Resolutions and Duality for Monomial Ideals

by

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To my parents, Ralph and Avis Miller

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symbol	definition/reference	page
elev	the elevation of an injective module, Definition 4.7	50
ϵ	deformation of a monomial ideal, Definition 5.42	73
$\overline{F}$	$\{1,\dots,n\}\setminus F$	9
$\mathbb{F}(-n)$	homological shift of $\mathbb{F}$ up by n, Definition 3.12	37
$\mathbb{F}_X$	flat complex supported on cellularly labelled X, Definition 3.12	37
$\mathbb{F}^{(X,Y)}$	relative cocellular free complex, Definition 5.13	64
$\Gamma_I(N)$	$\{z \in N \mid I^{r(z)}\cdot z = 0 \text{ for some } r(z)\in\mathbb{N}\}$	
H^i_I	i^{th} right derived functor of Γ_I	
$H^0_{\mathfrak{m}}(-)$	elements annihilated by a power of $\mathfrak{m}$ (also called $\Gamma_{\mathfrak{m}}$)	
$\widetilde{H}$	reduced homology or cohomology	
$\underline{\text{Hom}}$	$\mathbb{Z}^n$-graded Hom functor	13
$H(M;\mathbf{x})$	$\mathbb{Z}^n$-graded Hilbert series of M	58
hull(I)	hull complex of the ideal I	65
$\mathbb{I}$	complex of injective modules	
I	a monomial ideal	
$\widetilde{I}$	$I+\langle x_1^D,\dots,x_n^D\rangle$	73
I_ϵ	deformation of I by ϵ, Definition 5.42	73
$\sqrt{I}$	radical of the ideal I	11
$I^{[\mathbf{a}]}$	Alexander dual of the ideal I with respect to $\mathbf{a}$, Definition 1.2	7
$I^\star$	tight Alexander dual of the ideal I, Definition 1.2	7
I_Δ	Stanley-Reisner (or face) ideal of Δ	9
L_I	lcm-lattice of I	55
Λ^*	the free transpose of the monomial matrix Λ, Definition 3.12	37
lcm(m,m')	least common multiple of monomials m and m'	
link$_v X$	link of v in X	46
$M^{\mathbf{a}}$	Alexander dual of M with respect to $\mathbf{a}$, Definition 1.36	21
$\mathcal{M}$	category of $\mathbb{Z}^n$-graded modules	14
$\mathcal{M}^{\mathbf{a}}_+$	positively $\mathbf{a}$-determined $\mathbb{Z}^n$-graded modules, Definition 1.25	17
$\mathcal{M}^{\mathbf{a}}$	$\mathbf{a}$-determined $\mathbb{Z}^n$-graded modules, Definition 1.25	17
$\mathcal{M}^{\mathbf{a}}_-$	negatively $\mathbf{a}$-determined $\mathbb{Z}^n$-graded modules, Definition 1.25	17
$\overline{\mathcal{M}}^{\mathbf{a}}$	$\mathbf{a}$-bounded $\mathbb{Z}^n$-graded modules, Definition 1.25	17
$\mathcal{M}_{[F]}$	category of $\mathbb{Z}^F$-graded $S_{[F]}$-modules, Definition 3.17	38
$\mathfrak{m}$	maximal ideal $\langle x_1,\dots,x_n\rangle$	
$\mathfrak{m}^{\mathbf{b}}$	$\langle x_i^{b_i} \mid b_i \geq 1\rangle$, never the Alexander dual of $\mathfrak{m}$ with respect to $\mathbf{b}$	6
$\mu_{i,\mathbf{b}}(F,M)$	i^{th} Bass number of M at $\mathfrak{m}^F$ in degree $\mathbf{b}$, Definition 4.16	52
$\mu_{i,\mathbf{b}}(M)$	i^{th} Bass number of M at $\mathfrak{m}$ in degree $\mathbf{b}$, Definition 4.16	52
$\mathbb{N}$	nonnegative integers	
$N[\mathbf{b}]$	$\mathbb{Z}^n$-graded shift of N back by $\mathbf{b}$	13
$N_{[F]}$	restriction of N to $\mathbb{Z}^F$, Definition 3.17	38
$N_{(F)}$	homogeneous localization of N at $\mathfrak{m}^F$, Definition 3.17	38

symbol	definition/reference	page
$N^\vee$	Matlis dual of N	13
$P_{\mathbf{a}}$	positive extension with respect to **a**, Definition 1.30	18
P_t	hull polyhedron with real parameter t	65
proj. dim	projective dimension	48
$\mathbb{Q}$	the rational numbers	
$\mathbb{R}$	the real numbers	
$\mathbb{R}$	right derived functor	92
reg	regularity	48
rise	the rise of a complex of injectives, Definition 4.7	50
S	$k[x_1, \ldots, x_n]$	6
$S[m^{-1}]$	localization obtained by inverting the monomial m	
supp(**b**)	the support $\{i \mid b_i \neq 0\}$ of $\mathbf{b} \in \mathbb{Z}^n$	8
supp. reg	support-regularity, Definition 4.4	49
$\mathbb{T}_{\bullet}(I)$	Taylor resolution of S/I, Definition 6.1	88
Tot	total complex of a bigraded complex	92
$\underline{\mathrm{Tor}}$	$\mathbb{Z}^n$-graded Tor functor	14, 17
$\lvert X \rvert$	underlying unlabelled cell complex of X	62
X_U	$X_{\not\succeq \mathbf{1}}$, Definition 5.6	61
$X_{\preceq \mathbf{b}}$	$\{G \in X \mid \mathbf{a}_G \preceq \mathbf{b}\}$, Definition 5.6	61
$X_{\not\succeq \mathbf{b}}$	$\{G \in X \mid \mathbf{a}_G \not\succeq \mathbf{b}\}$, Definition 5.6	61
$X_{\mathbb{Z}}$	$\{G \in X \mid \mathbf{a}_G \in \mathbb{Z}^n\}$, Definition 5.6	61
$X_{\mathbf{1}-\mathbb{Z}}$	relabelled cellular pair, Definition 5.13	64
$\mathbf{x}^{\mathbf{b}}$	$x_1^{b_1} \cdots x_n^{b_n}$	6
$\mathbb{Z}_*$	$\mathbb{Z} \cup \{*\}$	24
ω_S	canonical module $S[-\mathbf{1}]$	58
$\omega_{S/I}$	canonical module $\underline{\mathrm{Ext}}_S^d(S/I, \omega_S)$ of S/I	90
$(-)_{\succeq \mathbf{b}}$	elements of degree $\succeq \mathbf{b}$, Definition 3.1	33
$\preceq$	partial order on $\mathbb{Z}_*^n$	6, 24
$*F$	$\sum_{i \in F} *\mathbf{e}_i$	24

