TABLEAU COMPLEXES

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ABSTRACT. Let X, Y be finite sets and T a set of functions from $X \to Y$ which we will call "tableaux". We define a simplicial complex whose facets, all of the same dimension, correspond to these tableaux. Such *tableau complexes* have many nice properties, and are frequently homeomorphic to balls, which we prove using vertex decompositions [BP79].

In our motivating example, the facets are labeled by semistandard Young tableaux, and the more general interior faces are labeled by Buch's set-valued semistandard tableaux. One vertex decomposition of this "Young tableau complex" parallels Lascoux's transition formula for vexillary double Grothendieck polynomials [La01, La03]. Consequently, we obtain formulae (both old and new) for these polynomials. In particular, we present a common generalization of the formulae of Wachs [Wa85] and Buch [Bu02], each of which implies the classical tableau formula for Schur polynomials.

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1. INTRODUCTION

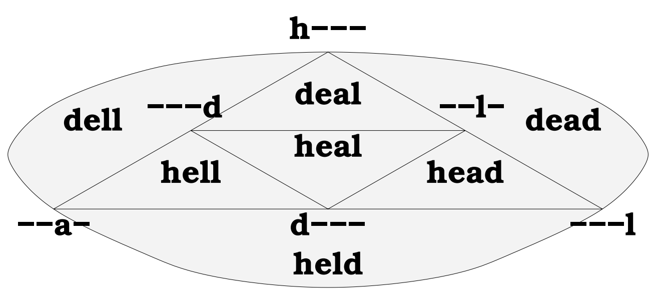
1.1. **Statement of results.** Let X and Y be two finite sets. We will call functions from X to Y **tableaux**; we think of each tableau $f : X \to Y$ as a labeling of the points of X by elements of Y. Formally, we identify a tableau f with its corresponding set $\{(x \mapsto y) : f(x) = y\} \subseteq X \times Y$ of ordered pairs, whose projection to X is bijective.

We specify a subset T of "special" tableaux. Also, let $E \subseteq X \times Y$ be a relation containing every $f \in T$. There are obvious minimal and maximal choices of E, namely $\bigcup T := \bigcup_{f \in T} f$ and $X \times Y$, but it will be convenient to not restrict E.

Our motivating example is when X is the set of boxes in partition λ and $Y = \{1, \ldots, n\}$, so a tableau $f : X \to Y$ is a Young tableau of shape λ and with entries bounded above by n (without any other demands on the labeling), and T is the special subset of semistandard Young tableaux. In a moment (Section 1.2), we will describe this case in detail.

Define the simplicial complex $\Delta_E(X \xrightarrow{T} Y)$, which we call a **tableau complex**, as follows. Consider the collection of subsets of E as a simplex under reverse inclusion; thus the vertices are the complements $(x \not\mapsto y) := E \setminus \{(x \mapsto y)\}$ rather than the elements $(x \mapsto y) \in E$. We view the faces of this simplex as **set-valued tableaux**, thought of as relations $F : X \Rightarrow Y$, in which every element $x \in X$ is labeled by a set of elements $F(x) \subseteq Y$. A set-valued tableau F' is a face of F whenever $F' \supseteq F$, meaning that $F'(x) \supseteq F(x)$ for all $x \in X$. (This set-theoretic containment is always intended when we say that one set-valued tableau contains another, even when both are being considered as faces of a simplicial complex, where containment among faces goes the opposite way.) The tableau complex is defined by its facets (maximal faces), which we declare to be the tableaux $f \in T$. (In this paper, the terms "function" and "tableau", when unadorned by "set-valued", mean single-valued functions in the usual sense.) The face with no vertices, which we call the **empty face**, is the set-valued tableau E.

Example. Consider the tableau complex in which $X = \{1, 2, 3, 4\}$ and Y is the English alphabet, and where T consists of all the English words in [dh]e[al][dl]. In detail, $E = \{(1 \mapsto d), (1 \mapsto h), (2 \mapsto e), (3 \mapsto a), (3 \mapsto l), (4 \mapsto d), (4 \mapsto l)\}$. For simplicity, in the following figure, at the vertices we indicate the *unused* letters of E. For example, the vertex common



to "hell", "heal", "head" and "held" is $(1 \not\mapsto d) = E \setminus \{(1 \mapsto d)\}$. This complex is a 2-ball, but if "deld" were a word, it would label the outer face, and this complex would be a 2-sphere.

Theorem A. *The following hold for an arbitrary tableau complex* $\Delta = \Delta_E(X \xrightarrow{T} Y)$.

1. Δ *is* ***pure****, meaning that its facets (the tableaux* $f \in T$*) all have the same dimension.*

2. *The codimension of a face* F *in* Δ *is the number* $\sum_{x\in X}(|F(x)|-1)$ *of its "extra" values. For example, faces of codimension* 1, *which we call* ***ridges***, *are set-valued tableaux taking two values for precisely one* $x \in X$ *and taking one value at all other points of* X.
3. *Each ridge is a face of at most two facets. In particular, if a tableau complex is shellable then it is homeomorphic to a ball or sphere.*
4. *The link of a face in a tableau complex is again a tableau complex.*

All of these results are proved in Section 2.1 except the last (the statement about links), which is Proposition 2.3.

We shall see that abstractly, a tableau complex is a (multicone on a) top-dimensional subcomplex of a join of boundaries of simplices. Tableau complexes can also be characterized, among pure complexes, by the extremal property given toward the end of Section 3.

Although tableau complexes are not generally (shellable) balls or spheres, we can give conditions that guarantee this conclusion. The next theorem thus defines the main class of tableau complexes of interest in this paper. Except for the claim about interior faces, which is Proposition 2.2, it is a simpler-to-state special case of Theorem 2.8.

Theorem B. *Let* X *be a poset, and* Y *totally ordered. Let* Ψ *be a set of pairs* (x_1, x_2) *in* X *with* $x_1 < x_2$. *Let* T *be the set of tableaux* $f : X \to Y$ *such that*

- *if* $x_1 \leqslant x_2$, *then* $f(x_1) \leqslant f(x_2)$, *and*
- *if* $(x_1, x_2) \in \Psi$, *then* $f(x_1) < f(x_2)$;

thus T *consists of the order-preserving tableaux from* X *to* Y *that are strictly order-preserving on the pairs in* Ψ. *Let* $E \supseteq \bigcup T$. *Then the tableau complex* $\Delta_E(X \xrightarrow{T} Y)$ *is*

1. *homeomorphic to a ball or sphere;*
2. *vertex-decomposable, as defined in* [BP79], *and hence shellable; and*
3. *a manifold with (possibly empty) boundary whose interior faces are those set-valued tableaux* F *such that every tableau* $f \subseteq F$ *lies in* T.

If Y is taken to be a set of natural numbers, then the tableaux in Theorem B are P-*partitions* [St98], where $X = P$. In our context, however, this point of view is misleading for a couple of reasons. First, the condition that Y be totally ordered can be relaxed in a natural way, as we will see during the proof. Second, P-partitions naturally form a set that is infinite and possesses additive structure; both of these properties are unnatural from the point of view of tableau complexes. More deeply, P-partitions correspond naturally to the basis elements of the Stanley-Reisner ring of a certain simplicial complex (a Gröbner degeneration of the cone of P-partitions) rather than to the facets.

We give three formulae for the Hilbert series of the Stanley-Reisner ring, the third one based on an explicit shelling of tableau complexes. For proofs, see Section 4, where the statements break the products over $\nu \supseteq F$ and $\nu \not\supseteq F$ further into products over $x \in X$.

Theorem C. *Let* $\Delta = \Delta_E(X \xrightarrow{T} Y)$ *be a tableau complex, and recall that the vertices of* Δ *are set-valued tableaux* $(x \mapsto\!\!\!\to y) \subseteq E$.

1. *The Hilbert series, in variables* $\{t_\nu : \nu$ *is a vertex of* $\Delta\}$, *equals* $K_\Delta / \prod(1-t_\nu)$, *where the denominator product is over all vertices* ν *of* Δ, *and the numerator is the* K-*polynomial*

$$K_\Delta = \sum_F \prod_{\nu \supseteq F} t_\nu \prod_{\nu \not\supseteq F} (1-t_\nu),$$

the sum being over all set-valued tableaux $\mathsf{F} \subseteq \mathsf{E}$ *such that* $\mathsf{f} \subseteq \mathsf{F}$ *for some* $\mathsf{f} \in \mathsf{T}$.

