{Q} \)-algebras \( R \).

1.5 Reductive groups

We now assume that \( k \) is a field and let \( k^{\text{sep}} \leq \overline{k} \) be a separable (resp. algebraic) closure of \( k \). Much of the theory simplifies in this case.

One has the following definition:

**Definition 1.9.** An (affine) **algebraic group** over \( k \) is an affine group scheme of finite type over \( k \).

In other words \( G \) is algebraic if \( G \) is represented by a quotient of \( k[x_1, \ldots, x_n] \) for some \( n \) (namely \( \mathcal{O}(G) \)). There is a natural action of \( G \) on \( \mathcal{O}(G) \); if we think of \( \mathcal{O}(G) \) as functions on \( G \), then the action is that induced by right translation. For any \( k \)-subspace of \( \mathcal{O}(G) \) invariant under \( G \) we obtain a representation of \( G \), and this representation can be chosen to be faithful. If one fills in the details of this discussion one obtains the following theorem [Wat79, §3.4]:

**Theorem 1.5.1** An algebraic group over \( k \) admits a faithful representation, and hence is linear. \( \square \)

We now discuss the Jordan decomposition. Let \( M_n \) denote the scheme of \( n \) by \( n \) matrices over \( k \). One reference for the following is [Wat79, Chapter 9].

**Definition 1.10.** Let \( k \) be a perfect field. An element \( x \in M_n(k) \) is said to be **semi-simple** if there exists \( g \in \text{GL}_n(k) \) such that \( g^{-1}xg \) is diagonal, **nilpotent** if there exists \( n \in \mathbb{N} \) such that \( x^n = 0 \), and **unipotent** if \( (x-I) \) is
nilpotent. For an arbitrary linear group we say that an element $x \in G(k)$ is **semi-simple**, (resp. **unipotent**) if $r(x)$ is so for some faithful representation $r : G \to \text{GL}_V$.

Of course one has to check that these notions are independent of the choice of representation $r$, but this can be done.

**Theorem 1.5.2 (Jordan decomposition)** Let $G$ be an algebraic group over a perfect field $k$. Given $x \in G(k)$ there exist $x_s, x_u \in G(k)$ such that $x_s$ is semi-simple, $x_u$ is unipotent, $x = x_s x_u = x_u x_s$. Moreover, this decomposition is unique. □

A nice feature of algebraic groups is that they are often smooth (see [Wat79, Chapter 11]):

**Theorem 1.5.3** An algebraic group $G$ over a field $k$ is smooth if $k$ has characteristic zero or if $k$ is perfect of characteristic $p > 0$ and $G$ is reduced.

We will also require the notion of connectedness. An affine scheme is **connected** if the only idempotents in $\mathcal{O}(X)$ are 0 and 1. A group scheme is connected if its underlying topological spaces it connected. Though it is not true for general affine schemes, a group scheme is connected if and only if $G(\overline{k})$ is irreducible [Wat79, §6.6]. Using this fact one shows that if $k$ is a subfield of $\mathbb{C}$, then $G$ is connected if and only if $G(\mathbb{C})$ is connected as a topological space.

We now come to the notions of solvable and unipotent groups. An algebraic group over $k$ is **unipotent** if every representation of $G$ has a fixed vector. This is equivalent to the requirement that $G(\overline{k})$ consists of unipotent elements [Wat79, §8.3].

To define solvable groups we recall that the **derived subgroup** $G^{\text{der}} := D G$ of an algebraic group $G$ is the intersection of all normal subgroups $N \leq G$ such that $G/N$ is commutative. One has

$$G^{\text{der}}(\overline{k}) = \{ xyx^{-1} y^{-1} : x, y \in G(\overline{k}) \}.$$ 

In analogy with the case of abstract groups, we define

$$D^n G := D(D^{n-1} G)$$

inductively for $n \geq 1$ and say that $G$ is **solvable** if $D^n G$ is the trivial group for $n$ sufficiently large.

**Definition 1.11.** Let $G$ be a smooth algebraic group. The **unipotent radical** $R_u(G)$ of $G$ is the maximal connected unipotent normal subgroup of $G$. The **(solvable) radical** is the maximal connected normal solvable subgroup of $G$.

We remark that since a unipotent group is always solvable we have $R_u(G) \leq R(G)$.  

Definition 1.12. A smooth connected algebraic group $G$ is said to be reductive if $R_u(G_{\mathbb{F}}) = \{1\}$ and semi-simple if $R(G_{\mathbb{F}}) = \{1\}$.

The most basic example of a reductive group is $GL_n$. It is not semi-simple since its center is normal and nontrivial. The subgroup $SL_n \leq GL_n$ is semi-simple (which implies reductive).

Remark 1.2. If $k$ is a perfect field, then $R_u(G_{\mathbb{F}}) = R_u(G_{\mathbb{F}})_{\mathbb{F}}$. However, this is false in general for nonperfect fields. For more details see [CGP10].

The group of upper triangular matrices in $GL_n$ is not reductive (as it is solvable). It is unipotent. We remark that unipotent groups are always upper-triangularizable; i.e. for any unipotent group $G$ there exists a representation $r : G \rightarrow GL_V$ such that the image of $G$ consists of upper triangular matrices by the Lie-Kolchin theorem [Wat79, Theorem 10.2].

Suppose that $k$ is a perfect field and $G$ is a reductive group over $k$. Then

$$G = Z_G G^\text{der}$$

where $Z_G \leq G$ is its center and $G^\text{der} \leq G$ is the derived subgroup of $G$. We note that since $G$ is reductive, $G^\text{der}$ is semi-simple. We also note that

$$Z_G \cap G^\text{der}$$

is the (finite) center of $G^\text{der}$. For example, when $G = GL_n$ the derived subgroup is $SL_n$ and the center $Z_G$ consists of the diagonal matrices.

We close this section by recalling the following theorem of Mostow [Mos56], which states that we can always break an algebraic group in characteristic zero into a reductive and unipotent part:

**Theorem 1.5.4** Let $G$ be an algebraic group over a characteristic zero field. Then there is a reductive subgroup such that $M \leq G$

$$G = MR_u(G).$$

All such subgroups are conjugate under $R_u(G)$. $\square$

The decomposition $G = MR_u(G)$ is called a **Levi decomposition** and $M$ is called a **Levi subgroup**. The Levi decomposition is often written $G = MN$, where $N$ is the unipotent radical.