Abstract

Resolutions and Duality for Monomial Ideals
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Alexander duality is made into a functor, based on Matlis duality, which extends the notion for radical monomial ideals in a polynomial ring S to any finitely generated $\mathbb{N}^n$-graded module. Monomial matrix notation is introduced and exploited. Alexander duality functors are shown to work on $\mathbb{Z}^n$-graded free and injective resolutions as well as their monomial matrices, with numerous applications. Using injective resolutions, theorems of Eagon-Reiner and Terai are generalized to all $\mathbb{N}^n$-graded modules: the projective dimension of M equals the *support-regularity* $\max\{|\operatorname{supp}(\mathbf{b})|-i \mid \mathbf{b} \in \mathbb{N}^n \text{ and } \beta_{i,\mathbf{b}} \neq 0\}$ of its Alexander dual, where $\operatorname{supp}(\mathbf{b}) = \{j \mid b_j \neq 0\}$ and $\beta_{i,\mathbf{b}}$ is the i^{th} Betti number in degree $\mathbf{b}$. Thus M is Cohen-Macaulay if and only if its Alexander dual has *support-linear* free resolution (i.e. support size of every minimal generator = support-regularity). Duality for resolutions interacts with cellular resolutions, lcm-lattices, and convex geometry, yielding: multiple equivalent characterizations of generic ideals; Cohen-Macaulay criteria for cogeneric ideals; convex-geometric injective resolutions; and planar graph minimal resolutions for trivariate ideals. Relations between Bass and Betti numbers as well as Hilbert series of dual ideals are derived, the latter from an interpretation of functoriality of Alexander duality in equivariant K-theory. For $\mathbb{Z}^n$-graded local cohomology functors $H^i_I(-)$ on radical monomial ideals $I \subset S$, Alexander duality implies a direct generalization of local duality, which is the case $I = \mathfrak{m}$ maximal. The connection is made in the derived category with Greenlees-May duality. A new flat complex for calculating local cohomology with monomial support is introduced. The module structures of $H^i_I(S)$ and $H^{n-i}_{\mathfrak{m}}(S/I)$ are proved equivalent, as are Terai's and Hochster's formulas for their repsective Hilbert series.

Preface

In the 1960s and 1970s, M. Hochster and R. Stanley pioneered the study of squarefree monomial ideals in polynomial rings in terms of simplicial homology. Among their innovations were the first applications of a combinatorial Alexander duality in this algebraic context. The development of an algebraic theory for so-called *face* or *Stanley-Reisner* rings might be thought of as the first intensive study of monomial ideals as intrinsically interesting objects. Hochster and Stanley used free resolutions and local cohomology among other devices as tools to separate out fine points like a chromatograph. Building on their foundations, the recent efforts of many authors have forged new kinds of theories, frequently focusing on the resolutions themselves as the objects of study. These theories are revealing deep and sometimes unexpected new relations between algebra, combinatorics, topology, and geometry.

As a first example, take the notion of *cellular resolution*, due to D. Bayer and B. Sturmfels [BS98]. The idea is to make a free resolution of a monomial ideal I (an algebraic object) out of a regular cell complex X (a topological object). Motivation for this type of construction came from a collaboration of Bayer and Sturmfels with I. Peeva [BPS98] describing the case where I is *generic* and the faces of the *Scarf complex* X are elements in a ranked poset of monomials in I (a combinatorial object). But even in this case, where X is simplicial, it is the realization of X as a *hull complex* (a geometric object) which endows it with desirable properties, via convexity. In the case of 3 variables, for instance, cellular resolutions are essentially planar graphs, and the convexity can be interpreted as yet another formulation of Steinitz's theorem [MS99].

Consider also Alexander duality, which was mentioned briefly above as a combinatorial phenomenon for squarefree ideals. Alexander duality is more commonly thought of as a topological duality between the homology of a subset of a sphere and the cohomology of the complement. However, it was applied algebraically in a landmark theorem of J. Eagon and V. Reiner [ER98], as well as in a subsequent generalization by N. Terai [Ter97], relating the length of the minimal free resolution of a squarefree ideal I (projective dimension of I) to the "width" of the minimal free resolution of the Alexander dual ideal (regularity of $I^\star$). We will show in Section 4.1 how the Eagon-Reiner theorem works for any monomial ideal or $\mathbb{N}^n$-graded module, by defining Alexander duality for such modules (Chapter 1), and indeed for resolutions (Chapter 3).

It is in this latter context that Alexander duality reasserts its topological heritage: topological duality (homology versus cohomology) for cellular resolutions is an instance of the new algebraic duality between injective resolutions of Alexander dual ideals (Chapter 5). Even the combinatorial duality for ideals between links and restrictions in Stanley-Reisner simplicial complexes, which has been ubiquitous ever since its exploitation by Hochster [Hoc77], shows up now for resolutions in duality for Scarf complexes (Example 3.32).

