TORIC DEGENERATION OF SCHUBERT VARIETIES AND GELFAND–TSETLIN POLYTOPES

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ABSTRACT. This note constructs the flat toric degeneration of the manifold $\mathcal{F}\ell_n$ of flags in $\mathbb{C}^n$ from [GL96] as an explicit GIT quotient of the Gröbner degeneration in [KM03]. This implies that Schubert varieties degenerate to reduced unions of toric varieties, associated to faces indexed by rc-graphs (reduced pipe dreams) in the Gelfand–Tsetlin polytope. Our explicit description of the toric degeneration of $\mathcal{F}\ell_n$ provides a simple explanation of how Gelfand–Tsetlin decompositions for irreducible polynomial representations of GL_n arise via geometric quantization.

INTRODUCTION

A number of recent developments at the intersection of algebraic geometry and combinatorics have exploited degenerations of certain varieties related to linear algebraic groups. Sometimes the varieties involved have been classical flag and Schubert varieties [Kog00, Vak03a, Vak03b], and other times they have been closely related affine varieties [KM03, KMS03]. In all cases, it has been vital not only to construct an appropriate family degenerating the primal variety, but also to identify combinatorially all components occurring in the degenerate limit. Indeed, this is what geometrically produces (or reproduces) various combinatorial formulae (for classical objects such as Littlewood–Richardson coefficients [Ful97], or for universally defined polynomials discovered more recently [LS82a, LS82b, Ful99, BF99, Buc02]): components of the special fiber correspond to combinatorial summands in the desired formula.

Independently motivated degenerations of similar flavors have appeared in areas related to representation theory, particularly standard monomial theory [GL96, Chi00, Cal02]. The primal varieties there have been generalized flag and Schubert varieties in arbitrary type, with the limiting fibers being toric varieties, or reduced unions thereof.

Our goal in this note is to create a single geometric framework relating some of the more natural degenerations above. To this end, we express the flat toric degeneration of the manifold $\mathcal{F}\ell_n$ of flags in $\mathbb{C}^n$ from [GL96], which is a special case of the degenerations in [Cal02], as a quotient of the Gröbner degeneration of $n \times n$ matrices M_n in [KM03]. The quotient is constructed by deforming the action of the lower triangular matrices $B \subset GL_n$ on the space M_n of matrices. More precisely, we construct an explicit action of B on $M_n \times \mathbb{C}$ and define the GIT quotient $B \backslash\!\!\backslash (M_n \times \mathbb{C})$, which is the total family $\mathcal{F}$ of the desired degeneration and still fibers over the line $\mathbb{C}$.

This degeneration can also be thought of simply as a Plücker embedding of the Gröbner degeneration from [KM03]. The closure of the image of this embedding is

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the family $\mathcal{F}$ of projective varieties over the affine line $\mathbb{C}$, whose fiber $\mathcal{F}(1)$ over $1 \in \mathbb{C}$ is the flag variety $\mathcal{F}\ell_n$, and whose fiber $\mathcal{F}(0)$ over $0 \in \mathbb{C}$ is the normal toric variety associated to the *Gelfand–Tsetlin polytope* from representation theory [GT50, GS83].

Two consequences result from our explicit description of the family $\mathcal{F}$. First, since $\mathcal{F}$ is derived from Gröbner degeneration, it induces a subfamily degenerating every Schubert variety in $\mathcal{F}\ell_n$. Therefore—and this is the main point—applying the combinatorial characterization of degenerated matrix Schubert varieties in [KM03], we characterize in Theorem 8 *which faces* of the Gelfand–Tsetlin toric variety occur in degenerate Schubert varieties of $\mathcal{F}\ell_n$. Namely, these faces correspond by [Kog00] to combinatorial diagrams called *rc-graphs* (or *reduced pipe dreams*) [FK96b, BB93].

Our second consequence is a simple explanation (Section 5) for how the classical *Gelfand–Tsetlin decomposition* of irreducible polynomial representations of GL_n into one-dimensional weight spaces arises geometrically from toric degeneration $\mathcal{F}$ of the flag manifold. The idea is to think of Gelfand–Tsetlin decomposition as geometric quantization on the total space of the family $\mathcal{F}$, directly extending the manner in which the Borel–Weil theorem is geometric quantization at the fiber $\mathcal{F}\ell_n = \mathcal{F}(1)$. The point is that sections of line bundles over $\mathcal{F}\ell_n$, which constitute irreducible representations of GL_n by Borel–Weil, canonically acquire at the special fiber of $\mathcal{F}$ an action of the torus densely embedded in the toric variety $\mathcal{F}(0)$. This produces a basis for sections over $\mathcal{F}\ell_n$ indexed by integer points of the Gelfand–Tsetlin polytope.

The methods in this note can be extended to partial flag manifolds in type A, but we believe the most exciting prospects for future research lie in extensions to other types. In particular, [GL96] and [Cal02] describe a number of degenerations of generalized flag varieties to toric varieties. Under these degenerations, Schubert varieties become unions of toric subvarieties. In "nice" cases [Lit98] (this is a technical term), the degenerate toric variety has an easily described moment polytope, in terms of generalizations of Gelfand–Tsetlin patterns. Identification of the components in degenerations of Schubert varieties could therefore provide arbitrary-type combinatorial generalizations of pipe dreams, such as those suggested by [FK96a] in type B.

The approach of degenerating generalized flag manifolds $P\backslash G$ instead of degenerating groups G sidesteps the use of an equivariant partial compactification of G to a vector space, which is suggested by [KM03] but is something we do not know how to do for arbitrary linear algebraic groups. Furthermore, one should be able to obtain positive combinatorial formulae for Schubert classes in arbitrary type (though perhaps not a definition of Schubert *polynomial*) by summing over components in torically degenerated Schubert varieties. The shapes taken by such combinatorial formulae would echo the manner in which [Kog00] geometrically decomposes Schubert classes, agreeing with the combinatorially positive formula in [BJS93, FS94] for the Schubert polynomials of Lascoux and Schützenberger [LS82a].

Part of our purpose in writing this note was to make toric degenerations of flag and Schubert varieties as in [GL96, Chi00, Cal02] accessible to an audience unaccustomed to specialized language surrounding arbitrary finite type root systems and standard monomial theory. In particular, we thought it important to include an equivalent characterization of the family $\mathcal{F}$ in the language of sagbi bases (Theorem 5), whose elementary definition and properties we review in Section 2. This version of the toric degeneration of $\mathcal{F}\ell_n$ is implicit in [GL96], but so much so as to be difficult to locate. In addition, although the identification of the limiting fiber $\mathcal{F}(0)$ as the Gelfand–Tsetlin toric variety can be derived using results of [Cal02] and [Lit98], we include the appropriate combinatorial arguments for the reader's convenience (Section 3).