2. *If* Δ *is homeomorphic to a ball or a sphere, then writing* $|\mathsf{F}| = \sum_{x \in X} |\mathsf{F}(x)|$ *and* $|X|$ *for the size of* X, *the* K*-polynomial can be expressed an alternating sum*

$$K_\Delta = \sum_{F} (-1)^{|F|-|X|} \prod_{\nu \not\supseteq F} (1 - t_\nu)$$

over the set-valued tableaux $\mathsf{F} \subseteq \mathsf{E}$ *such that every tableau* $\mathsf{f} \subseteq \mathsf{F}$ *satisfies* $\mathsf{f} \in \mathsf{T}$.

3. *Assume furthermore the hypotheses of Theorem B, and set* $\mathsf{E} = \bigcup \mathsf{T}$. *Then there is a shelling for* Δ *such that the minimal new face when the facet* $\mathsf{f} \in \mathsf{T}$ *is added during the shelling is an explicitly described set-valued tableau* $\mathsf{N}(\mathsf{f}) \supseteq \mathsf{f}$. *Consequently,*

$$K_\Delta = \sum_{f \in T} \prod_{\nu \not\supseteq f} (1 - t_\nu) \prod_{\nu \supseteq N(f)} t_\nu.$$

As a result of Theorem C.3, we get a positive combinatorial rule to compute the h-vector $(h_0, h_1, \dots)$ of Δ: if $\eta(f) = |E \setminus N(f)| - 1$, then h_j counts the number of $f \in T$ with $\eta(f) = j$.

1.2. **Young tableau complexes.** We now describe the prototypical example of a tableau complex, and an application to computing vexillary Grothendieck polynomials, or equivalently, Hilbert series formulae for vexillary determinantal varieties.

Let $\lambda \subseteq \mathbb{N}^2$ be an English partition, or equivalently, a Young shape with its origin at its upper-left corner. A **set-valued Young tableau** [Bu02] is a filling of the boxes of λ, each with a nonempty finite set of natural numbers. The set in each box is typically expressed as a strictly increasing list. When the set in every box is a singleton, what results is an (ordinary) Young tableaux. If $|\tau|$ denotes the number of entries in a set-valued tableau τ, and $|\lambda|$ is the number of boxes in the partition, then $|\tau| \geqslant |\lambda|$. Moreover, $|\tau| = |\lambda|$ only for tableaux. (Tableaux are assumed ordinary unless the term "set-valued" is written.)

A set-valued tableau τ is called **semistandard** if for every pair b_1, b_2 of boxes of τ,

- each entry of b_1 is weakly less than each entry of b_2 whenever b_1 lies left of b_2, and
- each entry of b_1 is strictly less than each entry of b_2 whenever b_1 lies above b_2.

One can speak of one set-valued tableau **containing** another (of the same shape λ) if for each box of λ, the set of numbers in one set-valued tableau contains the corresponding set in the other. In these terms, for τ to be semistandard, one needs that every tableau contained in τ is semistandard in the usual sense. More generally, we define a set-valued tableau to be **limit semistandard** if *some* tableau it contains is semistandard. For example, the first of the following set-valued tableaux is semistandard, the second is limit semistandard, and the third is neither:

1,3	3	3,6
4,5	5	9
8		

1,4	2,3	3
2	4,5	4
4		

2,4	3,4	3
2,3	3,5	4
6		

Hereafter, we will not bother to write the commas in our examples; no confusion will result because we only use numbers that are at most 9.

The **union** of two set-valued tableaux of the same shape λ simply assigns to each box of λ the union of the two sets associated to it. Moreover, if either set-valued tableau is limit

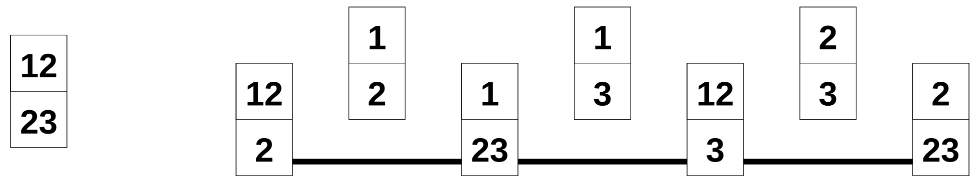


FIGURE 1. A Young tableau complex. At left is the empty-face tableau.

semistandard, so is the union. The intersection is not always defined, however, because of the requirement that every box of λ be nonempty.

In addition to the partition λ, fix a maximum entry value $n \in \mathbb{N}_+ = \{1, 2, 3, \ldots\}$. Define the **empty-face tableau** $E_{\lambda,n}$ associated to λ and n as the union of all the semistandard tableaux with shape λ and all entries at most n.

Consider the partition λ as a poset in which $(i, j) \leqslant (i', j')$ whenever $i \leqslant i'$ and $j \leqslant j'$; thus each box is less than the boxes southeast of it. Writing $[n] = \{1, \ldots, n\}$, we get a tableau complex $\Delta_{E_{\lambda,n}}(\lambda \xrightarrow{T} [n])$, in the sense of Theorem B: take Ψ to be the set of pairs (upper box, lower box) in which one box sits atop another in λ, so T is the set of semistandard Young tableaux on λ with maximum value n. Observing that $\Delta_{E_{\lambda,n}}(\lambda \xrightarrow{T} [n])$ depends only on λ and n, we denote this **Young tableau complex** by $\Delta(\lambda, n)$. See Figure 1 for an example. The special case of Theorem B for Young tableau complexes is as follows.

Corollary. *The Young tableau complex $\Delta(\lambda, n)$ is homeomorphic to a shellable ball or sphere, and its interior faces are labeled by Buch's semistandard set-valued Young tableaux* [Bu02].

Example 1.1. Let $\lambda = (2, 1)$ and $n = 3$. Then $\Delta_{\lambda,3}$ is a 3-dimensional ball. It has one interior vertex, missing the 2 in the upper left box. We draw the boundary 2-sphere in Figure 2.

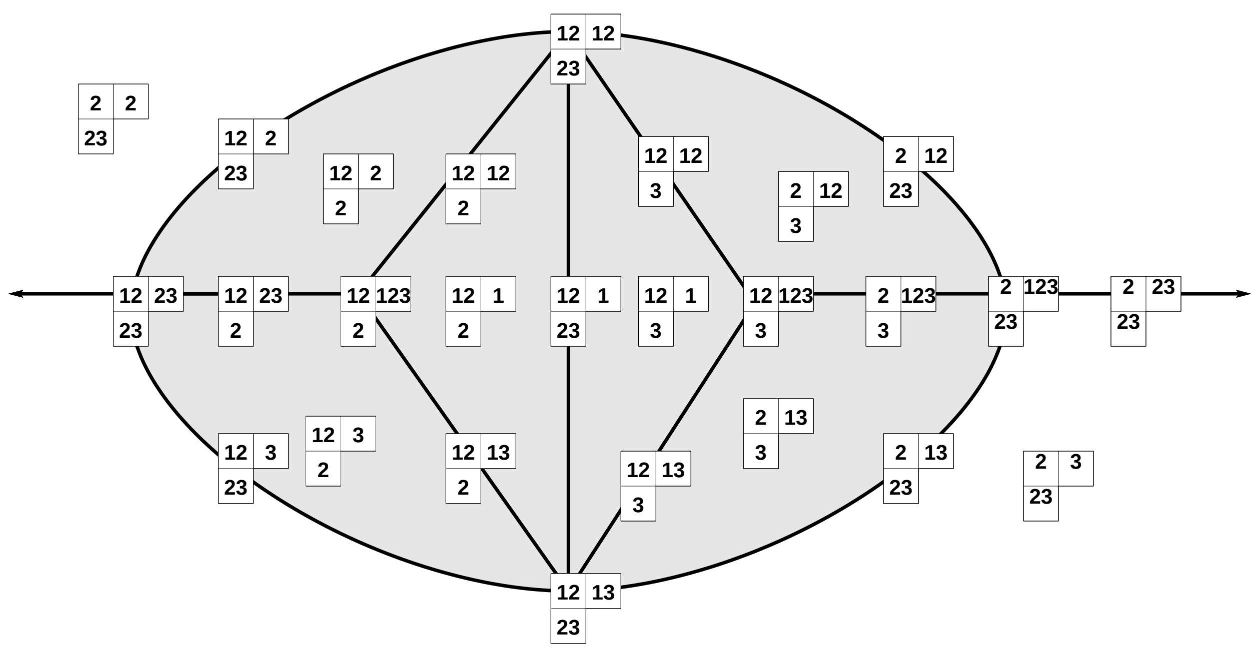


FIGURE 2. A triangulated 2-sphere of properly limit semistandard tableaux. The two edges with arrows meet around the back side of the sphere.