More importantly, though, is the realization that free resolutions are not the paragons of homological information. Settling for a free resolution, even a minimal one, means "forgetting" a lot of useful combinatorial as well as algebraic information. In many ways injective resolutions are to free resolutions as projective varieties are to affine varieties. In this analogy, the role of affine space is played by the combinatorial object $\mathbb{Z}^n$, whose projective version has some extra "hyperplanes at infinity" that arise from the addition of a single element $*$ to $\mathbb{Z}$. Injective resolutions measure the behavior of modules in $\mathbb{Z}^n$-degrees as they approach infinity, and the kind of information recorded by injective resolutions past that of free resolutions is analogous to the information carried by the divisor at infinity of an affine variety.

Indeed, in certain combinatorial topological cellular resolutions this is *precisely* what happens: the injective resolution corresponds to a polyhedral subdivision of a simplex, while the free resolution corresponds to only the interior faces in the triangulation (Corollary 5.28). This construction has algebraic applications to numerical invariants of monomial ideals whose irreducible components are essentially random (Section 5.4). In addition, this good behavior allows for better control over the graphs that appear as minimal resolutions of ideals in 3 variables (Section 5.5), which are essentially 3-connected and planar. And when ideals in any number of variables are considered whose generators are random, the "properness" of injective resolutions lends a rigidity giving rise to a host of equivalent characterizations of *genericity*, thereby justifying the naturality of the definition (Section 5.3).

Of course, the applicability of injective resolutions is not limited to cellular or combinatorial situations. In fact, one of the motivations for developing the general duality theory for resolutions in Chapter 3 was the formulation and proof of the generalization of the Eagon-Reiner and Terai theorems to arbitary $\mathbb{N}^n$-graded modules. The statement, Theorem 4.5, does not involve injective resolutions in any way, but the proof involves precisely the kinds of boundary phenomena observed in the cellular cases above. This is because the indecomposable injective modules correspond to the faces of a simplex. Roughly speaking, an injective resolution of a module M has smaller faces at the beginning and larger faces towards the end. The projective dimension of M is measured by how fast the faces grow in size: the faster the growth, the larger the projective dimension. In contrast, the free resolution of M consists of precisely those injective summands that correspond to the biggest face of the simplex. But it is the boundary faces of the simplex that get measured in the free resolution of the Alexander dual module $M^{\mathbf{a}}$, and this is the phenomenon observed by Eagon-Reiner and Terai in its residual form on free resolutions.

More or less implicit in all of this discussion is that one can actually compute and write down injective resolutions of $\mathbb{Z}^n$-graded modules. The reduction to a $\mathbb{Z}^n$-grading leaves only finitely many indecomposable injectives, at least up to graded shifts, and the maps between direct sums of them are given simply by matrices over the ground field k (Section 2.1). For example, cellular and cocellular monomial matrices can be defined just by requiring that these scalar entries be the (co)boundary complex of a regular cell complex (Section 2.3). In general, injective resolutions can be expressed using *monomial matrices*, which record the graded shifts and the scalar entries in a single array. Moreover, the duality theorem for resolutions (Theorem 3.23) reduces computation of injective resolutions to computation of free resolutions of finite-length modules, which is easily done by computer.

Quite a different interaction of computation with monomial ideals is furnished by sheaf cohomology on a toric variety, which can be calculated algorithmically using local cohomology supported on a squarefree monomial ideal I [EMS00]. Therefore, one wants to know the structure of local cohomology modules $H^{\bullet}_{I}(M)$ for various modules M over the Cox homogeneous coordinate ring of the toric variety, which is just a polynomial ring. If M happens to be $\mathbb{Z}^n$-graded (as many of the relevant modules are), then the local cohomology modules are graded by $\mathbb{Z}^n$, as well; and in these cases, it is easier to grasp the structure of the local cohomology. For instance, if M is the Cox ring itself, one can express the module structure of the local cohomology in terms of well-known invariants of I—these results of Terai and M. Mustaţă [Mus00a, Ter99] are presented in Section 6.2. Along with a new and simple proof, they are shown to be equivalent via a *generalized Alexander duality* to Hochster's original result on local cohomology of the quotient by I with support on the maximal ideal and H. G. Gräbe's refinement of Hochster's formula [Grä84].

More generally, for all $\mathbb{Z}^n$-graded modules M, the local cohomology modules satisfy a form of local duality which is a good deal stronger than the usual Grothendieck-Serre local duality (Section 6.3). It turns out that this generalized duality is a $\mathbb{Z}^n$-graded analog of Greenlees-May duality [GM92], which is an adjointness in the derived category between taking support on an ideal and completion with respect to it. This result says that Alexander duality can (and should) be examined at the level of derived categories, and gives rise to a number of interesting questions.

Greenlees-May duality was originally proved as a kind of universal coefficient theorem for the purpose of studying equivariant K-theory in topology [GM92]. Oddly enough, monomial ideals also have another (related?) connection to equivariant K-theory: because monomial ideals are torus-invariant, the subschemes they determine represent torus-equivariant K-homology classes. Furthermore, a principal monomial ideal represents a torus-equivariant line bundle, and thus determines a torus-equivariant K-cohomology class. In this context, Alexander duality for monomial ideals, which was already seen above to represent various kinds of topological duality, can also be viewed as a change of basis between equivariant K-homology and equivariant K-cohomology of affine space (Section 4.4).

This may seem a spurious notion until one realizes that calculations of equivariant theories for spaces such as flag manifolds and toric varieties can be reduced to calculations on affine space. In the case of flag manifolds the reduction is by Gröbner deformation, and (surprisingly) the duality theorem of Eagon and Reiner [ER98] recovers a positivity result for so-called Grothendieck polynomials, which represent the K-classes of Schubert varieties [KM00]. For toric varieties, on the other hand, one uses the idea behind the Cox homogeneous coordinate ring: a toric variety is the quotient of an open subset of affine space by the free action of a torus. What kind of open set? The complement of a subspace arrangement which is the zero set of ... a monomial ideal.