Organization. Our deformation of the Borel action on M_n is constructed in Section 1. Then Section 2 uses the results of [GL96] to show that the quotient $\mathcal{F} = B\backslash\!\backslash(M_n \times \mathbb{C})$ is a flat family. Section 3 proves that the zero fiber of $\mathcal{F}$ is the Gelfand–Tsetlin toric variety. The connection to [KM03] is exhibited in Section 4 by explaining how Schubert varieties behave inside of $\mathcal{F}$. The final section deals with geometric quantization of the degeneration, by analyzing sections of line bundles over the family $\mathcal{F}$, thereby constructing Gelfand–Tsetlin decompositions geometrically.

1. Degenerating the Borel action

Thinking about GL_n as a subset of $n \times n$ matrices M_n allows us to think about the flag manifold $\mathcal{F}\ell_n = B\backslash GL_n$ as a GIT quotient of M_n by B whose precise definition is given in Section 2. In this section we construct a degeneration of the action of B on M_n, and in the next section we explain what happens to the GIT quotient under this degeneration.

The group $(GL_n)^n$ has a left action on M_n columnwise: if $Z \in M_n$ has columns $Z_1, \ldots, Z_n$, then $\gamma = (\gamma_1, \ldots, \gamma_n) \in (GL_n)^n$ acts via

$$\gamma Z \quad = \quad \text{matrix with columns } \gamma_1 Z_1, \ldots, \gamma_n Z_n. \tag{1}$$

The torus of $(GL_n)^n$ under the left action coincides with the standard torus inside $GL(M_n)$, scaling separately each entry of any given matrix.

Let $B_\Delta \subset (GL_n)^n$ be the image of the *lower* triangular Borel subgroup $B \subset GL_n$ under the n-fold diagonal embedding in $(GL_n)^n$, so $B_\Delta = \{(b, \ldots, b) \mid b \in B\}$.

For every one-parameter subgroup $T \cong \mathbb{C}^*$ inside the torus of $(GL_n)^n$ consisting of sequences of diagonal matrices, denote by $\tilde{\tau} \in T$ the element corresponding to the complex number $\tau \in \mathbb{C}^*$. Given a matrix $\omega = (\omega_{ij})$ of integers, let the one-parameter subgroup $T(\omega)$ consist of sequences of diagonal matrices, with the j^{th} component of $\tilde{\tau}$ being the diagonal matrix $\tilde{\tau}_j = \operatorname{diag}(\tau^{\omega_{1j}}, \ldots, \tau^{\omega_{nj}})$ for the j^{th} column of ω.

For the rest of the paper, fix the matrix ω whose entries equal

$$\begin{aligned} \omega_{ij} &= 3^{n-i-j} && \text{if } i + j \leq n, \\ \text{and} \quad \omega_{ij} &= 0 && \text{if } i + j > n. \end{aligned} \tag{2}$$

For instance, when $n = 5$, we get the following 5×5 matrix:

$$\omega \quad = \quad \begin{bmatrix} 27 & 9 & 3 & 1 & 0 \\ 9 & 3 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Consider the family $\mathcal{B}^* \subset B^n \times \mathbb{C}^*$ of subgroups of B^n with fiber

$$B(\tau) \quad = \quad \tilde{\tau}^{-1} B_\Delta \tilde{\tau} \tag{3}$$

over $\tau \in \mathbb{C}^*$, where $\tilde{\tau}$ lies in the one-parameter subgroup $T(\omega)$ corresponding to ω. When $n = 5$, for instance, this multiplies the entries in B^n by powers of τ as follows.

$$\left(\begin{bmatrix} 1 &&&& \\ \tau^{18} & 1 &&& \\ \tau^{24} & \tau^{6} & 1 && \\ \tau^{26} & \tau^{8} & \tau^{2} & 1 & \\ \tau^{27} & \tau^{9} & \tau^{3} & \tau & 1 \end{bmatrix}, \begin{bmatrix} 1 &&&& \\ \tau^{6} & 1 &&& \\ \tau^{8} & \tau^{2} & 1 && \\ \tau^{9} & \tau^{3} & \tau & 1 & \\ \tau^{9} & \tau^{3} & \tau & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 &&&& \\ \tau^{2} & 1 &&& \\ \tau^{3} & \tau & 1 && \\ \tau^{3} & \tau & 1 & 1 & \\ \tau^{3} & \tau & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 &&&& \\ \tau & 1 &&& \\ \tau & 1 & 1 && \\ \tau & 1 & 1 & 1 & \\ \tau & 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 &&&& \\ 1 & 1 &&& \\ 1 & 1 & 1 && \\ 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}\right)$$

The family $\mathcal{B}^*$ extends to a family over all of $\mathbb{C}$:

Definition 1. The family $\mathcal{B} \subset B^n \times \mathbb{C}$ has fiber $B(\tau)$ over $\tau \in \mathbb{C}^*$, and fiber $B(0)$ consisting of sequences $(b_1, \ldots, b_n) \in B^n$, where b_j is obtained from the matrix b_n by setting to 0 all entries in columns $1, \ldots, n-j$ strictly below the main diagonal.

When $n = 5$, elements in the special fiber $B(0)$ look heuristically like:

$$\left(\begin{bmatrix} b &&&& \\ & b &&& \\ && b && \\ &&& b & \\ &&&& b \end{bmatrix}, \begin{bmatrix} b &&&& \\ & b &&& \\ && b && \\ &&& b & \\ &&& b & b \end{bmatrix}, \begin{bmatrix} b &&&& \\ & b &&& \\ && b && \\ && b & b & \\ && b & b & b \end{bmatrix}, \begin{bmatrix} b &&&& \\ & b &&& \\ & b & b && \\ & b & b & b & \\ & b & b & b & b \end{bmatrix}, \begin{bmatrix} b &&&& \\ b & b &&& \\ b & b & b && \\ b & b & b & b & \\ b & b & b & b & b \end{bmatrix}\right)$$

Lemma 2. *There is a canonical algebraic group isomorphism $B \times \mathbb{C} \to \mathcal{B}$ over $\mathbb{C}$.*

Proof. Use that $B \cong B_\Delta$ by sending $b \mapsto b_\Delta = (b, \ldots, b)$. For $\tau \neq 0$ the isomorphism is now by (3), sending b_Δ to $\tilde{\tau}^{-1} b_\Delta \tilde{\tau}$. For $\tau = 0$, the map sets to 0 all entries in columns $1, \ldots, n-j$ strictly below the main diagonal in the j^{th} entry of $(b, \ldots, b)$.