When vertex-decomposing a Young tableau complex—that is, writing it as the union of the star and the deletion of a single vertex—the two subcomplexes are **flagged** Young tableau complexes, in which a vector $\vec{n}$ bounds the sizes of the entries in the rows of λ. This suggests that we ought to work in that level of generality; see Figure 3.

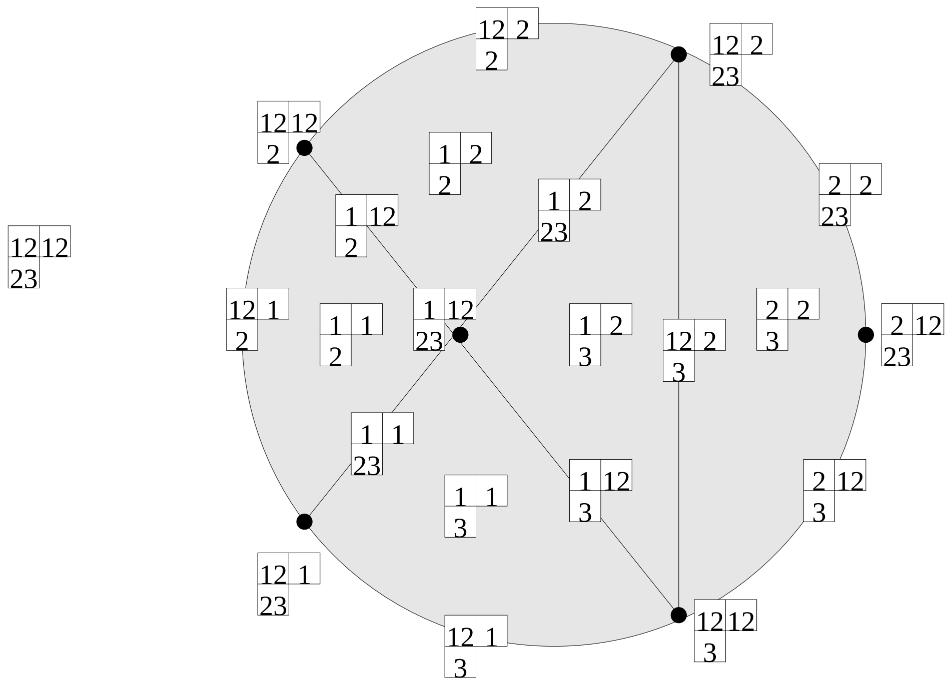


FIGURE 3. The empty-face tableau and simplicial complex for $\lambda = (2,1)$ and $\vec{n} = (2,3)$.

Flagged Young tableaux are used to compute the Schubert polynomials for vexillary permutations [Wa85], and one choice of vertex decomposition parallels the "transition formula" for their double Grothendieck polynomials [La01, La03]. Hence we are able to give set-valued-tableaux-based formulae for double Grothendieck polynomials of vexillary permutations, via Theorem C. The second formula in the following corollary, which appeared already in [KMY05], is a common generalization of Buch's and Wachs's formulae, each of which specializes to the usual tableau formula for Schur polynomials. The other two parts give new formulae for these polynomials. See Section 5 for proofs.

Corollary. *Let $\pi \in S_n$ be a vexillary permutation with associated partition λ and flagging $\vec{n}$. Each of the following is a formula for the double Grothendieck polynomial $\mathfrak{G}_\pi(\mathbf{x},\mathbf{y})$.*

1. *As a sum over the set $LSVT(\lambda,\vec{n})$ of limit semistandard tableau associated to $(\lambda,\vec{n})$,*
$$\mathfrak{G}_\pi(\mathbf{x},\mathbf{y}) = \sum_{\tau \in LSVT(\lambda,\vec{n})} \prod_{b\in\lambda} \prod_{i\in\tau(b)} (1 - x_i y^{-1}_{i+j(b)}) \prod_{h \in E_{\lambda,\vec{n}}(b)\setminus\tau(b)} x_h y^{-1}_{h+j(b)},$$
where $E_{\lambda,\vec{n}} = \bigcup T$ is the union of all semistandard tableaux $\tau \in SSYT(\lambda,\vec{n})$, and $j(b) = c(b) - r(b)$ is the difference of the row and column indices of the box $b \in \lambda$.
2. *As a sum over the set $SVT(\lambda,\vec{n})$ of semistandard set-valued tableau associated to $(\lambda,\vec{n})$,*
$$\mathfrak{G}_\pi(\mathbf{x},\mathbf{y}) = \sum_{\tau\in SVT(\lambda,\vec{n})} (-1)^{|\tau|-|\lambda|} \prod_{b\in\lambda}\prod_{i\in\tau(b)} (1 - x_i y^{-1}_{i+j(b)}).$$

3. *As a sum over the set $SSYT(\lambda, \vec{n})$ of semistandard Young tableaux of shape λ flagged by $\vec{n}$,*

$$\mathfrak{G}_\pi(\mathbf{x}, \mathbf{y}) = \sum_{\tau \in SSYT(\lambda, \vec{n})} \prod_{b \in \lambda} \prod_{i \in \tau(b)} (1 - x_i y^{-1}_{i+j(b)}) \prod_{h \in E_{\lambda, \vec{n}}(b) \setminus N_\tau(b)} x_h y^{-1}_{h+j(b)},$$

where N_τ is the tableau obtained by adding to each box b all numbers in $E_{\lambda,\vec{n}}(b)$ either smaller than the entry of $\tau(b)$, or larger than that entry provided that replacing the entry with the larger number would not give a tableau in $SSYT(\lambda, \vec{n})$.

The second of these formulae for vexillary double Grothendieck polynomials was based on the algebraic geometry of matrix Schubert varieties [KMY05]. It was that geometry that first motivated us to fit Young tableaux into a simplicial complex.

2. Properties of tableau complexes

2.1. **Generalities and boundary faces.** Recall that the vertices of a tableau complex consist of the complements $(x \not\mapsto y)$ of single elements of E.

Proposition 2.1. *Let $\Delta = \Delta_E(X \xrightarrow{T} Y)$ be a tableau complex, and assume that $(x \mapsto y) \in E$.*

1. *Δ is pure, and its facets are labeled by the tableaux in T.*
2. *Writing $|F| = \sum_{x \in X} |F(x)|$, the codimension of a face F in $\Delta_E(X \xrightarrow{T} Y)$ equals $|F| - |X|$.*
3. *Each ridge is contained in at most two facets. In particular, if $\Delta_E(X \xrightarrow{T} Y)$ is shellable then it is homeomorphic to a ball or sphere.*
4. *$(x \not\mapsto y)$ is a **cone vertex** (meaning it lies in every facet) if and only if $f(x) \neq y$ for every $f \in T$. In particular, $E = \bigcup T$ exactly when $\Delta_E(X \xrightarrow{T} Y)$ has no cone vertices.*
5. *$(x \not\mapsto y)$ is a **phantom vertex**, meaning $(x \not\mapsto y) \notin \Delta$, precisely if $f(x) = y$ for all $f \in T$.*

Proof. Only statement 3 is not immediate from the definitions. A ridge is a set-valued tableau taking one extra value. By the pigeonhole principle, since every $x \in X$ gets at least one $y \in Y$, there exists exactly one x with two values from Y, all others being 1. Such a set-valued tableau can contain at most two tableaux from T. The statement about being a ball or sphere now follows from [BLSWZ99, Proposition 4.7.22]. □

Cone vertices are in some sense uninteresting: a simplicial complex can be canonically reconstructed from its set of cone vertices and its **core**, which is the subcomplex with the cone vertices removed. In particular, the whole complex is a ball or sphere if and only if its core is a ball or sphere. It is convenient for inductive purposes not to assume that $E = \bigcup T$, although we will generally assume $E = \bigcup T$ in examples.