This brings us to our last unexpected connection between resolutions of monomial ideals and topology. The combinatorics determining the homological algebra of a monomial ideal I is strikingly similar to the combinatorics determining the cohomology of the complement Y of a complex subspace arrangement. The theorem from topology, due to M. Goresky and R. MacPherson [GM88], computes the cohomology of Y from the *intersection lattice* consisting of all intersections of subspaces in the arrangement. The observation made by V. Gasharov, I. Peeva, and W. Welker [GPW00], is that the free resolution of I depends analogously on the *lcm-lattice* consisting of all least common multiples of the minimal generators of I. That a similar type of result holds for injective resolutions is proved using Alexander duality in Section 4.3.

Note on references: Much of this material has been distributed, submitted, or published elsewhere. For instance, most of Chapters 2 and 3 are taken nearly verbatim from [Mil00]. To avoid endless citations, I have refrained from specifying the alternate location of such material. Exceptions are when there are coauthors involved, or when an old argument is missing or has been made obsolete, in which case I cite the relevant material for comparison.

Ezra N. Miller
29 February 2000

Chapter 1

Alexander duality

Given a monomial ideal I, there is an obvious "minimal" way of describing it: by listing its minimal monomial generators. On the other hand, there is another perfectly good minimal description of I which, instead of prescribing the monomials that *are* in I, eliminates the monomials *not* in I from competition: the irredundant irreducible decomposition. For a single given I these descriptions are not *a priori* particularly related (although resolutions can be thought of as providing rules for transforming between the two). However, there is another monomial ideal $I^\star$ whose minimal generators correspond to the irreducible components of I; and miraculously this is enough to guarantee that the irreducible components of $I^\star$ correspond to the minimal generators of I. This fact, which is intuitive for 3-dimensional monomial ideals because of a common optical illusion, actually provides a simple proof of the uniqueness of irredundant irreducible decompositions (Theorem 1.7).

Various previously defined families of ideals are Alexander dual to each other. Of particular interest are the generic and cogeneric ideals, and the squarefree ideals. The latter of these correspond to simplicial complexes, where the notion of Alexander duality is seemingly well-understood. In the simplicial world, Alexander duality is a combinatorial phenomenon, the simplicial complex Δ being an order ideal in the lattice of subsets of $\{1, \ldots, n\}$. The Alexander dual $\Delta^\star$ is given by the complement of the order ideal, which is an order ideal in the opposite lattice. For squarefree monomial ideals, then, all is well since the only monomials we care about are represented precisely by the lattice of subsets of $\{1, \ldots, n\}$. If we want to think of Alexander duality for general monomial ideals in order-theoretic terms, we instead consider the larger lattice $\mathbb{Z}^n$, by which we mean the poset with its natural partial order $\preceq$. Then a monomial ideal I can be regarded as a dual order ideal in $\mathbb{Z}^n$, and $I^\star$ is constructed (roughly) from the complementary set of lattice points, which is an order ideal; see Section 1.3.

The general form of Alexander duality to which this chapter aspires is obtained by combining the algebraic and combinatorial perspectives. Combinatorially each point in our lattice $\mathbb{Z}^n$ is just a point, labelled by a monomial. But algebraically, that point has extra structure: the monomial spans a 1-dimensional vector space. When we carry out lattice complementation to define the duality combinatorially, we must also take vector space duals if a satisfactory algebraic theory is to result. And once the lattice points are vector spaces, there is no reason to restrict them to be 1-dimensional. Thus we arrive at a duality which is *functorial*.

The functoriality of taking dual vector spaces in some sense solves all of the algebraic problems with defining Alexander duality for modules that aren't radical monomial ideals. However, there are still some combinatorial loose ends. In the squarefree case, we are satisfied to deal only with subsets of $\{1, \ldots, n\}$—that is, with monomials which are products of distinct variables. But when we carry out the duality with these we are automatically carrying along infinitely many other monomials along for the ride. In the general case, the duality will again be occurring combinatorially in a finite space, but we have to specify how the infinitely many remaining monomials are to join the parade. These considerations force us to choose beforehand which finite space to consider (Section 1.4), and how the rest of $\mathbb{Z}^n$ is to follow suit (Section 1.5). Only then can we define the Alexander duality functors (Section 1.6), and use the algebraic duality to prove their uniqueness.

1.1 Generators versus irreducible components

Let S be the $\mathbb{Z}^n$-graded k-algebra $k[x_1, \ldots, x_n]$, where k is a field. A *monomial* in S is a product of powers of variables in S; an *irreducible monomial ideal* is generated by powers of variables. Each of these objects may be specified uniquely by a single vector $\mathbf{b} = (b_1, \ldots, b_n) = \sum_i b_i \mathbf{e}_i \in \mathbb{N}^n$, where $\mathbf{e}_i$ is the i^{th} standard basis vector, using the notation

$$\text{monomial} \quad \mathbf{x}^{\mathbf{b}} = x_1^{b_1} \cdots x_n^{b_n} \qquad \text{and} \qquad \text{irreducible ideal} \quad \mathfrak{m}^{\mathbf{b}} = \langle x_i^{b_i} \mid b_i \geq 1 \rangle \,.$$

More generally, a $\mathbb{Z}^n$-graded ideal of S, which is precisely an ideal generated by monomials, is called a *monomial ideal*. From now on,

all ideals in this dissertation will be monomial ideals unless otherwise stated.

It is well-known that every monomial ideal has a unique minimal generating set of monomials. This is a consequence of the partial order on monomials by divisibility: the minimal generators of I are the monomials that are minimal in the set of all monomials in I. It is less well-known, however, that every monomial ideal I can be expressed uniquely as a finite irredundant intersection $I = \bigcap_{\mathbf{b} \in B} \mathfrak{m}^{\mathbf{b}}$ of irreducible monomial ideals, meaning that omitting any of the $\mathfrak{m}^{\mathbf{b}}$ leaves an ideal strictly containing I. Disregarding the uniqueness for the moment, any such intersection is called an *irredundant irreducible decomposition* of I, and the $\mathfrak{m}^{\mathbf{b}}$ are called *irreducible components* of I. The starting point for Alexander duality is to illustrate why the two uniqueness statements above are actually equivalent (Theorem 1.7), thereby also giving a new proof of the latter one.