Elementary computation shows that if b is a lower triangular matrix, then the matrix $\tilde{\tau}_j^{-1} b \tilde{\tau}_j$ has no negative powers of τ, and setting τ to 0 has the effect of setting to 0 all entries in columns $1, \ldots, n-j$ strictly below the main diagonal of b. Hence the $\tau = 0$ case above really is obtained from the $\tau \neq 0$ case by taking limits as $\tau \to 0$. □

The family $\mathcal{B}$ of groups acts fiberwise on $M_n \times \mathbb{C}$, but Lemma 2 allows us to view this fiberwise action as a single action of B on the total space $M_n \times \mathbb{C}$. The actions on all fibers $M_n \times \tau$ are isomorphic for $\tau \in \mathbb{C}^*$, in the sense that the map $Z \times 1 \mapsto \tilde{\tau}^{-1} Z \times \tau$ identifies $M_n \times 1$ with $M_n \times \tau$ equivariantly with respect to the actions of B on the fibers over 1 and τ. However, when τ equals zero, B acts on the j^{th} column as the product of an $n-j$ dimensional torus (in the upper-left corner) and a smaller Borel group with j columns (in the lower-right corner).

The action of $B = B(0)$ on $M_n \times 0$ commutes with an $\binom{n}{2}$ dimensional torus action, which scales all entries lying strictly above the main antidiagonal in each $n \times n$ matrix. We shall see that this torus acts on the degenerated $\mathcal{F}\ell_n$ to make it a toric variety.

2. Degeneration of Plücker coordinates

Degenerating the action of B on M_n via the action of $B \times \mathbb{C}$ in Lemma 2 on $M_n \times \mathbb{C}$ induces a degeneration of the GIT quotient of M_n by B. Using the results of Gonciulea and Lakshmibai [GL96], we show that the GIT quotient $B \backslash\!\backslash (M_n \times \mathbb{C})$, when defined appropriately, flatly degenerates the flag manifold $\mathcal{F}\ell_n$ to a toric variety.

Let U be the lower triangular matrices with 1's on the diagonal (the unipotent radical of B). As can be done in the general setting of B actions, we define the GIT quotient of M_n by B to be the "multiple Proj" of the ring of U-invariant functions on M_n. Let us be more precise in the present case.

For a subset $J \subset \{1, \ldots, n\}$ of size k, define $\Delta_J(Z)$ to be the k-minor of an $n \times n$ matrix Z whose columns are given by the set J and whose rows $1, \ldots, k$ are top-justified. Writing $\mathbb{C}[\mathbf{z}]$ with $\mathbf{z} = (z_{ij})_{i,j=1}^n$ for the coordinate ring of M_n, the set of *Plücker coordinates* consists of all minors having the form

$$p_J \;=\; \Delta_J(n \times n \text{ matrix of variables } \mathbf{z}).$$

They generate the ring $\mathbb{C}[\mathbf{p}] \subset \mathbb{C}[\mathbf{z}]$ of U invariant functions on M_n. This invariant ring $\mathbb{C}[\mathbf{p}]$ can be expressed as a quotient

$$\mathbb{C}[\mathbf{x}^1] \otimes \cdots \otimes \mathbb{C}[\mathbf{x}^n] \;\twoheadrightarrow\; \mathbb{C}[\mathbf{p}]$$

of the tensor product over $\mathbb{C}$ of n polynomial rings $\mathbb{C}[\mathbf{x}^k]$, where $\mathbf{x}^k$ is a set of variables x_J indexed by the size k subsets of $\{1, \ldots, n\}$. Thus the spectrum of $\mathbb{C}[\mathbf{p}]$ is a subvariety of the vector space underlying the exterior algebra $\bigwedge^* \mathbb{C}^n$ of $\mathbb{C}^n$. This gives rise to the *multiple* Proj of $\mathbb{C}[\mathbf{p}]$, by which we mean the corresponding subscheme of $\prod_{k=1}^n \mathbb{P}(\bigwedge^k \mathbb{C}^n) = \prod_{k=1}^n \mathrm{Proj}(\mathbb{C}[\mathbf{x}^k])$. This *Plücker embedding* of the flag manifold $\mathcal{F}\ell_n$ is the GIT quotient of M_n by B.

Now let us turn to the GIT quotient of $M_n \times \mathbb{C}$ by the action of B constructed in the previous section. In this context, we are thinking of $M_n \times \mathbb{C}$ as a (trivial) family over $\mathbb{C}$, and we wish to quotient out by the fiberwise action of the family $\mathcal{B}$ of groups parametrized by $\mathbb{C}$. By definition, this GIT quotient is the multiple Proj of the $\mathbb{C}[t]$-algebra of U-invariant functions on $M_n \times \mathbb{C}$. To describe some of these invariant functions, we need a preliminary result.

Think of the matrix ω as a weighting on the coordinate ring $\mathbb{C}[\mathbf{z}]$ of M_n under which each variable z_{ij} has weight ω_{ij}. Also, let $\Delta_{I,J}$ denote the minor with rows I and columns J. The *antidiagonal* of such a minor is the product of all entries along the main antidiagonal in the corresponding square matrix.

Lemma 3. *If every variable dividing the antidiagonal term of the minor $\Delta_{I,J}(Z)$ in the generic matrix Z lies on or above the main antidiagonal of Z, then the unique $\mathbf{z}$-monomial in $\Delta_{I,J}(Z)$ with the lowest weight is its antidiagonal term.*

Proof. It suffices to prove the lemma when I and J have cardinality 2, because every $\mathbf{z}$-monomial in each minor can be made into the antidiagonal term by successively replacing 2×2 diagonals with 2×2 antidiagonals. In the 2×2 case, let $I = \{i, i+k\}$ and $J = \{j, j+\ell\}$ with $i, j, k, \ell \geq 1$. The weights on the two terms in $\Delta_{I,J}(Z)$ satisfy

$$3^{n-i-j} + 3^{n-i-k-j-\ell} > 3^{n-i-k-j} + 3^{n-i-j-\ell},$$

which proves the lemma. □

Denote by $\tilde{t}$ the n-tuple of $n \times n$ diagonal matrices whose j^{th} diagonal entry in the i^{th} matrix is $t^{\omega_{ij}}$, and define $\tilde{t}Z$ for $\tilde{t} = \gamma$ as in (1) for the matrix Z of variables. In addition, for $J = \{j_1, \cdots, j_k\}$, let

$$\omega_J \;=\; \sum_{i=1}^{k} \omega_{i,n+1-j_i}$$

be the sum of weights along the antidiagonal of the square submatrix in rows $1, \ldots, k$ and columns J of ω. Then, as an immediate consequence of Lemma 3, we conclude that the polynomials

$$q_J \quad = \quad t^{-\omega_J} \Delta_J(\tilde{t}Z) \tag{4}$$

are U-invariants in $\mathbb{C}[\mathbf{z}, t]$ under the action of B resulting from Lemma 2. The power $t^{-\omega_J}$ precisely makes the antidiagonal term of q_J have coefficient ± 1.