Proposition 2.2. *Assume that $\Delta_E(X \xrightarrow{T} Y)$ is homeomorphic to a ball or sphere. A face F of $\Delta_E(X \xrightarrow{T} Y)$ lies on the boundary of $\Delta_E(X \xrightarrow{T} Y)$ if and only if there exists a tableau $g : X \to Y$ such that $g \subseteq F$ but $g \notin T$.*

Proof. By definition, a face of a simplicial ball lies in the boundary if and only if it is a face of a boundary ridge. A ridge itself lies in the boundary if and only if it is a face of precisely one facet. Of the two tableaux contained in a given ridge, at least one must lie in T, because a ridge is a face of $\Delta_E(X \xrightarrow{T} Y)$. Hence a ridge is a boundary face if and only if the unique other tableau it contains does not lie in T.

Now let F be an arbitrary face of $\Delta_E(X \xrightarrow{T} Y)$. If every function $f \subseteq F$ lies in T, then every ridge with F as a face is a union of two tableaux from T, so F is interior. On the other hand, suppose that $g \subseteq F$ for some tableau $g \notin T$, and let $f \in T$ be a facet having F as a face, so $f \subseteq F$. Then $f \cup g \subseteq F$ is a set-valued tableau that has F as a face. Some of the elements $x \in X$ are assigned two distinct Y-values by $f \cup g$. If deleting the value $f(x)$ from $(f \cup g)(x)$ yields a face G of $\Delta_E(X \xrightarrow{T} Y)$, then induction on the codimension implies that G lies on the boundary, and hence $F \supseteq G$ does, as well. If no such x exists, so deleting the value $f(x)$ from $f \cup g$ always results in a set-valued tableau that is not a face of $\Delta_E(X \xrightarrow{T} Y)$, then f is the unique tableau in T with $f \cup g$ as a face; thus $f \cup g$ is a face of only one facet (namely f), and hence $f \cup g$ is a boundary face with F as a subface. □

2.2. **Safe vertices in tableau complexes.** Given a simplicial complex Δ with a vertex v, define the **star** and **deletion** of v to be

$$\mathrm{star}_v \Delta = \{C \in \Delta : C \cup v \in \Delta\} \quad \text{and} \quad \mathrm{del}_v \Delta = \{C \in \Delta : v \notin C\}.$$

Then $\Delta = \mathrm{star}_v \Delta \cup \mathrm{del}_v \Delta$. The star has an obvious cone vertex, namely v itself, and its deletion from the star is called the **link** of v in Δ. More generally, the link of a face C in a simplicial complex Δ is defined as

$$\mathrm{link}_C \Delta = \{D \in \Delta : D \cap C = \varnothing, D \cup C \in \Delta\}.$$

By convention, the vertex set of this link does not include the (now phantom) vertices of C.

Proposition 2.3. *Let F be a face of $\Delta = \Delta_E(X \xrightarrow{T} Y)$. Let $T_{\mathrm{link}} = \{f \in T : f \subseteq F\}$ be the set of facets of Δ having F as a subface. Then the link of F in Δ is isomorphic to $\Delta_F(X \xrightarrow{T_{\mathrm{link}}} Y)$.*

Proof. It follows from the definitions that the faces of both $\mathrm{link}_F \Delta$ and $\Delta_F(X \xrightarrow{T_{\mathrm{link}}} Y)$ are the set-valued tableaux contained in F and containing a tableau from T. □

Proposition 2.4. *Let $\Delta_E(X \xrightarrow{T} Y)$ be a tableau complex. Let $T_{\mathrm{star}} = \{f \in T : f(x) \neq y\}$. Then $\mathrm{star}_{(x \mapsto\!\!\!\not\ y)} \Delta_E(X \xrightarrow{T} Y) = \Delta_E(X \xrightarrow{T_{\mathrm{star}}} Y)$.*

Proof. Since $\Delta_E(X \xrightarrow{T} Y)$ is pure, the star of $(x \mapsto\!\!\!\not\ y)$ is the union of the (closures of) facets that have $(x \mapsto\!\!\!\not\ y)$ as a vertex. These facets are exactly the tableaux $f \in T_{\mathrm{star}}$. □

Call a vertex $(x \mapsto\!\!\!\not\ y)$ of $\Delta_E(X \xrightarrow{T} Y)$ **safe** if for every $f \in T$, changing the label on x from $f(x)$ to y yields a tableau that is again in T. While the star of a vertex in a pure complex is always pure, the deletion might not be.

Proposition 2.5. *The deletion $\mathrm{del}_{(x \mapsto\!\!\!\not\ y)} \Delta$ of the vertex $(x \mapsto\!\!\!\not\ y)$ from the simplicial complex $\Delta = \Delta_E(X \xrightarrow{T} Y)$ is pure if and only if either $(x \mapsto\!\!\!\not\ y)$ is a cone vertex or $(x \mapsto\!\!\!\not\ y)$ is safe.*

Proof. If $(x \mapsto\!\!\!\not\ y)$ is a cone vertex then Δ is the cone over $\mathrm{del}_{(x \mapsto\!\!\!\not\ y)} \Delta$. A simplicial complex is pure if and only if the cone over it is, so we assume that $(x \mapsto\!\!\!\not\ y)$ is not a cone vertex.

Given a set-valued tableau F, let $\mathrm{del}_{(x \mapsto\!\!\!\not\ y)} F$ denote the set-valued tableau that sends $a \mapsto F(a)$ for $a \neq x$ and sends $x \mapsto F(x) \cup \{y\}$. In particular, $\mathrm{del}_{(x \mapsto\!\!\!\not\ y)} F = F$ if and only if $y \in F(x)$. The definitions imply that $\mathrm{del}_{(x \mapsto\!\!\!\not\ y)} \Delta$ consists of the set-valued tableaux $\mathrm{del}_{(x \mapsto\!\!\!\not\ y)} F$ for $F \in \Delta$, and the facets of $\mathrm{del}_{(x \mapsto\!\!\!\not\ y)} F$ have the form $\mathrm{del}_{(x \mapsto\!\!\!\not\ y)} f$ for tableaux $f \in T$. Since $(x \mapsto\!\!\!\not\ y)$ is not a cone vertex, at least one facet of Δ sends x to y. Thus the deletion is pure if and only if, for all tableaux $f \in T$, the set-valued tableau $\mathrm{del}_{(x \mapsto\!\!\!\not\ y)} f$ contains a tableau $g \in T$ satisfying $g(x) = y$. The desired result follows because when $f(x)$ does not already equal y, the only possibility for g is obtained by changing $f(x)$ to y. □

Corollary 2.6. *Let $(x \mapsto\!\!\!\!\!\not\;\; y)$ be a safe vertex of the tableau complex $\Delta = \Delta_E(X \xrightarrow{T} Y)$. If $T_{\mathrm{del}} = \{f \in T : f(x) = y\}$, then $\mathrm{del}_{(x \not\mapsto y)}\Delta = \Delta_E(X \xrightarrow{T_{\mathrm{del}}} Y)$.* □

2.3. **Tableau complexes on posets.** At this point we make some additional assumptions to guarantee a ready supply of safe vertices. The following theorem is stated much more generally than our motivating examples require; we hope that this Bourbakiesque level of generality helps to indicate which assumptions are leading to which conclusions.

The key to our geometric conclusions (shellable ball or sphere) is the notion of **vertex-decomposable** simplicial complex in the sense of [BP79]. By definition, every simplex is vertex-decomposable, and an arbitrary simplicial complex is vertex-decomposable if and only if it is pure and has a vertex whose deletion and link are both vertex-decomposable.

Lemma 2.7 ([BP79])**.** *A simplicial complex Δ is shellable if both the deletion $\mathrm{del}_v\Delta$ and the star $\mathrm{star}_v\Delta$ of a vertex v are shellable. Hence all vertex-decomposable simplicial complexes are shellable.*

Proof. ([BP79]) Construct a shelling of Δ by concatenating shellings of the deletion and star of v (in that order), the latter being the cone over a shelling of the link. □

Theorem 2.8. *Let X and Y be finite partially ordered sets. For each $x \in X$, let Y_x be a totally ordered subset of Y. Fix a set Ψ of pairs (x_1, x_2) from X such that $x_1 < x_2$. Let T be the set of tableaux $f : X \to Y$ such that*

- *$f(x) \in Y_x$ for all $x \in X$;*
- *f is weakly order-preserving, i.e. $x_1 \leqslant x_2$ implies that $f(x_1) \leqslant f(x_2)$; and*
- *if $(x_1, x_2) \in \Psi$ then $f(x_1) < f(x_2)$.*

Let $E \subseteq X \times Y$ contain $\bigcup T$. Then $\Delta_E(X \xrightarrow{T} Y)$ is homeomorphic to a ball or sphere, and it is vertex-decomposable.