The elementary form of Alexander duality is an operation taking monomial ideals to monomial ideals in such a way that minimal generators get transformed into irreducible ideals. Of necessity, this involves a somewhat strange inversion of $\mathbb{N}^n$-graded degrees. Recall that the set $\mathbb{Z}^n$ is a partially ordered set (poset), with $\mathbf{b} \preceq \mathbf{c} \Leftrightarrow b_i \leq c_i$ for all $i \in \{1, \ldots, n\}$.

Definition 1.1 *Given two vectors* $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$ *with* $\mathbf{b} \preceq \mathbf{a}$, *let* $\mathbf{a} \smallsetminus \mathbf{b}$ *denote the vector whose* i^{th} *coordinate is*

$$a_i \smallsetminus b_i := \begin{cases} a_i + 1 - b_i & \text{if } b_i \geq 1 \\ 0 & \text{if } b_i = 0\,. \end{cases}$$

Now we are able to make the central definition of this section.

Definition 1.2 (Alexander duality) *Given an ideal I, let $\mathbf{a}_I$ be the exponent on the least common multiple of the minimal generators of I. For any $\mathbf{a} \succeq \mathbf{a}_I$, the* Alexander dual ideal $I^{[\mathbf{a}]}$ with respect to $\mathbf{a}$ *is defined by*

$$I^{[\mathbf{a}]} = \bigcap \{\mathfrak{m}^{\mathbf{a} \smallsetminus \mathbf{b}} \mid \mathbf{x}^{\mathbf{b}} \text{ is a minimal generator of } I\}\,.$$

For the special case when $\mathbf{a} = \mathbf{a}_I$, define the tight *Alexander dual $I^\star := I^{[\mathbf{a}_I]}$.*

The first result on Alexander duality provides a relation to linkage (see [Vas98, Appendix A.9] for a brief introduction to linkage and references, or [SV86] for more details). Its corollary is fundamental, and will be indispensable for the proof of Theorem 1.7. Recall that $(J : I) = \langle m \in S \mid m \cdot I \subseteq J\rangle$ for ideals I and J of S, and is a monomial ideal if I and J are (because of the $\mathbb{Z}^n$-grading). Let $\mathbf{1} = (1, \ldots, 1) \in \mathbb{N}^n$.

Proposition 1.3 *If $\mathbf{a} \succeq \mathbf{a}_I$ then $I^{[\mathbf{a}]}$ is the unique ideal generated in degrees $\preceq \mathbf{a}$ satisfying $(\mathfrak{m}^{\mathbf{a}+\mathbf{1}} : I) = I^{[\mathbf{a}]} + \mathfrak{m}^{\mathbf{a}+\mathbf{1}}$.*

Proof: Let $\operatorname{Min}(I)$ be the set of exponents on the minimal generators of I. Then $(\mathfrak{m}^{\mathbf{a}+\mathbf{1}} : I) = \bigcap_{\mathbf{b} \in \operatorname{Min}(I)} (\mathfrak{m}^{\mathbf{a}+\mathbf{1}} : \mathbf{x}^{\mathbf{b}})$. But $\mathbf{x}^{\mathbf{c}} \cdot \mathbf{x}^{\mathbf{b}} \in \mathfrak{m}^{\mathbf{a}+\mathbf{1}} \Leftrightarrow \mathbf{b} + \mathbf{c} \not\preceq \mathbf{a} \Leftrightarrow \mathbf{c} \not\preceq \mathbf{a} - \mathbf{b} \Leftrightarrow \mathbf{x}^{\mathbf{c}} \in \mathfrak{m}^{\mathbf{a}+\mathbf{1}-\mathbf{b}}$. Thus, taking all intersections over $\mathbf{b} \in \operatorname{Min}(I)$,

$$\bigcap (\mathfrak{m}^{\mathbf{a}+\mathbf{1}} : \mathbf{x}^{\mathbf{b}}) \;=\; \bigcap \mathfrak{m}^{\mathbf{a}+\mathbf{1}-\mathbf{b}} \;=\; \bigcap \Big(\mathfrak{m}^{\mathbf{a} \smallsetminus \mathbf{b}} + \mathfrak{m}^{\mathbf{a}+\mathbf{1}}\Big) \;=\; \Big(\bigcap \mathfrak{m}^{\mathbf{a} \smallsetminus \mathbf{b}}\Big) + \mathfrak{m}^{\mathbf{a}+\mathbf{1}} \;=\; I^{[\mathbf{a}]} + \mathfrak{m}^{\mathbf{a}+\mathbf{1}}$$

since $\mathbf{a} \smallsetminus (\mathbf{a} \smallsetminus \mathbf{b}) = \mathbf{b}$ for all $\mathbf{b} \preceq \mathbf{a}$. The minimal generators of $I^{[\mathbf{a}]}$ all divide $\mathbf{x}^{\mathbf{a}}$ because each can be written as a least common multiple of generators of the ideals $\{\mathfrak{m}^{\mathbf{a} \smallsetminus \mathbf{b}} \mid \mathbf{b} \in \operatorname{Min}(I)\}$. Moreover, any ideal J generated in degrees $\preceq \mathbf{a}$ may be recovered from $J + \mathfrak{m}^{\mathbf{a}+\mathbf{1}}$ as the ideal generated by all monomials in $J + \mathfrak{m}^{\mathbf{a}+\mathbf{1}}$ dividing $\mathbf{x}^{\mathbf{a}}$. □

Corollary 1.4 $(I^{[\mathbf{a}]})^{[\mathbf{a}]} = I$.