Definition 4. Define the family $\mathcal{F}$ inside the product $\prod_{k=1}^{n} \mathbb{P}(\bigwedge^k \mathbb{C}^n) \times \mathbb{C}$ over the line $\mathbb{C} = \mathrm{Spec}(\mathbb{C}[t])$ as the multiple Proj of the subalgebra $\mathbb{C}[q_J \mid J \subseteq \{1, \ldots, n\}]$ of $\mathbb{C}[\mathbf{z}, t]$.

We wish to state the main result in this section in terms of sagbi bases. Recall that a *term order* on $\mathbb{C}[\mathbf{z}]$ is a multiplicative total order on monomials with $1 \in \mathbb{C}[\mathbf{z}]$ being smaller than any other monomial; see [Eis95, Chapter 15]. A set $\{f_1, \ldots, f_r\} \subset \mathbb{C}[\mathbf{z}]$ is a *sagbi basis* if the initial term $\mathsf{in}(f)$ of every polynomial f in the subalgebra $\mathbb{C}[f_1, \ldots, f_r]$ lies inside the *initial subalgebra* generated by the initial terms $\mathsf{in}(f_1), \ldots, \mathsf{in}(f_r)$. The initial subalgebra is generated by monomials, so its multiple Proj is a toric variety; following the conventions of [Stu96], we do not assume that toric varieties are normal. Choosing a weight order inducing the given term order [Eis95, Chapter 15] allows us to express the original algebra and its initial algebra as the fibers over 1 and 0 of a flat family of subalgebras of $\mathbb{C}[\mathbf{z}]$. In fact, $\{f_1, \ldots, f_r\}$ form a sagbi basis if and only if this degeneration to the initial subalgebra is flat.

The term orders on the coordinate ring $\mathbb{C}[\mathbf{z}]$ of M_n that interest us are *antidiagonal* and *diagonal*. By definition, the leading term of any minor in the matrix of variables under an (anti)diagonal term order is its (anti)diagonal term, namely the product of all entries on the (anti)diagonal of the corresponding square submatrix. Initial terms of polynomials other than minors will not be important in what follows.

Theorem 5. *The polynomials q_J from (4) generate the $\mathbb{C}[t]$-algebra of U-invariant functions inside $\mathbb{C}[\mathbf{z}, t]$, so $\mathcal{F}$ is the GIT quotient family $B \backslash\!\backslash (M_n \times \mathbb{C})$ flatly degenerating the flag manifold $\mathcal{F}\ell_n = \mathcal{F}(1)$ to a toric variety $\mathcal{F}(0)$. In fact, the Plücker coordinates p_J constitute a sagbi basis for any diagonal or antidiagonal term order.*

Proof. We first show that the second sentence implies the first. We may as well assume by symmetry (reflecting left-to-right) that the term order is antidiagonal.

Setting $t = 1$ in (4) obviously yields the Plücker coordinates p_J. Therefore the polynomials q_J generate the U-invariants in $\mathbb{C}[\mathbf{z}, t^{\pm 1}]$ over the coordinate ring $\mathbb{C}[t^{\pm 1}]$ of $\mathbb{C}^*$, because the Plücker coordinates p_J generate the U-invariants over $t = 1$, and the family of U-invariants is trivial by scaling outside $t = 0$. Intersecting the U-invariants in $\mathbb{C}[\mathbf{z}, t^{\pm 1}]$ with $\mathbb{C}[\mathbf{z}, t]$ yields U-invariants in $\mathbb{C}[\mathbf{z}, t]$, because a polynomial function on $M_n \times \mathbb{C}$ is U-invariant if and only if its restriction to $M_n \times \mathbb{C}^*$ is. Therefore, we must show that the polynomials q_J generate as a $\mathbb{C}[t]$ algebra the intersection with $\mathbb{C}[\mathbf{z}, t]$ of the subalgebra they generate inside $\mathbb{C}[\mathbf{z}, t^{\pm 1}]$. This follows from the sagbi property.

For the proof of the second sentence, we assume the term order is diagonal, to agree with [GL96]. Let H be the set of all subsets of $\{1, \ldots, n\}$. Following [GL96], define a partial order on H as follows. For $I = \{i_1 < \cdots < i_k\}$ and $J = \{j_1 < \cdots < j_\ell\}$ set

$$I \geq J \iff k \leq \ell \text{ and } i_s \geq j_s \text{ for } 1 \leq s \leq k.$$

This makes H a distributive lattice. It is shown in [GL96] that if $k \leq \ell$, then the meet and join of I and J are characterized by

$$\begin{aligned} I \wedge J &= (\min(i_1, j_1), \ldots, \min(i_k, j_k), j_{\ell+1}, \ldots, j_\ell) \\ \text{and} \quad I \vee J &= (\max(i_1, j_1), \ldots, \max(i_k, j_k)). \end{aligned}$$

In the degeneration from [GL96, Theorems 5.2 and 10.6], the fiber over 1 is the subalgebra of $\mathbb{C}[\mathbf{z}]$ generated by Plücker coordinates. The initial algebra in [GL96] maps surjectively to the 'diagonal' semigroup algebra generated by the diagonals, because the degenerated Plücker relations $x_I x_J - x_{I\wedge J} x_{I\vee J}$ in [GL96] hold on diagonals of Plücker coordinates. But the initial algebra in [GL96] is a domain of Krull dimension $\binom{n+1}{2}$, as is the diagonal semigroup algebra (whose exponents generate the additive group of upper-right-triangular integer matrices), so the surjection is an isomorphism. Hence the diagonal semigroup and Plücker algebras have equal Hilbert series, by flatness of the degeneration in [GL96]. Since the diagonal semigroup algebra is certainly contained inside the initial algebra, which has the same Hilbert series by flatness again, the diagonal semigroup algebra must equal the initial algebra. □

Remark 6. For each nonzero τ, the fiber $\mathcal{F}(\tau)$ equals $B(\tau)\backslash GL_n(\tau)$, where $GL_n(\tau)$ is the set of $n \times n$ matrices Z with nonzero τ-determinant $\det(\tilde{\tau} Z)$. When $\tau = 0$, we can define $GL_n(0)$ as the set of matrices with nonzero entries along the antidiagonal. The quotient $B(0)\backslash GL_n(0)$ is still well defined, but no longer equal to $\mathcal{F}(0)$: geometrically, under the zeroth Plücker map, whose coordinates are obtained by setting $t = 0$ in (4), the variety $GL_n(0)$ has image equal to an open cell inside the toric variety $\mathcal{F}(0)$. □