Proof. We need only prove vertex-decomposability, for then the ball or sphere conclusion is a consequence of Proposition 2.1.3 and Lemma 2.7. To demonstrate vertex-decomposability, we need only find, for each tableau complex satisfying the hypotheses of the theorem, a vertex whose deletion and link both satisfy the hypotheses.

Suppose that $\Delta_E(X \xrightarrow{T} Y)$ has a cone vertex $(x \not\mapsto y)$. Viewing $(x \not\mapsto y)$ as a subset of $X \times Y$, we find that $(x \not\mapsto y)$ already contains $\bigcup T$ by Proposition 2.1.4, so $\Delta_{(x \not\mapsto y)}(X \xrightarrow{T} Y)$ satisfies the conditions of the theorem. This simplicial complex is the link by Proposition 2.3, and it equals the deletion because $(x \not\mapsto y)$ is a cone vertex.

Now assume that $\Delta_E(X \xrightarrow{T} Y)$ has no cone vertices. If all of the vertices of $\Delta_E(X \xrightarrow{T} Y)$ are phantom, then there is only one facet and we are done. Otherwise, there exists a non-phantom vertex $(m \not\mapsto y)$. Choose one with maximal possible m, and let y_m be the maximum element of Y_m. Since there are no cone points, the values of all tableaux in T are fixed at elements $x > m$: for each $x > m$ and all $f \in T$ there is some $y_x \in Y$ such that $f(x) = y_x$. Therefore, as $(m \not\mapsto y_m)$ is itself not a cone vertex, we get $y_m \leqslant y_x$ for all $x > m$, and $y_m < y_x$ if $(m, x) \in \Psi$. It follows that the vertex $(m \not\mapsto y_m)$ is safe: we can safely change the label on m from $f(m)$ to y_m to get another tableau satisfying the three conditions to be in T because $y_m \geqslant f(m)$ for all $f \in T$.

(We used that Y_m has a maximum element for this, but not that it is totally ordered.)

Most of the work has now been done in Section 2.2: if $T_{\mathrm{star}} = \{f \in T : f(m) \neq y_m\}$ and $T_{\mathrm{del}} = \{f \in T : f(m) = y_m\}$, then the star and deletion of $(m \not\mapsto y_m)$ are $\Delta_E(X \xrightarrow{T_{\mathrm{star}}} Y)$

and $\Delta_E(X \xrightarrow{T_{del}} Y)$, respectively, by Proposition 2.4 and Corollary 2.6. The star and deletion satisfy the three conditions from the statement of the theorem, with the same X, Y, and Ψ, but with Y_m changed either to $Y_m \setminus \{y_m\}$ or else to $\{y_m\}$, respectively. Given that the star satisfies the hypotheses of the theorem, arguing as in the second paragraph of the proof shows that the link does, as well.

(To work inductively, we need $Y_m \setminus \{y_m\}$ to again have a maximum element; this is why we required Y_m to be totally ordered. In addition, our new choice of m for the link must have a maximum element in its Y_m; this is why we need *every* Y_x to have a maximum.) $\square$

Before this theorem, we never needed to compare $f(x_1)$ and $f(x_2)$ for $x_1 \neq x_2$; in some sense, it would have been more natural for the tableaux to take values in separate sets Y_x. Now that we used a partial order on Y to define our set T of tableaux, we have finally made such comparisons.

Example 2.9. More generally than in Section 1.2, let X be the set of boxes in a skew-shape λ/μ, and each $Y_x = Y = \{1, \ldots, n\}$. Partially order X by asking that each box is less than the boxes southeast of it. Let Ψ be the set of pairs $\{$(upper box, lower box)$\}$ where one box is atop another. Then T is the set of semistandard Young tableaux of shape λ/μ with maximum value n, and $\Delta_{\bigcup_{f\in T} f}(X \xrightarrow{T} Y)$ is the **Young skew tableau complex**.

The faces of this complex are labeled with *set-valued Young skew tableaux*, which were also introduced in [Bu02]. Buch's definition of "semistandard" set-valued Young tableaux exactly matches our criterion, Proposition 2.2, for a face to be interior.

(In fact the Ψ machinery was unnecessary to model semistandardness; we could just take $\Psi = \varnothing$, subtract $r-1$ from the values in the rth row, and adjust the sets Y_x to get a set combinatorially equivalent to semistandard Young tableaux. But the formulation with Ψ is clearer, more general, and no more difficult.)

Example 2.10. Let X be a poset, $\Psi = \varnothing$, and $Y = \{0, 1\}$. Then the tableaux correspond to partitioning X into a lower and an upper order ideal (the 0 and 1 parts), or equivalently to antichains in X (the maximum elements labeled 0). By Theorem 2.8, this tableau complex is homeomorphic to a ball (or sphere, if X is totally unordered).

Remark 2.11. Other classes of vertex-decomposable complexes include the greedoid complexes [BKL85] and subword complexes [KM03]; see [KM03, Remark 2.6] for an extended discussion. Tableau complexes are different from each of these. For example, the Young tableau complex for the vertical domino with entries at most 5 is not a greedoid complex if the ground set is taken to be the vertex set. To show the difference between subword and tableau complexes, consider the Young tableau complex for the 2×2 square shape with entries at most 3; it has dimension 3 and eight vertices, none of which is a cone vertex. On the other hand, deleting all cone points from the subword complex in [KMY05] having the same tableaux for facets yields a simplicial complex of dimension 2 with seven vertices.

It is worth noting that the phrase "ball or sphere" essentially always really means "ball". To get a sphere, there must be no cone vertices, so $E = \bigcup T$. But even then, every ridge lies in two facets, so every vertex must be safe; in other words, the possible T-tableau values at each $x \in X$ are independent. We spell this out further in Section 3. For now, here is a characterization of the interior faces, which includes all of the faces in the case of a sphere. As a matter of notation, if Y_1 and Y_2 are two subsets of a poset Y, write

$Y_1 \leqslant Y_2$ if $y_1 \leqslant y_2$ for all $y_1 \in Y_1$ and $y_2 \in Y_2$. Similarly, write $Y_1 < Y_2$ if strict inequality holds. The following is an immediate consequence of the definitions and Proposition 2.2.

Corollary 2.12. *Assume the notation and hypotheses of Theorem 2.8. A face* F *is interior to* $\Delta_E(X \xrightarrow{T} Y)$ *if and only if*

- $F(x_1) \leqslant F(x_2)$ *whenever* $x_1 < x_2$*; and*
- $F(x_1) < F(x_2)$ *whenever* $(x_1 < x_2) \in \Psi$.

2.4. **Shelling poset tableau complexes.** The next theorem will help us describe the h-vectors and Hilbert series of poset tableau complexes and their Stanley-Reisner rings.

Theorem 2.13. *Assume the notation and hypotheses of Theorem 2.8, and choose a linear extension* ε *of the partial ordering on* X*. Lexicographically order* T *by comparing* $f_1, f_2 \in T$ *at the* ε*-largest element* $m \in X$ *where they differ. Placing the one with the larger label on* m *first yields a total order on the facets of* $\Delta_E(X \xrightarrow{T} Y)$ *that is a shelling.*

Proof. Remember that $f_1(m)$ and $f_2(m)$ are comparable, since Y_m is totally ordered. Therefore the procedure in the statement of the theorem yields a total order on the facets. We will show that this total order is the shelling produced by applying Lemma 2.7 recursively as in the proof of Theorem 2.8. At each stage in that proof, we either vertex decompose at a cone vertex or we choose a maximal element $m \in X$ among those supporting non-phantom vertices $(m \mapsto\!\!\!\!\not\;\; y)$. Vertex decomposing at a cone vertex does not alter the set T of facet tableaux, so it does not matter in which order we delete cone vertices. Only the order in which we choose the maximal elements m matters. Use the ε-order: since Lemma 2.7 puts the deletion $(f(m) = y_m)$ first before the star $(f(m) \neq y_m$, which is equivalent to $f(m) < y_m$ because y_m is maximum in $Y_m)$, the resulting shelling is as desired. □

3. Characterizations of tableau complexes

Let Δ be a pure simplicial complex on a vertex set V. Declare that $W \subseteq V$ is a **pure factor** of Δ if the number $|f \cap W|$ of vertices in the intersection of f with W is the same for all facets $f \in \Delta$. For example, a singleton $\{v\}$ is a pure factor if and only if v is a phantom or cone vertex, with $|f \cap \{v\}| = 0$ or $|f \cap \{v\}| = 1$, respectively. If W is a pure factor, then its complement $V \setminus W$ is a pure factor, too. For any set $W \subseteq V$ of vertices, write $\Delta|_W$ for the **full subcomplex** $\mathrm{del}_{V \setminus W}\Delta$ **supported on** W.