### 1.6 Lie Algebras

Now that we have defined reductive groups, we could ask for a classification of them, or more generally for a classification of morphisms $H \rightarrow G$ of reductive groups. The first step in this process is to linearize the problem using objects
known as Lie algebras. We will return to the question of classification in §1.7 and Theorem 1.8.1 below.

**Definition 1.13.** Let $k$ be a ring. A **Lie algebra** (over $k$) is a free $k$-module $\mathfrak{g}$ of finite rank together with a bilinear pairing

$$\cdot \otimes \cdot : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$$

satisfying the following assumptions:

(a) $[X, X] = 0$ for all $X \in \mathfrak{g}$.
(b) For $X, Y, Z \in \mathfrak{g}$,

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0. \quad (1.7)$$

Morphisms of Lie algebras are simply $k$-module maps preserving $\cdot \otimes \cdot$.

The pairing $\cdot \otimes \cdot$ is known as the **Lie bracket** of the Lie algebra $\mathfrak{g}$ and the identity (1.7) is known as the **Jacobi identity**. We remark that if $R$ is a $k$-algebra then $\mathfrak{g} \otimes_k R$ inherits the structure of a Lie algebra over $R$ in a natural manner.

Let $\text{LAG}_k$ denote the category of linear algebraic groups over $k$ and let $\text{LieAlg}_k$ denote the category of Lie algebras over $k$. There exists a functor

$$\text{Lie} : \text{LAG}_k \rightarrow \text{LieAlg}_k$$

defined by

$$\text{Lie} G = \ker(G(k[t]/t^2) \rightarrow G(k)).$$

From the definition just given it is not clear that $\text{Lie} G$ is a Lie algebra, or even a $k$-algebra, for that matter. At the moment it is only a set-valued functor. There is one important special case where it is easy to deduce a Lie algebra structure. Let $G = \text{GL}_V$ for a free finite rank $k$-module $V$. In this case it is not hard to see that

$$\text{Lie} G = \{ I_V + X t : X \in \text{End}_{k-\text{linear}}(V) \} \rightarrow \text{End}_{k-\text{linear}}(V)$$

$$I_V + X t \mapsto X.$$

This Lie algebra is traditionally denoted $\text{gl}_V$, or simply $\text{gl}_n$ when $V = \mathbb{R}^n$.

One defines the bracket via

$$[X, Y] = X \circ Y - Y \circ X$$

and verifies that it satisfies the axioms for a Lie bracket.

In order to deduce that $\text{Lie} G$ is a Lie algebra in general we essentially just embed $\text{Lie} G$ in $\text{gl}_V$, but in a functorial way. The first step is to give $\text{Lie} G$ a $k$-algebra structure. Let
\[ \epsilon : \mathcal{O}(G) \rightarrow k \]

be the coidentity, i.e. the homomorphism corresponding to the identity element of \( G(k) \).

**Lemma 1.6.1** There is a canonical bijection
\[ \text{Lie} G \cong \text{Hom}_{k\text{-linear}}(\ker(\epsilon)/\ker(\epsilon)^2, k). \]

In particular \( \text{Lie} \) is a functor from \( \text{LAG}_k \) to the category of free \( k \)-modules of finite rank.

**Remark 1.3.** Using the \( k \)-algebra structure of \( \mathfrak{g} \) given above, one checks that for any \( k \)-algebra \( R \)
\[ \text{Lie} G(R) := (\text{Lie} G) \otimes_k R \]

**Proof.** An element of \( \text{Lie} G \) is an \( k \)-algebra homomorphism
\[ \varphi : \mathcal{O}(G) \rightarrow k[t]/t^2 \]
such that the composite with the natural map \( k[t]/t^2 \rightarrow k \) is \( \epsilon \). Thus \( \varphi \) maps the ideal \( \ker(\epsilon) \) into \( t \) and hence factors through \( \mathcal{O}(G)/\ker(\epsilon) \). Since
\[ \mathcal{O}(G)/\ker(\epsilon) = k \oplus \ker(\epsilon)/\ker(\epsilon)^2 \] (see Exercise 1.9) we obtain a bijection
\[ \text{Lie} G \rightarrow \text{Hom}_{k\text{-linear}}(\ker(\epsilon)/\ker(\epsilon)^2, k) \]
\[ \varphi \mapsto \varphi|_{\ker(\epsilon)/\ker(\epsilon)^2} \]
as claimed. We leave it to the reader to check the functoriality assertion. \( \square \)

The group \( G \) always admits a representation
\[ \text{Ad} : G \rightarrow \text{Aut}_{\text{Lie} G}. \]

To define it, note that for any \( k \)-algebra \( R \) the morphism \( R \rightarrow R[t]/t^2 \) giving \( R[t]/t^2 \) its \( R \)-algebra structure gives rise to a map \( G(R) \rightarrow G(R[t]/t^2) \). Thus the conjugation action of \( G(R) \) on itself gives rise to a conjugation action of \( G(R) \) on \( G(R[t]/t^2) \) which preserves \( \text{Lie} G(R) \). This is the action denoted \( \text{Ad} \).

We finally give the reader a definition of the bracket on \( \text{Lie} G \). For this we note that by functoriality of \( \text{Lie} \) as a \( k \)-module valued functor the morphism \( \text{Ad} \) gives rise to a morphism of free \( k \)-modules of finite rank
\[ \text{ad} : \text{Lie} G \rightarrow \text{Lie} \text{Aut}_{\text{Lie} G} = \mathfrak{gl}_{\text{Lie} G}. \]

We then define
\[ [A, X] = \text{ad}(A)(X). \]

One checks that this is indeed a Lie bracket and that when \( G = \text{GL}_V \) this recovers the earlier definition of the bracket.
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In practice, to compute the Lie algebra of a linear algebraic group $G$ one just chooses an embedding $G \rightarrow \text{GL}_V$ and computes $\text{Lie} G$ in terms of the conditions that cut the group $G$ from $\text{GL}_V$. One then obtains the bracket for free; it is just the restriction of the bracket on $\text{GL}_V$.