Proof: If J is any ideal which can be expressed as an intersection of finitely many irreducibles $J = \bigcap_{\mathbf{b} \in B} \mathfrak{m}^{\mathbf{b}}$, then

$$(\mathfrak{m}^{\mathbf{a}+\mathbf{1}} : J) = \langle \text{monomials } m \in S \mid m \cdot \mathfrak{m}^{\mathbf{b}} \subseteq \mathfrak{m}^{\mathbf{a}+\mathbf{1}} \text{ for some } \mathbf{b} \in B\rangle\,. \qquad (*)$$

Indeed, monomials in the right side certainly are in the left side; and if m is not in the right side then $m \notin \mathfrak{m}^{\mathbf{a}+\mathbf{1}}$, so m divides $\mathbf{x}^{\mathbf{a}}$, and $\mathbf{x}^{\mathbf{a}}/m \in J$ is a monomial demonstrating the fact that m is not in the left side. Therefore, if $\{\mathbf{x}^{\mathbf{b}} \mid \mathbf{b} \in B\}$ minimally generates I, then

$$\begin{aligned} (\mathfrak{m}^{\mathbf{a}+\mathbf{1}} : I^{[\mathbf{a}]}) &= (\mathfrak{m}^{\mathbf{a}+\mathbf{1}} : (I^{[\mathbf{a}]} + \mathfrak{m}^{\mathbf{a}+\mathbf{1}})) \\ &= (\mathfrak{m}^{\mathbf{a}+\mathbf{1}} : (\mathfrak{m}^{\mathbf{a}+\mathbf{1}} : I)) \\ &= (\mathfrak{m}^{\mathbf{a}+\mathbf{1}} : \textstyle\bigcap_{\mathbf{b} \in B} (\mathfrak{m}^{\mathbf{a}+\mathbf{1}} : \mathbf{x}^{\mathbf{b}})) \\ &= (\mathfrak{m}^{\mathbf{a}+\mathbf{1}} : \textstyle\bigcap_{\mathbf{b} \in B} \mathfrak{m}^{\mathbf{a}+\mathbf{1}-\mathbf{b}}), \end{aligned}$$

and this last colon ideal is $I + \mathfrak{m}^{\mathbf{a}+\mathbf{1}}$ by $(*)$. Proposition 1.3 completes the proof. □

Remark 1.5 Any ideal that can be expressed as an Alexander dual has at least one irreducible decomposition. Therefore, Corollary 1.4 proves *existence* of decompositions of monomial ideals as intersections of irreducible monomial ideals. This is independent of the fact that an irreducible monomial ideal cannot be expressed as an intersection of two arbitrary ideals properly containing it, which is the usual definition of "irreducible ideal".

The next lemma is for the proof of uniqueness of irredundant irreducible decompositions in Theorem 1.7. It also explains the odd definition of $\mathbf{a} \smallsetminus \mathbf{b}$.

Lemma 1.6 *If* $\mathbf{0} \preceq \mathbf{b}, \mathbf{c} \preceq \mathbf{a}$ *then* $\mathfrak{m}^{\mathbf{b}} \supseteq \mathfrak{m}^{\mathbf{c}}$ *if and only if* $\mathbf{a} \smallsetminus \mathbf{b} \succeq \mathbf{a} \smallsetminus \mathbf{c}$.

Proof: $\mathfrak{m}^{\mathbf{b}} \supseteq \mathfrak{m}^{\mathbf{c}}$ if and only if ($0 < b_i \leq c_i$ whenever $c_i \neq 0$). This occurs if and only if $a_i + 1 > a_i \smallsetminus b_i \geq a_i \smallsetminus c_i$ whenever $c_i \neq 0$; and if $c_i = 0$ then clearly $a_i \smallsetminus b_i \geq a_i \smallsetminus c_i$. □

Theorem 1.7 *Every ideal I has a unique irredundant irreducible decomposition given by*

$$I = \bigcap \{\mathfrak{m}^{\mathbf{a} \smallsetminus \mathbf{b}} \mid \mathbf{x}^{\mathbf{b}} \textit{ is a minimal generator of } I^{[\mathbf{a}]}\}$$

for any $\mathbf{a} \succeq \mathbf{a}_I$. *In other words, since* $\mathbf{a} \smallsetminus (\mathbf{a} \smallsetminus \mathbf{b}) = \mathbf{b}$,

$$I^{[\mathbf{a}]} = \langle \mathbf{x}^{\mathbf{a} \smallsetminus \mathbf{b}} \mid \mathfrak{m}^{\mathbf{b}} \textit{ is an irreducible component of } I \rangle \, .$$

Proof: The given intersection is equal to I by Corollary 1.4. Suppose J is obtained from $I^{[\mathbf{a}]}$ by omitting one of the minimal generators. Then $J^{[\mathbf{a}]}$ is the ideal obtained from I by leaving off one of the intersectands, and $J \neq I^{[\mathbf{a}]}$ implies that $J^{[\mathbf{a}]} \neq I$ by Corollary 1.4 again. Thus the irreducible decomposition is irredundant, for any $\mathbf{a} \succeq \mathbf{a}_I$. Now suppose we are given any irredundant irreducible decomposition $I = \bigcap_{\mathbf{b} \in B} \mathfrak{m}^{\mathbf{b}}$, and choose $\mathbf{a}$ so that $\mathbf{b} \preceq \mathbf{a}$ for all $\mathbf{b} \in B$. The ideals $\{\mathfrak{m}^{\mathbf{b}} \mid \mathbf{b} \in B\}$ are pairwise incomparable by irredundance, so the set $\{\mathbf{x}^{\mathbf{a} \smallsetminus \mathbf{b}} \mid \mathbf{b} \in B\}$ minimally generates some ideal by Lemma 1.6. Furthermore, the Alexander dual of the ideal in question is I by definition, whence the ideal is $I^{[\mathbf{a}]}$ by Corollary 1.4. It follows that $B = \{\mathbf{a} \smallsetminus \mathbf{c} \mid \mathbf{x}^{\mathbf{c}}$ is a minimal generator of $I^{[\mathbf{a}]}\}$. Therefore the decomposition is unique, and in particular independent of the choice of $\mathbf{a}$. □

Remark 1.8 Theorem 1.7 along with Proposition 1.3 provides a useful way to compute the irreducible components of I given its minimal generators: simply take those generators $\mathbf{x}^{\mathbf{b}}$ of $(\mathfrak{m}^{\mathbf{a}+\mathbf{1}} : I)$ whose exponents are $\preceq \mathbf{a}$, and replace each one by $\mathfrak{m}^{\mathbf{a} \smallsetminus \mathbf{b}}$. Of course, we can also compute the generators of I from its irreducible components this way: turn each component $\mathfrak{m}^{\mathbf{b}}$ into a generator $\mathbf{x}^{\mathbf{a} \smallsetminus \mathbf{b}}$ for $I^{[\mathbf{a}]}$, and compute I using Proposition 1.3.