Proposition 3.1. *Let* Δ *be a pure simplicial complex on the vertex set* V*, and suppose that* $V = V_1 \cup \cdots \cup V_k$ *is partitioned into a disjoint union of pure factors* $V_1, \ldots, V_k$*. Then* $\Delta|_{V_i}$ *is pure for each* $1 \leqslant i \leqslant k$*, and* Δ *is a top-dimensional subcomplex of their* ***join***

$$\{F_1 \cup \cdots \cup F_k : F_i \text{ is a face of } \Delta|_{V_i} \text{ for each } 1 \leqslant i \leqslant k\}.$$

Proof. Since Δ is pure, it follows from the definitions that $\Delta|_{V_i}$ is pure. On the other hand, it also follows by definition that a subset $f \subseteq V_1 \cup \cdots \cup V_k$ is a facet of the join of the complexes $\Delta|_{V_i}$ if and only if $f \cap V_i$ is a facet of the individual complex $\Delta|_{V_i}$ for each i. Since the vertex set V is the disjoint union $V_1 \cup \cdots \cup V_k$, every facet of Δ has this property. □

Lemma 3.2. *Let* Δ *be a tableau complex* $\Delta_{\bigcup T}(X \xrightarrow{T} Y)$*. For each* $x \in X$*, define the subset* $V_x = \{(x \mapsto\!\!\!\!\not\;\; y) : (x \mapsto y) \in \bigcup T\}$ *of the vertex set of* Δ*. Then*

1. *the subsets* $\{V_x : x \in X\}$ *partition the vertex set of* Δ*;*

2. *each subset V_x is a pure factor; in fact, $|f \cap V_x| = |V_x| - 1$ for every facet tableau f; and*
3. *each induced complex $\Delta|_{V_x}$ is the boundary of the simplex on V_x.*

Proof. The first two numbered claims are immediate from the definitions. For the third, it follows from the second that $\Delta|_{V_x}$ is a union of some subset of the facets (each of size $|V_x| - 1$) in the boundary of the simplex on V_x. Each facet of $\Delta|_{V_x}$ avoids using some (unique) vertex of V_x. If any vertex of V_x does not occur this way, then it lies in every facet of $\Delta|_{V_x}$; in other words, it is a cone vertex of $\Delta|_{V_x}$. Since each facet of Δ survives after deleting $V \setminus V_x$ to give a facet of $\Delta|_{V_x}$, we conclude that Δ has a cone vertex, contradicting Proposition 2.1.4. Therefore every face of size $|V_x| - 1$ occurs in $\Delta|_{V_x}$. □

Theorem 3.3. *A pure complex is (isomorphic to) a tableau complex $\Delta_{\bigcup T}(X \xrightarrow{T} Y)$ if and only if it can be expressed as a top-dimensional subcomplex of a join of boundaries of simplices.*

Proof. That $\Delta_{\bigcup T}(X \xrightarrow{T} Y)$ can be expressed in the desired manner is an immediate consequence of Proposition 3.1 and Lemma 3.2. Now suppose that that Δ is a pure complex expressible as a top-dimensional subcomplex of the join of boundaries of simplices with vertex sets $V_1, \ldots, V_k$. Then the vertex set V of Δ is the disjoint union $V_1 \cup \cdots \cup V_k$. Let $X = \{1, \ldots, k\}$ and set $Y = V$. Each facet f of Δ defines a function $X \to Y$ taking i to the element $V_i \setminus f$. Using these as the set T of tableaux, we find that $\Delta = \Delta_{\bigcup T}(X \xrightarrow{T} Y)$. □

Remark 3.4. In particular, if a tableau complex has no boundary ridges (ridges contained in just one facet), then it is the join of a bunch of boundaries of simplices, and in particular it is a sphere. This, plus Proposition 2.3, gives another proof of Proposition 2.2.

Let the **codimension** of a pure complex Δ be the number of vertices outside any facet. The only way for the codimension to equal 0 is if V consists only of cone vertices (so V is a simplex). If V breaks up as a union $V_1 \cup \cdots \cup V_k$ of pure factors, then the codimension of Δ is the sum of the codimensions of the full subcomplexes supported on the V_i.

This suggests a characterization of tableau complexes by the following extremal property. Given a pure complex Δ with the cone and phantom vertices deleted, look for a pure factor W. Splitting into W and $V \setminus W$, the codimension of each full subcomplex can be no larger than that of Δ. By the theorem, Δ is a tableau complex if and only if we can split enough to whittle the codimensions of all of the full subcomplexes down to 1.

The situation is somewhat dual to order complexes of ranked posets. If P is a ranked poset, its **order complex** has vertex set P, and $Q \subset P$ defines a face if and only if Q is totally ordered. If P_r denotes the set of elements with a given rank r, then the induced complex on P_r is pure of dimension 0, rather than codimension 1 like a tableau complex. (If it seems unsatisfying for "codimension 1" to be dual to "dimension 0", then consider the latter more honestly as "affine-dimension 1".) Very few simplicial complexes are both order complexes and tableau complexes; we leave their characterization as an exercise for the reader.

Remark 3.5. Tableau complexes bear superficial similarities to matroid complexes. A simplicial complex is a matroid if and only if every subcomplex induced on a subset of the vertex set is pure. Theorem 3.3 says that a simplicial complex is a tableau complex if and only if the vertex set can be partitioned into subsets that are pure factors of codimension 1. In reality, there are matroid complexes that are not tableau complexes, and conversely. For example, we have already seen in Remark 2.11 that tableau complexes can fail to be greedoid complexes, of which matroid complexes are special cases. For an example the other way, the matroid for the complete graph K_4 on four vertices is the union of three segments joined at a point. If this were a tableau complex, then so would be the result of deleting the cone point, by Proposition 2.3. But a set of three points is not a tableau complex by Remark 3.4, and hence neither is the matroid for K_4.

4. Hilbert series and K-polynomials

In this section we collect some formulae for the Hilbert series of the Stanley-Reisner rings of tableau complexes. Our main reference for generalities on Betti numbers, Hilbert series, and K-polynomials (which are numerators of Hilbert series) is [MS04]. For notation, let $\Bbbk$ be a field, and set $S = \Bbbk[V]$, the polynomial ring in variables $v \in V$ indexed by a finite set V. This is the ambient ring for objects like the **Stanley-Reisner ideal** $I_\Delta = \langle \prod_{v \in F} v : F \text{ is not a face of } \Delta \rangle$ of a simplicial complex Δ with vertex set V, and the **Stanley-Reisner ring** S/I_Δ. We shall use the alphabet $\mathbf{t} = \{t_v : v \in V\}$ for finely graded Hilbert series and K-polynomials. When Δ is a tableau complex $\Delta_E(X \xrightarrow{T} Y)$, recall that V is the set $\{(x \not\mapsto y) : (x \mapsto y) \in E\}$ of complements of single elements of E.

Proposition 4.1. *The* K*-polynomial associated to the tableau complex* $\Delta = \Delta_E(X \xrightarrow{T} Y)$ *is*

$$K(S/I_\Delta; \mathbf{t}) = \sum_F \prod_{x \in X} \left(\prod_{y \in E(x) \setminus F(x)} t_{(x \not\mapsto y)} \prod_{y \in F(x)} (1 - t_{(x \not\mapsto y)}) \right),$$

the sum being over all set-valued tableaux $F \subseteq E$ *each containing some tableau* $f \in T$.

Proof. This formula is [MS04, Theorem 1.13] applied to tableau complexes, since the condition $y \in E(x) \setminus F(x)$ means that $(x \not\mapsto y)$ is a vertex of F and the condition $y \in F(x)$ means that $(x \not\mapsto y)$ is not a vertex of F. □

Our second formula uses the ball-or-sphere hypothesis; it therefore holds for (among other things) all poset tableau complexes. It will be simpler to prove the formula for general balls and spheres first.