**Example 1.5.** If $G = \text{SL}_n$, the special linear group of matrices in $\text{GL}_n$ of determinant 1, then

$$\det(I_n + Xt) = 1 + \text{tr}(X)t \pmod{t^2}$$

by Taylor expansion, so

$$\mathfrak{sl}_n := \text{Lie} \text{SL}_n = \{ X \in \mathfrak{gl}_n : \text{tr}(X) = 0 \}.$$

Notice how the use of the Taylor expansion amounted to differentiating some equation once and then taking the zero term; this method of determining Lie algebras works for at least all the classical groups, which are roughly subgroups of $\text{GL}_V$ defined as the subgroup fixing various perfect pairings on $V$ (see Exercise 1.9).

### 1.7 Tori

Throughout this section we assume that $k$ is a field with separable closure $k^{\text{sep}}$ and algebraic closure $\bar{k}$.

**Definition 1.14.** An **algebraic torus** or simply **torus** is a linear algebraic group $T$ over $k$ such that $T_{k^{\text{sep}}} \cong \mathbb{G}_m^n$ for some $n$. The integer $n$ is called the **rank** of the torus.

We remark that if $T$ is a torus of rank $n$ over $k$ then there is a finite degree extension $L/k$ over which $T$ **splits**, in other words, $T_L \cong \mathbb{G}_m^n$ [Con14, Lemma B.1.5].

**Definition 1.15.** A **character** of an algebraic group $G$ is an element of $X^*(G) = \text{Hom}(G, \mathbb{G}_m)$. A **co-character** or one parameter subgroup is an element of $X_*(G) = \text{Hom}(\mathbb{G}_m, G)$.

For $k$-algebras $k'$ one usually abbreviates $X^*(G)_{k'} := X^*(G_{k'})$, etc. We warn the reader that usually the notation $X^*(G)$ is used for what we would call $X^*(G)_{\bar{k}}$. Therefore, if there is danger of confusion we will often write $X^*(G)_k$ for $X^*(G)$, though strictly speaking this is redundant.

One indication of the utility of the notion of characters is the following theorem [Wat79, §7.3]:

**Theorem 1.7.1** The association
defines a contravariant equivalence of categories between the category of algebraic tori defined over \( k \) and finite rank \( \mathbb{Z} \)-torsion free \( \mathbb{Z}[\text{Gal}(k^{\text{sep}}/k)] \)-modules.

We now record a few examples of tori.

**Example 1.6.** We define a special orthogonal group \( \text{SO}_2 \leq \text{GL}_2 \) by stipulating that for \( \mathbb{Q} \)-algebras \( R \) one has

\[
\text{SO}_2(R) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in R \text{ and } a^2 + b^2 = 1 \right\}.
\]

Over any field containing a square root of \(-1\) one has

\[
\frac{1}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \begin{pmatrix} a - bi \\ a + bi \end{pmatrix}.
\]

It follows that \( \text{SO}_2(\mathbb{Q}(i)) \cong \mathbb{G}_m \).

**Example 1.7.** Let \( L/k \) be a separable extension and let

\[
N_{L/k} : \text{Res}_{L/k} \mathbb{G}_m \rightarrow \mathbb{G}_m
\]

be the norm map; it is given on points by \( x \mapsto \prod_{\tau \in \text{Hom}_k(L, k)} \tau(x) \). Then the kernel of \( N_{L/k} \) is an algebraic torus. When \( L = \mathbb{Q}(i) \) and \( k = \mathbb{Q} \) this torus is isomorphic to the group \( \text{SO}_2 \) of the previous example.

**Example 1.8.** If \( L/k \) is any (separable) field extension then \( \text{Res}_{L/k}(\mathbb{G}_m) \) is an algebraic torus. Moreover one can show that:

\[
X^*(\text{Res}_{L/k}(\mathbb{G}_m))_L \cong \bigoplus_{\tau} \mathbb{Z}_\tau
\]

where the summation runs over the embeddings \( \tau : L \rightarrow k^{\text{sep}} \) of \( L \) into a \( k \).

In the special case where \( L/k \) is Galois the left hand side admits a natural action of \( \text{Gal}(L/k) \), and as a representation of \( \text{Gal}(L/k) \)

\[
X^*(\text{Res}_{L/k}(\mathbb{G}_m))_L \otimes_{\mathbb{Z}} \mathbb{C}
\]

is isomorphic to the induced representation \( \text{Ind}_{1}^{\text{Gal}(L/k)}(1) \), where \( 1 \) denotes the trivial representation (of the trivial group).

The examples above illustrate that though an algebraic torus \( T \) satisfies \( T_{k^{\text{sep}}} \cong \mathbb{G}_m^n \), it may not be the case that \( T \cong \mathbb{G}_m^n \). This motivates the following definition:

**Definition 1.16.** An algebraic torus \( T \) over a field \( k \) is said to be **split** if \( T \cong \mathbb{G}_m^n \) over \( k \) or if equivalently \( X^*(T)_k \cong \mathbb{Z}^{\text{rank}(T)} \). An algebraic torus \( T \) is said to be **anisotropic** if \( X^*(T)_k = \{\text{id}\} \).
Any torus $T$ can be decomposed as $T = T^{\text{ani}}T^{\text{spl}}$ where $T^{\text{spl}}$ is the maximal split subtorus, $T^{\text{ani}}$ is the maximal anisotropic subtorus, and $T^{\text{ani}} \cap T^{\text{spl}}$ is finite.

**Definition 1.17.** A torus $T \leq G$ is **maximal** if $T_K$ is maximal among all tori of $G_K$.

**Theorem 1.7.2** Every connected algebraic group admits a maximal torus. All maximal tori in $G(\mathbb{k})$ are conjugate under $G(\mathbb{k})$.

**Proof.** In the generality we are considering the first result is due to Grothendieck (see, e.g. [Con14, Appendix A]). The second is [Bor91, IV.11.3]. \qed

In view of the second assertion of Theorem 1.7.2, the rank of a maximal torus of $G$ is an invariant of $G$; it is known as the **rank** of $G$.