3. Gelfand–Tsetlin polytopes

Next we show that the zero fiber $\mathcal{F}(0)$ is the normal toric variety associated to a Gelfand–Tsetlin polytope defined as follows. Let $\lambda = (\lambda_1 > \cdots > \lambda_n)$ be a nonincreasing sequence of nonnegative integers. An array $\Lambda = (\lambda_{i,j})_{i+j \leq n+1}$ of real numbers is a *Gelfand–Tsetlin pattern* for λ if $\lambda_{i,1} = \lambda_i$ for all $i = 1, \ldots, n$, and $\lambda_{i,j} \geq \lambda_{i,j+1} \geq \lambda_{i+1,j}$ for $i, j = 1, \ldots, n$. Equivalently, entries in Gelfand–Tsetlin patterns Λ decrease in the directions indicated by the arrows in diagram below, whose left column is λ:

$$\begin{array}{ccccccc} \lambda_{1,1} & \rightarrow & \lambda_{1,2} & \rightarrow & \lambda_{1,3} & \rightarrow & \cdots \\ \downarrow & \swarrow & \downarrow & \swarrow & & & \\ \lambda_{2,1} & \rightarrow & \lambda_{2,2} & \rightarrow & \cdots & & \\ \downarrow & \swarrow & & & & & \\ \lambda_{3,1} & \rightarrow & \cdots & & & & \\ \downarrow & & & & & & \\ \vdots & & & & & & \end{array} \tag{5}$$

The *Gelfand–Tsetlin polytope* P_λ is the convex hull of all integer Gelfand–Tsetlin patterns for λ. This polytope defines the *Gelfand–Tsetlin normal toric variety* together with its projective embedding. For background on normal toric varieties, see [Ful93].

Set $a_k = \lambda_k - \lambda_{k+1}$ for $k = 1, \ldots, n$, where by convention $\lambda_{n+1} = 0$, and assume $a_k \geq 1$ for all k. Recall that we expressed the flag manifold $\mathcal{F}\ell_n$ as a subvariety of the product $\mathbb{P}_1 \times \cdots \times \mathbb{P}_n$, where $\mathbb{P}_k = \mathbb{P}(\bigwedge^k \mathbb{C}^n)$. Since all a_k are strictly positive, the integer sequence $\mathbf{a} = (a_1, \ldots, a_n)$ corresponds to a choice of very ample line bundle $\mathcal{O}_{\mathcal{F}\ell_n}(\mathbf{a})$ on $\mathcal{F}\ell_n$, namely the result of tensoring together the pullbacks of the bundles $\mathcal{O}_{\mathbb{P}_1}(a_1), \ldots, \mathcal{O}_{\mathbb{P}_n}(a_n)$ to $\mathcal{F}\ell_n$. In fact, we get a choice of very ample line bundle $\mathcal{O}_{\mathcal{F}}(\mathbf{a})$ on the entire family $\mathcal{F}$ from Definition 4, to get an embedding of the family

$$\mathcal{F} \to \mathbb{P}_\lambda \times \mathbb{C} \quad \text{where} \quad \mathbb{P}_\lambda = \mathbb{P}\big(\otimes_{k=1}^{n} \operatorname{Sym}^{a_k}(\textstyle\bigwedge^k \mathbb{C}^n)\big). \tag{6}$$

The image of the zero fiber $\mathcal{F}(0)$ is a toric variety (though not a normal one, a priori) projectively embedded inside $\mathbb{P}_\lambda$ by a line bundle $\mathcal{O}_{\mathcal{F}(0)}(\mathbf{a})$.

Let α_I be the exponent vector on the antidiagonal monomial of the Plücker coordinate p_I. Such exponent vectors are elements in $\mathbb{Z}^{n^2}$ that look, for example, like

			1
	1		
1			

for p_{124} and

		1	
1			

for p_{13}.

The vector space of global sections of $\mathcal{O}_{\mathcal{F}(0)}(\mathbf{a})$ decomposes into a multiplicity-one direct sum of weight spaces. The set of weights occurring in this decomposition is

$$\Upsilon_\lambda = \big\{\text{sums } \textstyle\sum \alpha_I \text{ in which } a_k \text{ of the indexing sets } I \subseteq \{1, \ldots, n\} \text{ have size } k\big\}. \tag{7}$$

Proposition 7. *The projective embedding $\mathcal{F}(0) \to \mathbb{P}_\lambda$ is the projective embedding of the Gelfand–Tsetlin normal toric variety associated to the polytope P_λ.*

Proof. By standard results about projective embeddings of normal toric varieties [Ful93], it suffices to identify the set Π_λ of lattice points in P_λ with Υ_λ. This we shall do for all λ, not just those producing strictly positive $a_1, \ldots, a_n$.

The set Π_λ sits inside an integer lattice of rank $\frac{n(n-1)}{2}$, and we claim that the linear map $\phi : \mathbb{Z}^{n^2} \to \mathbb{Z}^{\frac{n(n-1)}{2}}$ given by

$$\lambda_{ij} \;=\; a_{i,j} + a_{i,j+1} + \cdots + a_{i,n+1-i}$$

induces the required bijection from Υ_λ to Π_λ.

To check this, consider the map $\psi : \mathbb{Z}^{\frac{n(n-1)}{2}} \to \mathbb{Z}^{n^2}$ given by

$$a_{ij} \;=\; \lambda_{i,j} - \lambda_{i-1,j}.$$

Notice that the composite maps $\phi\circ\psi$ and $\psi\circ\phi$ are identities on Π_λ and Υ_λ respectively. It remains to check $\phi(\Upsilon_\lambda) \subseteq \Pi_\lambda$ and $\psi(\Pi_\lambda) \subseteq \Upsilon_\lambda$; these are simple exercises in linear algebra that go as follows.

To check $\phi(\Upsilon_\lambda) \subseteq \Pi_\lambda$, consider an element $\alpha = \sum \alpha_I$ of Υ_λ. The $\lambda_{i,1}$ coordinate of $\phi(\alpha)$ is the sum of all a's in row i, and this equals $a_i + \cdots + a_n = \lambda_i$, which is the number of indices I of size at least i. The $\lambda_{i,j}$ coordinate of $\phi(\alpha)$ is the sum of the a's in the horizontal strip between $a_{i,j}$ and the antidiagonal. This is not greater than $\lambda_{i-1,j}$, which is the sum of the one-longer horizontal strip starting at $a_{i-1,j}$. On the other hand, $\lambda_{i,j}$ is at least $\lambda_{i-1,j+1}$. Indeed, $\lambda_{i-1,j+1}$ is the sum of the entries in the horizontal strip of the same length as for $\lambda_{i,j}$, but shifted down one row and moved one column to the left. Since α is the sum of α_I's, the sum of entries for $\lambda_{i-1,j+1}$ is at most that for $\lambda_{i,j}$. Thus $\phi(\Upsilon_\lambda)$ is a subset of Π_λ.