Proposition 4.2. *If* Δ *is a simplicial ball or sphere with vertex set* V*, then*

$$K(S/I_\Delta; \mathbf{t}) = \sum_F (-1)^{\operatorname{codim}(F)} \prod_{v \in V \setminus F} (1 - t_v),$$

where the sum is over all interior faces of Δ*. (All faces are interior if* Δ *is a sphere.)*

Proof. Consider the **Alexander dual** ideal $I_\Delta^\star = \langle \prod_{v \in V \setminus F} v : F \text{ is a face of } \Delta \rangle$, and start by calculating the K-polynomial of $I_\Delta^\star$. The coefficient on the monomial $\prod_{v \in V \setminus F} t_v$ in $K(I_\Delta^\star; \mathbf{t})$ is the alternating sum of the Betti numbers of $I_\Delta^\star$ in degree $\prod_{v \in V \setminus F} v$ [MS04, Proposition 8.23]. By Hochster's formula [MS04, Corollary 1.40], the i^{th} such Betti number equals the dimension $\dim_\Bbbk \widetilde{H}_{i-1}(\operatorname{link}_F \Delta; \Bbbk)$ of the reduced homology of the link of F in Δ, and it comes with a sign $(-1)^i$. If F is a boundary face, then the link of F is contractible; but if F is an interior face, then the link of F is a sphere of dimension $\operatorname{codim}(F) - 1$. Therefore $K(I_\Delta^\star; \mathbf{t}) = \sum_F (-1)^{\operatorname{codim}(F)} \prod_{v \in V \setminus F} t_v$, where the sum is over all interior faces F of Δ. The Alexander inversion formula [MS04, Theorem 5.14] now implies the desired result. □

Theorem 4.3. *If the tableau complex* $\Delta = \Delta_E(X \xrightarrow{T} Y)$ *is homeomorphic to a ball or sphere, then*

$$K(S/I_\Delta; \mathbf{t}) = \sum_F (-1)^{|F|-|X|} \prod_{x \in X} \prod_{y \in F(x)} (1 - t_{(x \mapsto\!\!\!\to y)}),$$

the sum being over all set-valued tableaux $F \subseteq E$ *such that every tableau* $f \subseteq F$ *lies in* T.

Proof. The factor $(-1)^{|F|-|X|}$ is the codimension of F. The condition $y \in F(x)$ means that $(x \mapsto\!\!\!\to y)$ lies in the vertex set of Δ but not F. The sum is over all interior faces by Proposition 2.2. Therefore the result is simply Proposition 4.2 for tableau complexes. □

A **shelling** of a pure d-dimensional simplicial complex Δ is an ordering of the facets $F_1, \ldots, F_k$ such that $F_i \cap (F_1 \cup \cdots \cup F_{i-1})$ has pure dimension $d-1$ for each $1 \leqslant i \leqslant k$. This guarantees that for each i, there is a unique minimal new face $N_i \in \Delta$ that is a face of F_i but not of $F_1, \ldots, F_{i-1}$. By convention, N_1 is the empty face.

Lemma 4.4. *Given a shelling of a simplicial complex* Δ *with new faces* $N_1, \ldots, N_k$ *as above,*

$$K(S/I_\Delta; \mathbf{t}) = \sum_{i=1}^{k} \prod_{\nu \notin F_i} (1 - t_\nu) \prod_{\nu \in N_i} t_\nu.$$

Proof. Use induction on the number k of facets of Δ. □

The $\mathbb{Z}$-graded coarsening of the Hilbert series to one variable t gives

$$H(S/I_\Delta, t) = \sum_{j=0}^{d} \frac{h_j t^j}{(1-t)^d}.$$

When Δ is shellable, the h**-vector** $(h_0, h_1, \ldots, h_d)$ consists of nonnegative integers. Moreover, the shelling gives a manifestly positive way to compute these numbers: h_j counts the number of dimension j faces among $N_1, \ldots, N_k$.

Theorem 4.5. *Resume the notation and hypotheses of Theorem 2.8, and set* $E = \bigcup T$. *Given a tableau* $f \in T$, *define* U_f *as the set of elements* $y \in Y$ *such that* $f(x) \leqslant y$ *and moving the label on* x *from* $f(x)$ *up to* y *yields a tableau in* T. *Then*

$$K(S/I_\Delta; \mathbf{t}) = \sum_{f \in T} \prod_{x \in X} \left((1 - t_{(x \mapsto\!\!\!\to f(x))}) \prod_{y \in U_f(x)} t_{(x \mapsto\!\!\!\to y)} \right).$$

Finally, if $\eta(f) = -|X| + \sum_{x \in X} |U_f(x)|$, *then* h_j *is the number of tableaux* $f \in T$ *with* $\eta(f) = j$.

Proof. The proof will be done once we produce a shelling of Δ for which $N_f = E(x) \setminus U_f(x)$ is the minimal new face at the stage when we add the facet f. Pick a linear extension ε of X and take the shelling order of the facets $f_1, f_2, \ldots$ of Δ as in Theorem 2.13. For $f := f_i$ we show that N_f is the minimal new face of f.

First, N_f is a set-valued tableau in Δ that is a face of f, since it contains f. Second, to see that N_f is not a face of any previous facet, we must show that f does not contain $f_1, \ldots, f_{i-1}$. Note that by construction, any other $g \in T$ contained in N_f must assign to each $x \in X$ either $f(x)$ or some $y < f(x)$. Such a facet tableau g must appear later than f in the facet ordering. The maximality of $N_f \subseteq E$ containing f and not containing $f_1, \ldots, f_{i-1}$ is also clear from the construction. Hence N_f is the minimal new face of f, as claimed. For the K-polynomial formula, apply Lemma 4.4. □

Example 4.6. For the tableau complex in Figure 3 (after Example 1.1), listing the facets in the order

$$\begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}$$

yields the shelling in the proof of Theorem 4.5. For the first of these tableaux, all of the sets $U(x)$ are singletons: there is no way to increase the number in any box while respecting the flagging, which requires that the entries in the top row are at most 2 and the lower entry is at most 3. For the second tableau above, $U(x) = \{1, 2\}$ for the upper-left corner x, because moving the 1 up to a 2 keeps the tableau semistandard. Similarly, all of the sets $U(x)$ for the third and fourth tableaux are singletons except for upper-right box and the bottom box, respectively, whose sets $U(x)$ are $\{1, 2\}$ and $\{2, 3\}$. In the last tableau above, only the two lower-right corners have non-singleton sets $U(x)$, and these are $\{1, 2\}$ and $\{2, 3\}$. For the above five tableau, the function η at the end of Theorem 4.5 takes the values $0, 1, 1, 1,$ and 2, respectively. Thus the simplicial complex in Figure 3 has h-vector $(1, 3, 1)$.

Our final K-polynomial formula in this section will arise again after Corollary 5.3.

Proposition 4.7. *If* $(x \mapsto\!\!\!\to y)$ *is a safe vertex of* $\Delta = \Delta_E(X \xrightarrow{T} Y)$*, then*

$$K(S/I_\Delta; \mathbf{t}) = t_{(x \mapsto\!\!\to y)} K(S/I_{\text{del}}; \mathbf{t}) + (1 - t_{(x \mapsto\!\!\to y)}) K(S/I_{\text{star}}; \mathbf{t}),$$

where I_{del} *and* I_{star} *are the Stanley-Reisner ideals for the deletion tableau complex* $\Delta_E(X \xrightarrow{T_{\text{del}}} Y)$ *and the star tableau complex* $\Delta_E(X \xrightarrow{T_{\text{star}}} Y)$ *from Propositions 2.4 and 2.5, respectively.*

Proof. Any vertex decomposition $\Delta = \text{del}_\nu\Delta \cup \text{star}_\nu\Delta$ gives an inductive formula

$$K(S/I_\Delta; \mathbf{t}) = t_\nu K(S/I_{\text{del}_\nu\Delta}; \mathbf{t}) + (1 - t_\nu) K(S/I_{\text{star}_\nu\Delta}; \mathbf{t})$$

for the K-polynomial. □

5. Applications to vexillary double Grothendieck polynomials

In this section, we apply Section 4 to obtain formulae for double Grothendieck polynomials for vexillary permutations. This gives formulae for the Hilbert series and K-polynomials of vexillary matrix Schubert varieties (also known as (one-sided) ladder determinantal varieties). See [KMY05] for a treatment of the related algebraic geometry.