For the remainder of this section $G$ is a connected reductive group and $T \leq G$ is a maximal torus.

**Definition 1.18.** The **Weyl Group** of $T$ in $G$ is $W(G,T) := N_G(T)/Z_G(T)$, where $N_G(T)$ is the normalizer of $T$ in $G$ and $Z_G(T)$ is the centralizer of $T$ in $G$.

This brief definition hides some important subtleties as we now explain. A possible reference is [Con14, §3]. The basic reference for quotients by reductive groups in the affine case is [MFK94, §0]. We note that $N_G(T)$, $Z_G(T) \leq G$ are algebraic subgroups of $G$ and $W(G,T)$ is the scheme-theoretic quotient of $N_G(T)$ by $Z_G(T)$. It would take us too far afield to adequately explain what this means, but in terms of representing rings, we note that $Z_G(T)$, being a subgroup of $N_G(T)$, acts on $\mathcal{O}[N_G(T)]$ and one has

$$W(G,T) := N_G(T)/Z_G(T) := \text{Spec}(\mathcal{O}[N_G(T)]^{Z_G(T)}).$$

It turns out that $W(G,T)$ is again a smooth (even étale) group scheme of finite type over $k$. Moreover one has

$$W(G,T)(\overline{k}) = N_G(T)(\overline{k})/Z_G(T)(\overline{k}).$$  \hspace{1cm} (1.9)

To describe $W(G,T)(R)$ for general $k$-algebras $R$ is involved, but at least for subfields $k \leq L \leq \overline{k}$ we can describe it as follows. Since $N_G(T)$ and $Z_G(T)$ are schemes over $k$ the set $N_G(T)(\overline{k})$ comes equipped with an action of $\text{Gal}(\overline{k}/k)$ preserving $Z_G(T)(\overline{k})$; hence we also obtain an action of $\text{Gal}(\overline{k}/k)$ on $W(G,T)(\overline{k})$. Then

$$W(G,T)(L) = W(G,T)(\overline{k})^{\text{Gal}(\overline{k}/L)}.$$

A nice discussion of this point, which is a special case of what is known as Galois descent, is contained in [Mum99, §II.4]. We wish to discuss these concepts in the special case where $G = \text{GL}_n$. Before beginning we record the following definition:
Definition 1.19. The group $G$ is said to be split if there exists a maximal torus of $G$ that is split.

If $T \leq \text{GL}_n$ is the torus of diagonal matrices then $T$ is split, and hence so is $\text{GL}_n$. For any $k$-algebra $R$ the Weyl group $W(T, \text{GL}_n)(R)$ can be identified with the group of permutation matrices and hence in this case $W(T, \text{GL}_n)(R)$ is $S_n$, the symmetric group on $n$ letters.

In general, if $L/k$ is an étale $k$-algebra of rank $n$ (for example a field extension of degree $n$) then choosing a basis for $k$ we obtain an embedding

$$\text{Res}_{L/k} \mathbb{G}_m \rightarrow \text{GL}_n.$$ 

In particular if $L/k$ is Galois then $W(\text{GL}_n, T)(k) \cong \text{Gal}(L/k)$. Every maximal torus in $\text{GL}_n$ arises in this manner for some $L/k$ (compare Exercise 7.5).

After spending all this time discussing tori we give one important motivation for introducing them. Suppose that $G$ is a split reductive group and that $T \leq G$ is a split maximal torus. Given any representation

$$r : G \rightarrow \text{GL}_V$$

it follows from the fact that $T$ is abelian that we can decompose $V$ as

$$V = \bigoplus_{\alpha \in X^*(T)} V_{\alpha}$$

(1.10)

where $V_{\alpha} = \{v \in V : r(t).v = \alpha(t).v\}$. The $\alpha$ occurring in this decomposition are known as weights and the $V_{\alpha}$ are known as weight spaces.

One has the following basic theorem:

Theorem 1.7.3 The collection of weights $\alpha$ together with the weight multiplicities $\dim V_{\alpha}$ determine the representation $V$ up to isomorphism. \hfill $\square$

It is not hard to see that the set of weights is invariant under the natural action of $W(T, G)(k)$ on $X^*(T)$ which leaves the multiplicities of the weight spaces $\dim V_{\alpha}$ fixed.

In fact, there is a partial ordering one can put on the space $X^*(T)$ such that each representation of $G$ admits a highest weight with respect to the ordering, unique up to the action of $W(T, G)(k)$. Moreover, one can characterize the set of highest weights that can occur. This is known as Cartan-Weyl highest weight theory.

One key fact one can take from this discussion is that the representation theory of $G$ is determined by the restriction of the representation to a sufficiently large subgroup (in this case a maximal torus). Happily, the representation theory of a maximal torus is fairly simple and can be studied via linear algebra. The idea of studying representations by studying their restriction to subgroups is an incredibly useful one; it will come up in various guises throughout this book.
1.8 Root data

Let $G$ be a split connected reductive group over a perfect field $k$ and let $T \leq G$ be a split maximal torus. Our next goal is to associate to such a pair $(G, T)$ a root datum $\Psi(G, T) = (X, Y, \Phi, \Phi^\vee)$. The root datum is a refinement of the related notion of a root system (also defined below) that Demazure introduced to systematically keep track of the central torus of $G$ [DG74, Expose XXII]. The root datum characterizes $G$, and in fact Demazure proves that it characterizes (split) $G$ even in the case where $k$ is replaced by $\mathbb{Z}$. This in some sense completed work of Chevalley who proved the analogous statement for semi-simple groups over fields [Che58].

Let $g$ denote the Lie algebra of $G$. As we saw in §1.6 one has an adjoint representation
\[ \text{Ad} : G \longrightarrow \text{GL}(g). \] (1.11)

For example, when $G = \text{GL}_n$ this is the usual action of $\text{GL}_n$ on the space of $n \times n$ matrices $\mathfrak{gl}_n$ by conjugation.