Remark 1.9 The dual $I^{[\mathbf{a}]}$ with respect to any $\mathbf{a} \succeq \mathbf{a}_I$ depends only on those coordinates of $\mathbf{a}$ in $\mathrm{supp}(\mathbf{a}_I)$, where $\mathrm{supp}(\mathbf{b}) = \{i \in \{1, \ldots, n\} \mid b_i \neq 0\} \subseteq \{1, \ldots, n\}$ is the *support* of any vector $\mathbf{b} \in \mathbb{Z}^n$. This is because $\mathbf{a} \smallsetminus \mathbf{b}$ only depends on the support of $\mathbf{b}$, and $\mathrm{supp}(\mathbf{b}) \subseteq \mathrm{supp}(\mathbf{a}_I)$ for all of the relevant $\mathbf{b}$.

1.2 Examples of Alexander dual ideals

Squarefree ideals

The original purpose of Definition 1.2 was to extend to all monomial ideals the notion of Alexander duality for *squarefree* monomial ideals I, generated by products of variables (otherwise known as *squarefree monomials*). In this case, the duality can first be defined via the associated simplicial complex $\Delta(I)$, called the *Stanley-Reisner complex* of I, on the set of vertices $\{1,\dots,n\}$. The faces of $\Delta(I)$ correspond to the squarefree monomials not in I:

$$\Delta(I) = \{F \subseteq \{1,\dots,n\} \mid \mathbf{x}^F \notin I\},$$

where F is identified with its characteristic vector $\sum_{i\in F} \mathbf{e}_i \in \{0,1\}^n \subset \mathbb{N}^n$, so $\mathbf{x}^F := \prod_{i\in F} x_i$. This construction has an inverse: given a simplicial complex Δ on the vertices $\{1,\dots,n\}$, the *Stanley-Reisner* or *face* ideal I_Δ of Δ is defined by the nonfaces of Δ:

$$I_\Delta = \langle \mathbf{x}^F \mid F \notin \Delta \rangle .$$

The definitions are set up so that $I_{\Delta(I)} = I$ and $\Delta(I_\Delta) = \Delta$.

A simplicial complex Δ is really a combinatorial object: an order ideal in the poset of subsets of $\{1,\dots,n\}$. There is already a well-known duality for posets, and when it is applied to Δ, we get the *Alexander dual simplicial complex* $\Delta^\star$, which consists of the complements of the nonfaces of Δ:

$$\Delta^\star = \{F \subseteq \{1,\dots,n\} \mid \overline{F} \notin \Delta\},$$

where $\overline{F} = \{1,\dots,n\} \setminus F$. It is this construction from which the origin of the name "Alexander duality" can be explained.

Perhaps the most usual form of Alexander duality is topological. Given a closed CW-complex X inside of an $(n-2)$-sphere S^{n-2}, there is an isomorphism between the reduced homology $\widetilde{H}_{i-1}(X;G)$ of X with coefficients in an abelian group G and the reduced cohomology $\widetilde{H}^{n-2-i}(S^{n-2}\setminus X;G)$ of the complement. In the context of simplicial complexes on n vertices, there is an $(n-2)$-sphere floating around, namely the full simplex on $\{1,\dots,n\}$ minus the interior face. The simplicial complex Δ can also be construed as a closed subset of this sphere, at least whenever Δ is not itself the entire simplex. The same comments apply to $\Delta^\star$, and with this in mind, there is an isomorphism

$$\widetilde{H}_{i-1}(\Delta;G) \cong \widetilde{H}^{n-2-i}(\Delta^\star;G) \tag{1.1}$$

for any group G. Why should the Alexander duality isomorphism hold with $\Delta^\star$ in place of $S^{n-2}\setminus\Delta$? It turns out that $\Delta^\star$ is actually homotopy-equivalent to this complement. In fact, the open set $S^{n-2}\setminus X$ inside of the boundary of the simplex deformation retracts in an easy way onto the full subcomplex of the first barycentric subdivision of S^{n-2} on the vertices not in Δ (this is the topological way to take the nonfaces of Δ); and this retract becomes $\Delta^\star$ when it is reflected through the barycenter of the full simplex on $\{1,\dots,n\}$ (this is the topological way to take the complement in $\{1,\dots,n\}$ of each nonface).

Now we have two possible ways of taking an Alexander dual of a squarefree monomial ideal $I = I_\Delta$: either take $I_\Delta^\star$ as in Definition 1.2 or $I_{\Delta^\star}$ as above. But the face ideal I_Δ may be equivalently described as

$$I_\Delta = \bigcap_{\overline{F} \in \Delta} \mathfrak{m}^F ,$$

since $\mathfrak{m}^F \supseteq I \;\Leftrightarrow\; F$ has at least one vertex in each nonface of $\Delta \;\Leftrightarrow\; \overline{F}$ is missing at least one vertex from each nonface of $\Delta \;\Leftrightarrow\; \overline{F}$ is a face of Δ.