Conversely, given a Gelfand–Tsetlin pattern Λ for λ we need to show that $\psi(\lambda)$ can be written as a sum $\sum \alpha_I$. (It will follow immediately that there are a_k indices I

of cardinality k, since the number of indices of cardinality at least k must be λ_k.) Derive a pattern Λ' from Λ by decreasing the last nonzero entry λ_{k,i_k} in each row by 1. Then Λ' is still a Gelfand–Tsetlin pattern. At the same time it is clear that $\psi(\Lambda) = \psi(\Lambda') + \alpha_I$ for the set $I = \{i_1 > \ldots > i_\ell\}$, where row ℓ of Λ is not zero but row $\ell+1$ of Λ is zero. Induction on the sum of the entries in Gelfand–Tsetlin patterns finishes the proof. □

4. Degenerating Schubert varieties

In this section we present our main theorem, which says that Schubert varieties degenerate inside the family $\mathcal{F}$ to unions of toric subvarieties given by rc-faces of the Gelfand–Tsetlin polytope. First, we must review the combinatorics involved.

Consider a finite subset R of $\{1,\ldots,n\} \times \{1\ldots,n\}$ and think of it as a network of pipes, which intersect at each $(i,j) \in R$ and do not intersect otherwise. Such subsets are called *diagrams* (but are also known as *pipe dreams*). For example, see Figure 1.

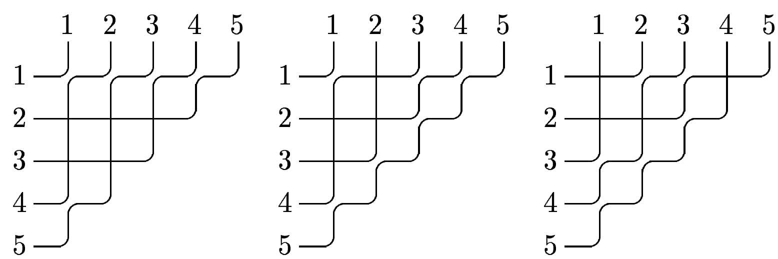


Figure 1. Diagrams that are given by $\{(2,1),(2,2),(2,3),(3,1),(3,2)\}$, $\{(1,2),(2,1),(2,2),(3,1)\}$ and $\{(1,1),(1,4),(2,1),(2,2)\}$.

Associate to each diagram R the permutation $w_R \in S_n$ such that the pipe entering row i exits column $w_R(i)$. For example, the permutations associated to the diagrams from Figure 1 are 15423, 14235 and 21534. The diagram R is an *rc-graph* (or *reduced pipe dream*) if no two strands intersect twice. The first and the third diagrams from Figure 1 are rc-graphs, while the second one is not. These diagrams were originally introduced in [FK96b]. They index the monomials in Schubert polynomials the same way that semistandard Young tableaux index monomials in Schur polynomials; see [BJS93], [FS94] and [FK96b] for details.

For an rc-graph R, let

- L_R be the coordinate subspace of M_n consisting of all matrices whose coordinates z_{ij} are zero for every crossing $(i,j) \in R$;
- F_R be the *rc-face* of the Gelfand–Tsetlin polytope given by setting $\lambda_{i,j} = \lambda_{i+1,j}$ for each $(i,j) \in R$; and
- T_R be the *rc-toric subvariety* of the Gelfand–Tsetlin toric variety with face F_R.

Next let us review some geometric ingredients for our main theorem and its proof. For a permutation $w \in S_n$, the *Schubert determinantal ideal* $I_w \subseteq \mathbb{C}[\mathbf{z}]$, defined by Fulton in [Ful92], is generated by all minors of size $1 + w_{qp}$ in the top left $q \times p$ submatrix $Z_{p\times q}$ of $Z = (z_{ij})$ for all q, p, where w_{qp} is the number of $i \leq q$ such that $w(i) \leq p$. The *matrix Schubert variety* for w [KM03] is by definition the zero set $\overline{X}_w$

of I_w. We denote by $X_w \subset \mathcal{F}\ell_n$ the *Schubert variety* obtained by projecting $\overline{X}_w \cap GL_n$ to $\mathcal{F}\ell_n$. This Schubert variety is the closure in $\mathcal{F}\ell_n$ of the B^+ orbit through the coset $B\overline{w}$, where the permutation matrix $\overline{w}$ has its nonzero entries at $(i, w(i))$, and B^+ is the Borel group of upper triangular matrices acting on the right of $\mathcal{F}\ell_n = B\backslash GL_n$.

In general, a flat family degenerating a variety Y does not induce degenerations on its subvarieties. However, Gröbner and sagbi degenerations of Y are canonically isomorphic to trivial families over $\mathbb{C}^*$. Any subvariety of Y defines an (isomorphically trivial) subfamily over $\mathbb{C}^*$, and hence a flat subfamily over all of $\mathbb{C}$ by taking the closure of this subfamily.

Theorem 8. *The quotient family $\mathcal{F} = B\backslash\!\backslash(M_n \times \mathbb{C})$ induces flat degenerations of Schubert subvarieties X_w of the complete flag manifold $\mathcal{F}\ell_n = \mathcal{F}(1)$ to reduced unions $\bigcup_{w(R)=w} T_R$ of toric subvarieties of the Gelfand–Tsetlin toric variety $\mathcal{F}(0)$.*

Proof. It was shown in [KM03] that the minors generating I_w constitute a Gröbner basis for any antidiagonal term order, and hence define a Gröbner degeneration of $\overline{X}_w$. Moreover, it was shown that any such degeneration—in particular (by Lemma 3) the one given by ω—degenerates the matrix Schubert variety $\overline{X}_w$ to the reduced union $\bigcup_{w(R)=w} L_R$ of rc-subspaces for w inside M_n. The image of the total space of this Gröbner degeneration under the family of Plücker embeddings given by coordinates (4) equals our family $\mathcal{F}$ by Theorem 5. On the other hand, the closure of the image of an rc-subspace L_R under the degenerated Plücker map obtained by setting $t = 0$ in (4) equals the corresponding rc-toric subvariety T_R by definition. □

The argument in the proof can be summarized as: the GIT quotient by B of the Gröbner degeneration in [KM03] equals the sagbi degeneration in Theorem 8.

Remark 9. The Schubert variety $X_w \subseteq \mathcal{F}\ell_n$ equals the intersection of the embedded subvariety $\mathcal{F}\ell_n \subset \prod \mathbb{P}(\bigwedge^k \mathbb{C}^n)$ with a set of hyperplanes, one hyperplane $p_I = 0$ for each subset $I = \{i_1, \ldots, i_k\}$ satisfying $k > w_{k,i_k}$, where w_{k,i_k} is the number of $i \leq k$ with $w(i) \leq i_k$. Intersecting the family $\mathcal{F}$ with the same set of hyperplanes produces the degeneration of X_w in Theorem 8. □

Remark 10. Using the involution on $\mathcal{F}\ell_n$ that switches the Plücker coordinate p_I with $p_{\bar{I}}$, where $\bar{I}$ is the complement of I, it can be shown that opposite Schubert varieties degenerate to unions of toric subvarieties associated to opposite rc-walls of the Gelfand–Tsetlin polytope. □

5. Gelfand–Tsetlin decomposition

This final section gives a geometric construction of the Gelfand–Tsetlin basis of an irreducible GL_n representation, by extending the Borel–Weil construction to the whole family $\mathcal{F}$. Our proof logically depends only on the Borel–Weil theorem. To introduce notation, we begin by reviewing the construction of Gelfand and Tsetlin [GT50].