Since $\text{Ad}(T)$ consists of commuting semi-simple elements the action of $T$ on $g$ is diagonalizable. For a character $\alpha \in X^*(T)$, let
\[ g_\alpha := \{ X \in g \mid \text{Ad}(t)X = \alpha(t)X \text{ for all } t \in T(k) \}. \] (1.12)

**Definition 1.20.** The nonzero $\alpha \in X^*(T)$ such that $g_\alpha \neq 0$ are called the roots of $T$ in $G$. We let $\Phi(G, T)$ be the (finite) set of all such roots $\alpha$, and call the corresponding $g_\alpha$ root spaces.

Just as in (1.10) we have a decomposition
\[ g = t \oplus \bigoplus_{\alpha \in \Phi(G, T)} g_\alpha \]
where $t := \text{Lie} T$. It turns out that each of the root spaces $g_\alpha$ are one dimensional.

One turns the set of roots $\Phi(G, T)$ into a combinatorial gadget as follows. Let
\[ V := \langle \Phi(G, T) \rangle \otimes_{\mathbb{Z}} \mathbb{R}, \]
where $\langle \Phi(G, T) \rangle \subset X^*(T)(k)$ denotes the ($\mathbb{Z}$-linear) span of $\Phi(G, T)$. The pair $(\Phi(G, T), V)$ satisfies remarkable symmetry properties that are axiomatized in the following:

**Definition 1.21.** Let $V$ be a finite dimensional $\mathbb{R}$-vector space, and $\Phi$ a subset of $V$. We say that $(\Phi, V)$ is a root system if the following three conditions are satisfied:
(a) $\Phi$ is finite, does not contain 0, and spans $V$;
(b) For each \( \alpha \in \Phi \) there exists a reflection \( s_\alpha \) relative to \( \alpha \) (i.e. an involution of \( V \) with \( s_\alpha(\alpha) = -\alpha \) and restricting to the identity on a subspace of \( V \) of codimension 1) such that \( s_\alpha(\Phi) = \Phi \);

(c) For every \( \alpha, \beta \in \Phi \), \( s_\alpha(\beta) - \beta \) is an integer multiple of \( \alpha \).

A root system \((\Phi, V)\) is said to be of rank \( \dim_\mathbb{R} V \), and is said to be reduced if for each \( \alpha \in \Phi \), \( \pm \alpha \) are the only multiples of \( \alpha \) in \( \Phi \).

We will not need the notion until §1.9, but a subset

\[ \Delta \subset \Phi \]  \hspace{1cm} (1.13)

is a base if it is a basis of \( V \) and each \( \alpha \in \Phi \) can be uniquely expressed as

\[ \sum_{i=1}^\ell c_i \alpha_i \]

where the \( c_i \)'s are all integers that have the same sign. We define

\[ \Phi^+ \subset \Phi \quad \text{(resp. } \Phi^- \subset \Phi) \]

to be the set of roots expressible as a sum of positive (resp. negative) linear combinations of the elements in the base. Then \( \Phi = \Phi^+ \uplus \Phi^- \).

Remark 1.4. One can also define a system of positive roots a priori and use it to define bases; see, for example, [Hum78, §10].

The Weyl group of \((\Phi, V)\) is the subgroup of \( \text{GL}(V) \) generated by the reflections \( s_\alpha \):

\[ W(\Phi, V) := \langle s_\alpha : \alpha \in \Phi \rangle \subseteq \text{GL}(V). \]

Proposition 1.8.1. If \((\Phi, V)\) is the root system associated with the split maximal torus \( T \leq G \) then \((\Phi, V)\) is reduced and

\[ W(\Phi, V) \cong W(G, T)(k). \]

It seems difficult to locate this result in the literature, so we provide the proof for those familiar with Galois cohomology (one nice reference for Galois cohomology is [Ser02]).

Proof. One has an exact sequence of pointed sets

\[ 1 \to Z_G(T)(k) \to N_G(T)(k) \to W(G, T)(k) \to \ker \left( H^1(k, Z_G(T)) \to H^1(k, N_G(T)) \right) \]

(compare Exercise 1.14). Since \( T \) is a maximal torus \( T = Z_G(T) \) [Hum75, §26.2], and by Hilbert’s theorem 90 we deduce that \( H^1(k, Z_G(T)) = 1 \). Thus \( W(G, T)(k) = N_G(T)(k)/Z_G(T)(k) \). On the other hand, it is known that if \( T \) is split then
Let \((\Phi, V)\) be the root system associated with \(T \subset G\). There exists a pairing
\[
(\ , \ ) : V \times V \to \mathbb{C}
\]
for which the elements in the Weyl group become orthogonal transformations. Thus if \(\alpha \in \Phi\) there exists a unique \(\alpha^\vee \in X_*(T)\) such that
\[
\langle - , \alpha^\vee \rangle := \alpha^\vee(-) = \frac{2(-,\alpha)}{(\alpha,\alpha)}
\]
as maps \(X^*(T) \to \mathbb{C}\). Let \(\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\}\) and \(V^\vee := \langle \Phi^\vee \rangle \otimes_{\mathbb{Z}} \mathbb{R}\).

**Lemma 1.8.1** The pair \((\Phi^\vee, V^\vee)\) is a root system.

A fundamental result (Theorem 1.8.1) is that the quadruple
\[
\Psi(G, T) = (X^*(T), X_*(T), \Phi, \Phi^\vee)
\]
attached to \(T \subset G\) contains enough information to characterize \(G\), at least over \(\overline{k}\). To be more precise suppose that we have a quadruple \((X, Y, \Phi, \Phi^\vee)\) consisting of a pair of free abelian groups \(X, Y\) with a perfect pairing \(\langle , \rangle : X \times Y \to \mathbb{Z}\) and a bijective correspondence
\[
\Phi \longrightarrow \Phi^\vee
\]
\[
\alpha \longmapsto \alpha^\vee.
\]
For each \(\alpha \in \Phi\) we let \(s_\alpha : X \to X\) and \(s_{\alpha^\vee} : Y \to Y\) be the endomorphism given by
\[
s_\alpha(x) := x - \langle x, \alpha^\vee \rangle \alpha
\]
\[
s_{\alpha^\vee}(y) := y - \langle y, \alpha \rangle \alpha^\vee.
\]

**Definition 1.22.** The quadruple \((X, Y, \Phi, \Phi^\vee)\) is a root datum if
(a) \(\langle \alpha, \alpha^\vee \rangle = 2\), and
(b) if for each \(\alpha \in \Phi\), then \(s_\alpha(\Phi) \subset \Phi\), and the group \(\langle s_\alpha \mid \alpha \in \Phi\rangle\) generated by \(\{s_\alpha\}\) is finite.

