

TORIC DEGENERATION OF SCHUBERT VARIETIES AND GELFAND–TSETLIN POLYTOPES

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ABSTRACT. This note constructs the flat toric degeneration of the manifold $\mathcal{F}\ell_n$ of flags in \mathbb{C}^n from [GL96] as an explicit GIT quotient of the Gröbner degeneration in [KM03]. This implies that Schubert varieties degenerate to reduced unions of toric varieties, associated to faces indexed by rc-graphs (reduced pipe dreams) in the Gelfand–Tsetlin polytope. Our explicit description of the toric degeneration of $\mathcal{F}\ell_n$ provides a simple explanation of how Gelfand–Tsetlin decompositions for irreducible polynomial representations of GL_n arise via geometric quantization.

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