For a dominant weight λ of GL_n, which by definition is a decreasing sequence $(\lambda_1 > \cdots > \lambda_n)$ of positive integers, let V^λ be the irreducible representation of GL_n with highest weight λ. For $n \geq i \geq 1$ identify GL_i with the subgroup of GL_n sitting in the bottom right $i \times i$ corner. As a GL_{n-1} representation, V^λ breaks up into a direct sum of irreducible components

$$V^\lambda = \bigoplus_{\mu \prec \lambda} V^\mu, \tag{8}$$

where the dominant weight $\mu = (\mu_1 > \cdots > \mu_{n-1})$ of GL_{n-1} satisfies $\mu \prec \lambda$ if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \cdots \geq \mu_{n-1} \geq \lambda_n,$$

so μ interpolates between λ. Iterating defines *partial Gelfand–Tsetlin decompositions*

$$V^\lambda = \bigoplus_{\Lambda_i} V^{\Lambda_i} \tag{9}$$

of V^λ into irreducible components for the action of GL_i, where Λ_i runs through chains $(\lambda \succ \lambda^{n-1} \succ \cdots \succ \lambda^i)$ with λ^j being a weight of GL_j. Gelfand and Tsetlin studied the decomposition of V^λ as a direct sum of one-dimensional subspaces V^Λ, one for each chain $\Lambda = (\lambda \succ \lambda^{n-1} \succ \cdots \succ \lambda^1)$. By definition, Λ lies in Π_λ, the set of integer Gelfand–Tsetlin patterns for λ. Hence we have the *Gelfand–Tsetlin decomposition*

$$V^\lambda = \bigoplus_{\Lambda \in \Pi_\lambda} V^\Lambda. \tag{10}$$

For a weight λ, consider the very ample line bundle $\mathcal{L} = \mathcal{O}_{\mathcal{F}}(\mathbf{a})$ from Section 3 over the family $\mathcal{F}$. Let $\mathcal{L}_\tau^\lambda$ be the restriction of this line bundle to the fiber over $\tau \in \mathbb{C}$. The Borel–Weil theorem states that the representation V^λ with highest weight λ is isomorphic to the space of algebraic sections of $\mathcal{L}_1^\lambda$ as a representation of GL_n:

$$V^\lambda = \Gamma(\mathcal{L}_1^\lambda). \tag{11}$$

At the same time, we have already seen that the space $\Gamma(\mathcal{L}_0^\lambda)$ of sections over the toric variety $\mathcal{F}(0)$ carries an action of a torus $\mathbf{T}$ of dimension $\binom{n}{2}$ under which $\Gamma(\mathcal{L}_0^\lambda)$ decomposes into one-dimensional weight spaces. Think of $\mathbf{T}$ as the product of one-dimensional tori T'_{ij} for $i+j \leq n$, each of which acts on M_n by scaling the (i,j) entry, which lies strictly above the main antidiagonal. The weight spaces for the action of $\mathbf{T}$ on $\Gamma(\mathcal{L}_0^\lambda)$ are indexed by the set Υ_λ from (7) in Section 3. In other words,

$$\Gamma(\mathcal{L}_0^\lambda) = \bigoplus_{A \in \Upsilon_\lambda} \mathbb{C}^A, \tag{12}$$

where $\mathbb{C}^A$ is a complex line on which $\mathbf{T}$ acts with weight A.

Let T_{ij} be the one-dimensional torus scaling simultaneously the entries of an $n \times n$ matrix in row i, between column $j+1$ and the antidiagonal column $n+1-i$. Then $\mathbf{T}$ can be thought of as the product of all tori T_{ij} with $i+j \leq n$. Under this direct product decomposition of $\mathbf{T}$, the discussion in the proof of Proposition 7 identifying Υ_λ with Π_λ implies the weight space decomposition

$$\Gamma(\mathcal{L}_0^\lambda) = \bigoplus_{\Lambda \in \Pi_\lambda} \mathbb{C}^\Lambda, \tag{13}$$

The weight space decompositions (12) and (13) are of course the same, and the two indexings correspond to two different choices of bases of the weight lattice of $\mathbf{T}$.

The family $\mathcal{F}$ is projective over the affine complex line $\mathrm{Spec}(\mathbb{C}[t])$, so the algebraic sections $\Gamma(\mathcal{L}^\lambda)$ form a finitely generated module over the coordinate ring $\mathbb{C}[t]$ of the base. The localization $\Gamma(\mathcal{L}^\lambda) \otimes_{\mathbb{C}[t]} \mathbb{C}[t^{\pm 1}]$ is a finitely generated free module over the coordinate ring $\mathbb{C}[t^{\pm 1}]$ of the complement of $0 \in \mathbb{C}$, by triviality of the family $\mathcal{F}$ outside the fiber over 0, and invariance of the vector space dimension of $\Gamma(\mathcal{L}^\lambda_\tau)$ as a function of τ. On the other hand, $\dim_{\mathbb{C}} \Gamma(\mathcal{L}^\lambda_0) = \dim_{\mathbb{C}}(\mathcal{L}^\lambda_\tau)$ for $\tau \neq 0$; indeed, both dimensions equal the number of lattice points in the Gelfand–Tsetlin polytope P_λ, by (10), (11), and (13). Hence we get the following.

Proposition 11. *$\Gamma(\mathcal{L}^\lambda)$ is free over $\mathbb{C}[t]$, and possesses a $\mathbb{C}[t]$-basis of sections each of which is equivariant for the action of the n-dimensional diagonal torus in GL_n. Restricting this basis to the 0 and 1 fibers results in a torus-equivariant isomorphism*

$$\Phi : \quad \Gamma(\mathcal{L}^\lambda_1) = V^\lambda \quad \longrightarrow \bigoplus_{\Lambda \in \Pi_\lambda} \mathbb{C}^\Lambda = \Gamma(\mathcal{L}^\lambda_0).$$

To understand the map Φ in terms of the inductive construction of Gelfand and Tsetlin, we present a construction of an n-parameter family extending $\mathcal{F}$. Let ω_i be the $n \times n$ matrix whose i^{th} column has entries $\omega_{1i}, \ldots, \omega_{ni}$ and whose other columns are zero (the integers ω_{ij} are defined in (2)). Write $T(\omega_i)$ for the one parameter subgroup of the torus of B^n associated to ω_i, and denote by $\tilde{\tau}_i$ the element of $T(\omega_i)$ corresponding to the complex number τ_i. Define the family

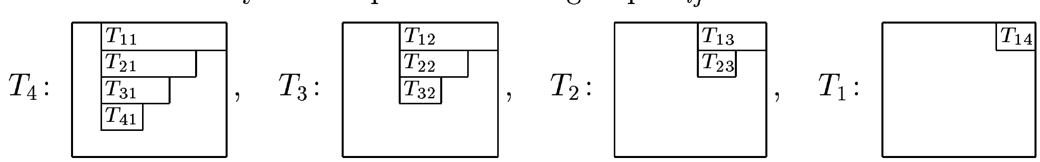
$$B(\tau_1, \cdots, \tau_n) \quad = \quad \tilde{\tau}_1^{-1} \cdots \tilde{\tau}_n^{-1} B_\Delta \tilde{\tau}_1 \cdots \tilde{\tau}_n$$

of subgroups of B^n. This family extends to zero values of τ_i and defines an action of B on $M_n \times \mathbb{C}^n$ as in Lemma 2.