We say that a root datum is **reduced** if \(\alpha \in \Phi\) only if \(2\alpha \notin \Phi\).

An isomorphism of root data \((X, Y, \Phi, \Phi^\vee) \xrightarrow{\sim} (X', Y', \Phi', (\Phi')^\vee)\) is a group isomorphism \(X \xrightarrow{\sim} X'\) inducing dual isomorphisms sending \(\Phi\) to \(\Phi'\) and \(\Phi^\vee\) to \((\Phi')^\vee\), respectively. We note that an isomorphism of reductive groups over \(k\) gives rise to an isomorphism of their underlying root data. Indeed, if \(G \to G'\) is an isomorphism of reductive groups over \(k\), then upon precomposing the
isomorphism with an inner automorphism of $G$ and postcomposing with an inner automorphism of $G'$ we can assume that the isomorphism maps a given maximal torus $T$ of $G$ to a given maximal torus $T'$ of $G'$, and hence induces an isomorphism of the underlying root data. The following result not only implies that every root datum comes from an algebraic group, it implies that every isomorphism of root data arises in this manner:

**Theorem 1.8.1 (Chevalley, Demazure)** Assume $k = \bar{k}$. The map

$$
\left\{ \text{isomorphism classes of connected reductive groups over } k \right\} \longrightarrow \left\{ \text{isomorphism classes of reduced root data} \right\}
G \mapsto \Psi(G, T)
$$

is bijective. Moreover, every isomorphism of root data $\Psi(G, T) \rightarrow \Psi(G', T')$ is induced by an isomorphism $G \rightarrow G'$ sending $T$ to $T'$, unique up to the conjugation actions of $T(k)$ and $T'(k)$.

This theorem tells us that if we can classify root data, then we can classify split reductive groups. In fact, Killing obtained a classification of the underlying root systems (up to some mistakes) in [Kil88] [Kil90]. Killing’s work was revisited and corrected in E. Cartan’s thesis. We will not discuss this theory more in this book, except to note that it is a triumph of mathematics.

If $(X, Y, \Phi, \Phi^\vee)$ is a root datum, then so is $(Y, X, \Phi^\vee, \Phi)$. The associated reductive algebraic group over $\mathbb{C}$ is denoted $\hat{G}$ and is called the **complex dual of** $G$. We note that there is an isomorphism

$$W(G, T)_k \longrightarrow W(\hat{G}, \hat{T})(\mathbb{C}) \quad (1.14)
$$

$s_\alpha \mapsto s_{\alpha^\vee}$.

The complex dual $\hat{G}$ is a key component in the definition of Langlands dual group (see §12.3).

One might ask if one could define in a natural way a morphism of root data, and thereby use root data to classify morphisms between reductive groups. If such a definition exists, we do not know it, and there are reasons to be pessimistic. However, it is the case that a great deal of information about morphisms between reductive groups can be deduced by considering root data. A systematic account of this for classical groups is given in Dynkin’s work [Dyn52].

We now record arguably the most basic example of a root datum:

**Example 1.9.** Let $G = GL_n$. The group of diagonal matrices

$$T(R) := \left\{ \begin{pmatrix} t_1 & \cdots & t_n \\ & \ddots & \\ & & t_n \end{pmatrix} \mid t_i \in R^\times \right\}
$$

is a maximal torus in $G$. The groups of characters and of cocharacters of $T$ are both isomorphic to $\mathbb{Z}^n$ via
1.9 Parabolic subgroups

We assume in this subsection that $G$ is a reductive group over a perfect field $k$ with algebraic closure $\overline{k}$.

Definition 1.23. A closed subgroup $B \leq G$ is a Borel subgroup if $B_{\overline{k}}$ is a maximal connected solvable subgroup of $G_{\overline{k}}$. A subgroup $P \leq G$ is a parabolic subgroup if $P_{\overline{k}}$ contains a Borel subgroup of $G_{\overline{k}}$.

The group $G_{\overline{k}}$ trivially has Borel subgroups, and hence $G$ always has at least one parabolic subgroup, namely $G$ itself. A proper parabolic subgroup is a parabolic subgroup of $G$ not equal to $G$. In general, a reductive connected algebraic group need not have proper parabolic subgroups. For example, if $B$ is a division algebra over $k$ and $G$ is the algebraic group defined by

$$G(R) = (B \otimes_k R)^{\times},$$

then $G$ does not have a Borel subgroup (Exercise 1.12).

It is useful to isolate two particular classes of groups:

Definition 1.24. A reductive group $G$ is said to be split if there exists a maximal split torus $T \subset G$ (over $k$); it is said to be quasi-split if it contains a Borel subgroup.
Note that $G$ is split only if it is quasi-split, but that the converse is not true. Indeed, take $G$ to be the unitary group $U(1,1)$ over the real numbers defined by
\[ G(R) = \{ g \in \text{GL}_2(\mathbb{C} \otimes_\mathbb{R} R) : \bar{g}^t (-1^{-1}) g = (-1^{-1}) \} \]
where the bar denotes complex conjugation. Then the subgroup of upper triangular matrices in $G$ is a Borel subgroup of $G$. Thus $G$ is quasi-split. It is not, however, split.

From the optic of finite-dimensional representation theory the behavior of a split reductive group is essentially as simple as a group over an algebraically closed field (compare §1.7). Quasi-split groups are a little more technical to handle, but the existence of the Borel subgroup makes the theory not much more difficult. The fact that a general reductive group does not have a Borel subgroup (over the base field) creates more substantial problems. In this case one has to do with a **minimal parabolic subgroup**. Of course, in the quasi-split case, a minimal parabolic subgroup is simply a Borel subgroup.