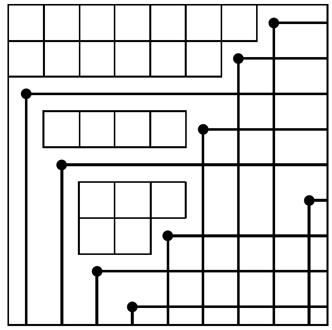
5.1. **Vexillary permutations and flaggings of partitions.**

Identify a permutation $\pi \in S_n$ with the square array having blank boxes in all locations except at $(i, \pi(i))$ for $i = 1, \ldots, n$, where we place **dots**. This defines the **dot-matrix** of π. We associate the **diagram**

$$D(\pi) = \big\{(p, q) \in \{1, \ldots, n\} \times \{1, \ldots, n\} : \pi(p) > q \text{ and } \pi^{-1}(q) > p\big\}$$

to π. Pictorially, if we draw a "hook" consisting of lines going east and south from each dot, then $D(\pi)$ consists of the squares not in the hook of any dot.

Example 5.1. Let $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 7 & 1 & 6 & 2 & 9 & 5 & 3 & 4 \end{pmatrix}$. Its dot-matrix and diagram are combined below:



In what follows, we will assume our permutations π are **vexillary**, also known as 2143-**avoiding**: there exist no indices $1 \leqslant a < b < c < d \leqslant n$ with $\pi(b) < \pi(a) < \pi(d) < \pi(c)$. We need some facts about vexillary permutations; further details consistent with the terminology and notation used here may be found in [KMY05] and the references therein.

Throughout we will identify a partition λ with its Young diagram. There is a partition λ associated to π: let the kth diagonal of λ (those boxes $\{(i, k+i)\}$) have as many boxes as the kth diagonal of $D(\pi)$, for each k. Indeed, this sets up a natural bijection between the boxes of λ and the boxes of $D(\pi)$, taking the jth box down in the kth diagonal to the jth box down in the kth diagonal. (The difference is that in $D(\pi)$ there may be spaces in between the boxes.) This bijection also defines a **flagging** $\vec{n}$ on the rows of λ. Namely, $n_i \in \mathbb{N}_+$ equals the row of $D(\pi)$ containing the box corresponding to the rightmost box of the i^{th} row of λ. We will thus speak interchangeably about π and the pair $(\lambda, \vec{n})$.

In Example 5.1, the permutation π is vexillary, $\lambda = (7, 6, 4, 3, 2)$, and $\vec{n} = (1, 2, 4, 6, 7)$.

We remark that $\vec{n}$ need not be a weakly increasing sequence. For instance, if $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 7 & 4 & 5 & 8 & 1 & 3 & 6 \end{pmatrix}$, then $\lambda = (5, 3, 2, 2, 1)$ and $\vec{n} = (2, 5, 4, 4, 5)$.

Call a set-valued tableau τ with shape λ $\vec{n}$**-flagged** if the maximum (so, the last) entry in each row is bounded above by the corresponding entry of $\vec{n}$.

Extend the definition of the **empty-face tableau** in the obvious way: it is the union of all the $\vec{n}$-flagged semistandard tableau on the shape λ. Let this set-valued tableau be denoted by $E_{\lambda,\vec{n}}$. (Note that $E_{\lambda,\vec{n}}(b)$ is an interval in the natural numbers $\mathbb{N}$: the smallest entry is the row position of $b \in \lambda$ while the largest entry is the position of the corresponding box of $D(\pi)$, under the bijection between λ and $D(\pi)$ described above.) This gives rise to a tableau complex $\Delta(\lambda, \vec{n})$ generalizing that described in Section 1.2:

Corollary 5.2. *For a partition λ and a flagging $\vec{n}$ associated to a vexillary permutation π, $\Delta(\lambda, \vec{n})$ is a simplicial ball, and its interior faces are the flagged semistandard set-valued Young tableaux.*

5.2. **Formulae for vexillary Grothendieck polynomials.** For each permutation $\pi \in S_n$ there is a **(double) Grothendieck polynomial**

$$\mathfrak{G}_\pi(x_1, \ldots, x_n, y_1, \ldots y_n) \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1^{\pm 1}, \ldots, y_n^{\pm 1}]$$

of Lascoux and Schützenberger [LS82]. The case that π is vexillary has been of specific interest; see [Fu92, KM01, KMY05]. We give tableau formulae in this setting.

Let $SVT(\lambda, \vec{n})$ denote the semistandard set-valued tableaux of shape λ and flagging $\vec{n}$. Similarly, denote by $SSYT(\lambda, \vec{n})$ and $LSVT(\lambda, \vec{n})$ the corresponding set of semistandard and limit semistandard tableaux, respectively. For a set-valued tableau τ, let $\tau(b)$ denote the set of entries in box b.

Corollary 5.3. *Let $\pi \in S_n$ be a vexillary permutation and $(\lambda, \vec{n})$ be the associated partition and flagging. Each of the following is a formula for the double Grothendieck polynomial $\mathfrak{G}_\pi(\mathbf{x}, \mathbf{y})$.*

$$\sum_{\tau \in LSVT(\lambda,\vec{n})} \prod_{b \in \lambda} \prod_{i \in \tau(b)} (1 - x_i y_{i+j(b)}^{-1}) \prod_{i \in E_{\lambda,\vec{n}}(b) \setminus \tau(b)} x_i y_{i+j(b)}^{-1}$$

$$\sum_{\tau \in SVT(\lambda,\vec{n})} (-1)^{|\tau|-|\lambda|} \prod_{b \in \lambda} \prod_{i \in \tau(b)} (1 - x_i y_{i+j(b)}^{-1})$$

$$\sum_{\tau \in SSYT(\lambda,\vec{n})} \prod_{b \in \lambda} \prod_{i \in \tau(b)} (1 - x_i y_{i+j(b)}^{-1}) \prod_{i \in E_{\lambda,\vec{n}}(b) \setminus N_\tau(b)} x_i y_{i+j(b)}^{-1}$$

Here, $j(b) = c(b) - r(b)$ is the difference of the column and row indices of the box $b \in \lambda$. Moreover, in the last formula, N_τ is the tableau obtained by adding to each box b all entries of $E(b)$ either smaller than the entry of $\tau(b)$, or larger than the entry of $\tau(b)$ but such that replacing $\tau(b)$ with this larger number would not give a tableau in $SVT(\lambda, \vec{n})$.

Proof. Formally, the second formula in the corollary is obtained as follows. Compute the K-polynomial of $\Delta_{\lambda,\vec{n}}$ via Theorem 4.3, using λ, b, and i here in place of X, x, and y there. Then, for each fixed b and i, substitute the expression $x_i y_{i+j(b)}^{-1}$ for the variable $t_{(b \mapsto i)}$.

It was shown in [KMY05] that the second formula equals the desired double Grothendieck polynomial. Therefore, applying the same substitution procedure to the results of Proposition 4.1 and Theorem 4.5 yields two more formulae for the same Grothendieck polynomial. These formulae are, respectively, the first and third formulae here. ☐

In the above proof we appealed to [KMY05] to confirm that our K-polynomials are in fact Grothendieck polynomials. Let us briefly sketch how this can be proved directly; we refer the reader to [KMY05] for terminology. To each vexillary permutation π there is an *accessible box* of λ. From this one can define two vexillary permutations $\pi_{\mathcal{P}}$ and $\pi_{\mathcal{C}}$. We obtain a safe vertex of the flagged Young tableau complex by removing the largest entry of $E_{\lambda,\vec{n}}$ that appears in the accessible box. The resulting deletion and star subcomplexes are naturally isomorphic to (multicones over) the flagged Young tableau complexes for $\pi_{\mathcal{P}}$ and $\pi_{\mathcal{C}}$ respectively. The recursion from Proposition 4.7 is precisely Lascoux's transition formula for vexillary Grothendieck polynomials [La01, La03] (after the substitution $t_{(x \mapsto y)} \mapsto x_i y_{i+j(b)}^{-1}$). Thus, since both polynomials satisfy the same recursion (and initial conditions), they are equal.

Specializations of these formulae are of interest. Suppose we set $y_j = 1$ for each j and replace x_i with $1 - x_i$ for each i. If we assume furthermore that π is Grassmannian, then we obtain Buch's formula [Bu02] for the single Grothendieck polynomial $\mathfrak{G}_\lambda(\mathbf{1} - \mathbf{x})$ [Bu02]. If instead we take the lowest degree terms of the polynomial, we obtain Wachs's formula for a flagged Schur polynomial [Wa85]. Making both of these specializations gives the classical tableau formula for an ordinary Schur polynomial.

We remark that it is also possible to use similar methods to extend these results to give set-valued skew tableau formulae for "321-avoiding permutations" (see Example 2.9).

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