Definition 12. The n-parameter family $\tilde{\mathcal{F}} = B \backslash\!\backslash (M_n \times \mathbb{C}^n)$ is the *degeneration in stages*. Denote by $\tilde{\mathcal{F}}_i$ its fiber over the point $(0, \ldots, 0, 1 \ldots, 1)$ with i entries equal to 1.

Observe that $\tilde{\mathcal{F}}_n = \mathcal{F}\ell_n$ is the flag manifold, and $\tilde{\mathcal{F}}_0$ is the Gelfand–Tsetlin toric variety by Theorem 8.

Let $\mathbf{T}_i$ be the torus $T_{n-1} \times \cdots \times T_i$ with $T_j = T_{1,n-j} \times \cdots \times T_{j,n-j}$, and set $\mathbf{G}_i = GL_i$, thought of in the bottom right corner again. For the $n = 5$ example of $\mathbf{T}_i$, each torus T_j scales the entries by its one-parameter subgroups T_{ij} in the indicated locations:



Each fiber $\tilde{\mathcal{F}}_i$ carries the action of the group $\mathbf{G}_i \times \mathbf{T}_i$. Moreover, for each i, a subfamily of $\tilde{\mathcal{F}}$ degenerates $\tilde{\mathcal{F}}_i$ to $\tilde{\mathcal{F}}_{i-1}$ flatly and $\mathbf{G}_{i-1} \times \mathbf{T}_i$ invariantly.

Associate to each dominant weight λ the very ample line bundle $\tilde{\mathcal{L}}^\lambda$ over $\tilde{\mathcal{F}}$, as we did for the family $\mathcal{F}$. Then, as in (6) for $\mathcal{F}$, we get an embedding of $\tilde{\mathcal{F}}$ into $\mathbb{P}_\lambda \times \mathbb{C}^n$. Let V_i^λ be the space of algebraic sections of the restriction of the line bundle $\tilde{\mathcal{L}}^\lambda$ to $\tilde{\mathcal{F}}_i$, treated as a representation of $\mathbf{G}_i \times \mathbf{T}_i$. Since $\tilde{\mathcal{F}}_i$ degenerates to $\tilde{\mathcal{F}}_{i-1}$ flatly and $\mathbf{G}_{i-1} \times \mathbf{T}_i$ invariantly, there are $\mathbf{G}_{i-1} \times \mathbf{T}_i$ invariant isomorphisms $\Phi_i : V_i^\lambda \to V_{i-1}^\lambda$ for $i = 1, \ldots, n$. These are analogous to the isomorphism Φ from Proposition 11, and constructed geometrically in the same way. In fact Φ equals the composition $\Phi_1 \circ \cdots \circ \Phi_n$, or equivalently $\Phi : V^\lambda = V_n^\lambda \xrightarrow{\Phi_n} V_{n-1}^\lambda \xrightarrow{\Phi_{n-1}} \cdots \xrightarrow{\Phi_1} V_0^\lambda$.

In what follows, we write $\Pi_\lambda(i)$ for the set of integer patterns

$$\Lambda_i \;=\; (\lambda \succ \lambda^{n-1} \succ \cdots \succ \lambda^i),$$

with λ^j being a weakly decreasing sequence of j nonnegative integers. For each pattern $\Lambda_i \in \Pi_\lambda(i)$, let $V_i^{\Lambda_i}$ be the irreducible representation of $\mathbf{G}_i$ with highest weight λ^i, and declare the torus $\mathbf{T}_i$ to act on every vector in $V_i^{\Lambda_i}$ with weight Λ_i.

Theorem 13. *The sections V_i^λ of the line bundle $\tilde{\mathcal{L}}^\lambda$ over the fiber $\tilde{\mathcal{F}}_i$ of the degeneration in stages decomposes into irreducible components for $\mathbf{G}_i \times \mathbf{T}_i$ as*

$$V_i^\lambda \;=\; \bigoplus_{\Lambda_i \in \Pi_\lambda(i)} V_i^{\Lambda_i}, \tag{14}$$

$$\textit{so} \quad V^\lambda \;=\; \bigoplus_{\Lambda_i \in \Pi_\lambda(i)} \Phi_n^{-1} \circ \cdots \circ \Phi_{i+1}^{-1}(V_i^{\Lambda_i})$$

is a partial Gelfand–Tsetlin decomposition. The Gelfand–Tsetlin decomposition is thus

$$V^\lambda \;=\; \bigoplus_{\Lambda \in \Pi_\lambda} \Phi^{-1}(\mathbb{C}^\Lambda).$$

Proof. It is enough to assume (14) is proved for i, and then prove it for $i-1$. Let $\Lambda_{i-1} \mapsto \Lambda_i$ be the map $\Pi_\lambda(i-1) \to \Pi_\lambda(i)$ forgetting λ^{i-1}. Since Φ_i is $\mathbf{G}_{i-1}$ equivariant, we get a decomposition of V_{i-1}^λ into irreducible components under $\mathbf{G}_{i-1}$:

$$V_{i-1}^\lambda \;=\; \bigoplus_{\Lambda_{i-1} \in \Pi_\lambda(i-1)} \; \bigoplus_{\Lambda_{i-1} \mapsto \Lambda_i} V^{\lambda^{i-1}}.$$

It remains to show that T_{i-1} acts on each irreducible $V^{\lambda^{i-1}}$ with weight λ^{i-1}.

After the identification of V^λ with $\Gamma(\mathcal{L}_1^\lambda)$, every highest weight vector of the $\mathbf{G}_{i-1}$ action on V^λ can be thought of as a monomial $\prod p_I$ in Plücker coordinates for subsets I whose columns in the range $n-i+2, \ldots, n$ are left justified. (These are the monomials invariant with respect to the right action of U_{i-1}^+, the upper triangular matrices inside $\mathbf{G}_{i-1}$ with 1's on the diagonal.) Now simply note that the weight of T_{i-1} on such a monomial coincides with the weight of the diagonal torus in $\mathbf{G}_{i-1}$. □

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