Basic facts about parabolic subgroups come up constantly in the theory of automorphic representations. For example, they are used to understand the structure of adelic quotients at infinity (see §2.7) and are used to describe the representation theory of a reductive group inductively (see §4.8, §8.2 and Chapter 10). Thus we record some of the basic structural facts about the set of parabolic subgroups of $G$ in this section. One reference is [Bor91, §20-21].

It is convenient to start with split tori in $G$. If there is no split torus contained in $G$ then $G$ is said to be **anisotropic**. In this case the only parabolic subgroup of $G$ is $G$ itself. Otherwise $G$ is said to be **isotropic**. If $G$ is isotropic then there exists a maximal split torus $T \leq G$ unique up to conjugation. Just as in the case when $G$ is split (discussed in §1.8), we can decompose $\mathfrak{g}$ under the adjoint action (1.11) into eigenspaces under $T$:
\[ \mathfrak{g} := \mathfrak{m} \oplus \bigoplus_{\alpha \in \Phi(G,T)} \mathfrak{g}_\alpha \]  
(1.16)
where $\mathfrak{g}_\alpha$ is defined as in (1.12). Here $\Phi(G, T)$ is set of nonzero weights occurring in the decomposition above, and $\mathfrak{m}$ is the 0 eigenspace. The set $\Phi(G, T)$ is usually referred to as the set of relative roots (or more precisely, roots relative to $k$ as opposed to $\overline{k}$), but we will avoid this terminology because we will later discuss relative trace formulae, which have nothing to do with this notion of relative. The set $\Phi(G, T) \subset X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ is a root system, but it is not in general reduced.

Just as all the maximal split tori in $G$ are conjugate under $G(k)$, the minimal parabolic subgroups are also conjugate under $G(k)$. Moreover, every minimal parabolic subgroup contains the centralizer $Z(T)$ of a maximal split torus $T \leq G$. Thus to describe all minimal parabolic subgroups it suffices to fix a maximal split torus $T \leq G$ and describe all minimal parabolic subgroups
containing $Z(T)$. Let $\Delta \subseteq \Phi(G, T)$ be a base. Then we can consider the set of positive roots $\Phi^+$ with respect to $\Delta$. Then there is a unique minimal parabolic subgroup $P_0 \geq Z(T)$ whose unipotent radical $N_0 \leq P_0$ satisfies

$$\text{Lie } N_0 = \bigoplus_{\alpha \in \Phi^+} g_\alpha.$$ 

**Theorem 1.9.1** The correspondence described above defines a bijection

$$\{\text{bases } \Delta \subseteq \Phi(G, T)\} \longrightarrow \{P_0 \geq Z(T) : P_0 \text{ minimal parabolic subgroups}\}.$$

If we fix a minimal parabolic subgroup $P_0$ then the parabolic subgroups containing $P_0$ are called **standard**. Every parabolic subgroup is conjugate under $G(k)$ to a unique standard parabolic subgroup, and two standard parabolic subgroups are $G(k)$-conjugate if and only if they are equal. Thus to describe all parabolic subgroups of $G$ it suffices to fix a minimal parabolic subgroup $P_0$ of $G$ and describe all standard parabolic subgroups (with respect to $P_0$).

To accomplish this, let $J \subseteq \Delta$ be a subset. Let

$$\Phi(J) := \{\alpha = c_\delta \delta : c_\delta \in \mathbb{Z}, c_\delta > 0 \text{ if } \delta \notin J\}$$

There is a unique parabolic subgroup $P_J \geq P_0$ with unipotent radical $N_J$ such

$$\text{Lie } N_J = \bigoplus_{\alpha \in \Phi(J)} g_\alpha.$$ 

**Theorem 1.9.2** There is a bijective correspondence

$$\{J \subset \Delta\} \longrightarrow \{\text{standard parabolic subgroups of } G\},$$

$$J \mapsto P_J.$$

The two bijections in theorems 1.9.1 and 1.9.2 allow us to define the notion of a parabolic subgroup opposite to a given parabolic subgroup. More precisely, suppose we are given a standard parabolic subgroup $P \leq G$ with unipotent radical $N$ containing the centralizer of a maximal split torus $T$ of $G: Z(T) \leq P \leq G$. Then one has an **opposite parabolic** $P^-$ constructed as follows: If $P = P_0$ is minimal, then we take $P_0^-$ to be the minimal parabolic subgroup attached to the base

$$-\Delta := \{\alpha \in \Phi : -\alpha \in \Delta\}.$$ 

If $P = P_J$, then we define $P^-$ to be the parabolic subgroup containing the minimal parabolic subgroup $P_0^-$ that is attached to the subset

$$-J := \{\alpha \in \Phi : -\alpha \in J\} \subseteq -\Delta.$$
It turns out that $P \cap P^- = M$, the Levi subgroup of $P$. The unipotent radical of $P^-$ is usually denoted $N^-$. 

**Example 1.10.** The subgroup $B \leq \text{GL}_n$ of upper triangular matrices is a Borel subgroup. Throughout this book when we speak of standard parabolic subgroups in $\text{GL}_n$ we will mean parabolic subgroups that are standard with respect to this choice of Borel subgroup. The base of $\Phi(G,T)$ and set of positive roots associated to $B$ are

$$\Delta := \{ e_{i,i+1} : 1 \leq i \leq n - 1 \}$$
$$\Phi^+ := \{ e_{i,j} : i < j \}$$

respectively. In particular there is a bijection

$$\{ \text{subsets of } \Delta \} \xrightarrow{\sim} \{ n_1, \ldots, n_d \in \mathbb{Z}_{>0} : \sum_{i=1}^d n_i = n \}$$

where $n_1$ is the first index such that $e_{n_1,n_1+1}$ is not an element of the subset, $n_2$ is the second index such that $e_{n_2,n_2+1}$ is not an element of the subset, etc. 

Unwinding these equivalences, we see that the standard parabolic subgroups of $\text{GL}_n$ correspond bijectively to ordered tuples of positive integers $n_1, \ldots, n_d$ such that $\sum_{i=1}^d n_i = n$. The corresponding parabolic subgroup is the product of $B$ and the block diagonal matrices of the form

$$\begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_d \end{pmatrix}$$

where $\gamma_i \in \text{GL}_{n_i}(\mathbb{R})$. We refer to this parabolic subgroup as the (standard) parabolic subgroup of type $(n_1, \ldots, n_d)$.

Finally, it is sometimes useful to generalize the notion of roots still further. If $T$ is any split torus in $G$, we can decompose $\mathfrak{g}$ into eigenspaces as before. We let $\Phi(G,T)$ be the set of nonzero eigenspaces. Suppose that $P \leq G$ is a parabolic subgroup and that $T \leq P$ is a maximal split torus in the center of $P$. In this case, the set of roots need not be a root system [Kna86, §V.5]. However, we can still define a partition $\Phi^+ \sqcup \Phi^- = \Phi(G,T)$ such that $\Phi^+$ is the set of roots contained in the unipotent radical of $P$. This is the set of **positive roots defined by $P$**. We will also require the notion of a positive Weyl chamber in this context. Let $\Lambda \subset X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ denote the union of the hyperplanes on which an element of $\Phi(G,T)$ vanishes. A **Weyl chamber** is a connected component of $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} \setminus \Lambda$; they are permuted transitively by $W(G,T)$. The **positive Weyl chamber** attached to $P$ is the unique Weyl
chamber whose elements are positive $\mathbb{R}$-linear combinations of the positive roots.

**Exercises**

1.1. Prove that a morphism of affine schemes induces a continuous map of the underlying topological spaces. Prove that if

$$\text{Spec}(A/I) \to \text{Spec}(A)$$

is an embedding then the image of the topological space of Spec($A/I$) in Spec($A$) is $V(I)$.

1.2. Assume that $X = \text{Spec}(\mathbb{C}[t_1, \ldots, t_n]/(f_1, \ldots, f_{n-d}))$ is a smooth scheme over the complex numbers. Show that $X(\mathbb{C})$ may be identified with a smooth complex manifold, namely the zero locus of the $f_i$.

1.3. Prove that giving a morphism of schemes $X \to Y$ is equivalent to giving a natural transformation of functors $X \to Y$. Here a natural transformation $X \to Y$ is just a collection of set-theoretic maps

$$X(R) \to Y(R)$$

for all $k$-algebras $R$ such that if $R' \to R$ is a $k$-algebra homomorphism then the following diagram commutes:

$$\begin{array}{ccc}
X(R') & \longrightarrow & Y(R') \\
\downarrow & & \downarrow \\
X(R) & \longrightarrow & Y(R).
\end{array}$$

In particular, every natural transformation of functors comes from a morphism of $k$-algebras $O(Y') \to O(Y)$.

1.4. Give examples of affine schemes of finite type over $\mathbb{C}$ that are nonreduced, reducible, and reduced and irreducible.

1.5. Prove that if $X \to Y$ is a closed immersion of affine schemes over $k$ then $X(R) \to Y(R)$ is injective for all $k$-algebras $R$.

1.6. Let $X, Y, Z$ be affine $k$-schemes equipped with morphisms $f : X \to Y$ and $g : Z \to Y$. Prove that

$$X \times_Y Z(R) = X(R) \times_{Y(R)} Z(R)$$

for $k$-algebras $R$. 
1.7. Let $G$ be an affine scheme over $k$. Let $\text{Id} : G \rightarrow G$ denote the identity morphism, let

$$p_i : G \times G \rightarrow G$$

denote the two projections, and let

$$\text{diag} : G \rightarrow G \times G$$

denote the diagonal map. We say that $G$ is a **group object in the category of $k$-schemes** if there exist morphisms of $k$-schemes

$$m : G \times G \rightarrow G$$
$$e : \text{Spec } k \rightarrow G$$
$$\text{inv} : G \rightarrow G$$

such that the following diagrams commute:

Prove that $G$ is a group scheme if and only if it is a group object in the category of $k$-schemes.

1.8. For $\mathbb{R}$-algebras $R$ define

$$U_n(R) := \{ g \in \text{GL}_n(\mathbb{C} \otimes_R R) : gg^t = I_n \}$$

where the bar denotes the action of complex conjugation. Show that $U_n$ is an algebraic group over $\mathbb{R}$, that $U_n(\mathbb{R})$ is compact, and that

$$U_n(\mathbb{C}) \cong \text{GL}_n(\mathbb{C}).$$

The group $U_n$ is called the definite unitary group over $\mathbb{R}$. 
1.9. Prove that

\[ O(G)/\ker(\epsilon) = k \oplus \ker(\epsilon)/\ker(\epsilon)^2 \]

as \( k \)-algebras.

1.10. Assume \( k \) is a field of characteristic not 2 and \( J \in \text{GL}_n(k) \) is an invertible symmetric or skew-symmetric matrix. If

\[ G(R) := \{ g \in \text{GL}_n(R) : g^t J g = J \} \]

then

\[ \text{Lie } G = \{ X \in \mathfrak{gl}_n : X^t J + J X = 0 \}. \]

1.11. Let \( k \) be a perfect field. Prove that the set of conjugacy classes of maximal tori \( T \leq \text{GL}_n/k \) is in natural bijection with étale \( k \)-algebras of degree \( n \).

1.12. If \( B \) is a division algebra over a field \( k \) and \( G \) is the algebraic group defined by

\[ G(R) = (B \otimes_k R)^\times, \]

then \( G \) has no proper parabolic subgroups (over \( k \)).

1.13. Show that for a perfect field \( k \) the set of \( \text{GL}_n(k) \)-conjugacy classes of parabolic subgroups of \( \text{GL}_n \) is in bijective correspondence with the partitions of \( n \). Here a partition of \( n \) is a nonincreasing set of positive integers whose sum is \( n \).

1.14. Let \( H \leq G \) be affine algebraic groups over a perfect field \( k \). Show that there is an exact sequence of pointed sets

\[ 1 \to H(k) \to G(k) \to G/H(k) \to \ker(H^1(k, H) \to H^1(k, G)). \]
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