Proposition 1.10 *For a simplicial complex Δ on $\{1, \ldots, n\}$ we have $I_\Delta^\star = I_{\Delta^\star}$.*

Proof: Remark 1.9 implies that $I^\star = I^{[\mathbf{1}]}$ if I is squarefree. Observe that $\mathbf{1} \smallsetminus \mathbf{b} = \mathbf{b}$ if $\mathbf{b} \in \{0,1\}^n$, and use Theorem 1.7. We get $I_\Delta^\star = \langle \mathbf{x}^F \mid \overline{F} \in \Delta \rangle = \langle \mathbf{x}^F \mid F \notin \Delta^\star \rangle = I_{\Delta^\star}$. (Strictly speaking, Theorem 1.7 can only be applied when the irreducible decomposition is irredundant—that is, we should express I_Δ as an intersection $\bigcap \mathfrak{m}^F$ over the *facets* $\overline{F} \in \Delta$, and $I_{\Delta^\star} = \langle \mathbf{x}^F \mid F$ is a *minimal* nonface of $\Delta^\star$.) □

Example 1.11 There are self-dual simplicial complexes, such as the two-dimensional simplicial complex on $\{1, 2, 3, 4\}$ consisting of an empty triangle and a single fourth vertex. There are also complexes which are isomorphic to their duals (after relabeling the vertices), but not equal. For example, the *stick twisted cubic* with ideal $I = \langle ab, bc, cd \rangle = \langle a, c \rangle \cap \langle b, c \rangle \cap \langle b, d \rangle \subset k[a, b, c, d]$, whose dual is $\langle ac, bc, bd \rangle$, has this property. □

Example 1.12 Let Δ be the boundary complex of a simplicial d-polytope Q with n vertices $v_1, \ldots, v_n$ and r facets $F_1, \ldots, F_r$. The irreducible decomposition of the face ideal I_Δ is

$$I_\Delta = \bigcap_{\text{facets } F_i} \langle x_j \mid v_j \notin F_i \rangle .$$

Since the facets of Q correspond to the vertices of the *polar polytope* X, and the vertices of Q correspond to the facets of X, the Alexander dual ideal $I_\Delta^\star$ is the *irrelevant ideal of the Cox homogeneous coordinate ring* of the toric variety with moment polytope Q [Cox95]. □

Remark 1.13 For squarefree ideals, Alexander duality is sort of a canonical thing to do. Using *polarization* [SV86, Chapter II], it is possible to make any monomial ideal into a squarefree one which shares many of its properties. However, $I^{[\mathbf{a}]}$ is not, for any $\mathbf{a}$, the depolarization of the Alexander dual of the polarization of I. For instance, when $I = \langle x^2, xy, y^2 \rangle$, the polarization is $I_{\text{polar}} = \langle x_1x_2, x_1y_1, y_1y_2 \rangle$, whose Alexander dual is $I_{\text{polar}}^\star = \langle x_1y_1, x_1y_2, x_2y_1 \rangle$. Removing the subscripts on x and y then yields the principal ideal $\langle xy \rangle$, whereas $I^\star = \langle xy^2, x^2y \rangle$, and $I^{[\mathbf{a}]}$ always has two minimal generators, for any $\mathbf{a}$.

Definition 1.2 succeeds in generalizing to arbitrary monomial ideals the definition of Alexander duality for squarefree ideals. The connection with the squarefree case is never lost, however, because the general definition does the same thing to the zero-set of I as the squarefree definition does.

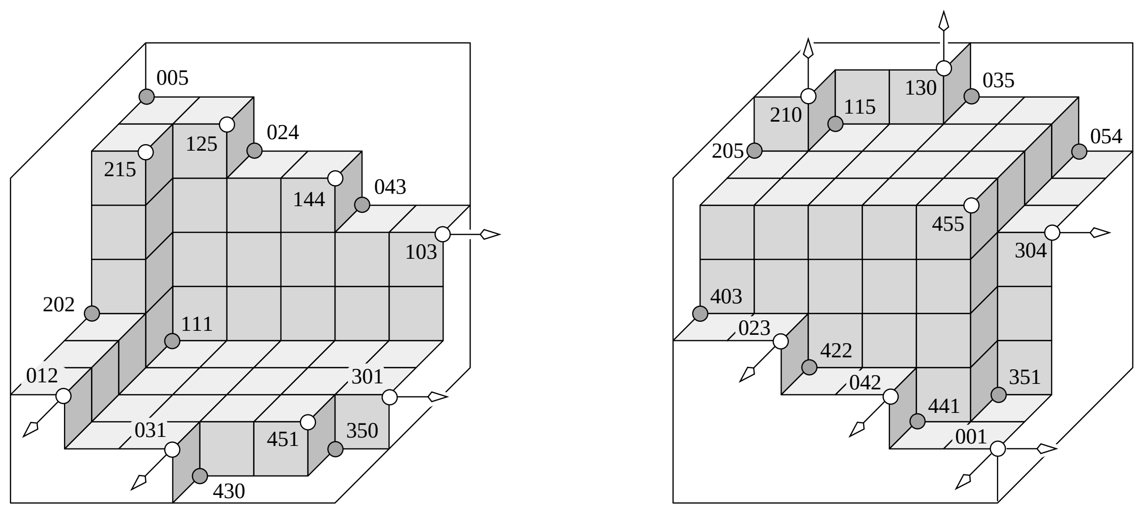
Proposition 1.14 *Taking Alexander duals commutes with taking radicals: $\sqrt{I^\star} = \sqrt{I}^{\,\star}$.*

Proof: Recall that $\mathrm{supp}(\mathbf{b}) = \{i \in \{1,\ldots,n\} \mid b_i \neq 0\}$. Identifying $\mathrm{supp}(\mathbf{b})$ with its characteristic vector in $\{0,1\}^n$ as usual, taking radicals of irreducible ideals can be expressed as $\sqrt{\mathfrak{m}^{\mathbf{b}}} = \mathfrak{m}^{\mathrm{supp}(\mathbf{b})}$. Since $\mathbf{0} \preceq \mathbf{b} \preceq \mathbf{a}_I$ whenever $\mathfrak{m}^{\mathbf{b}}$ is an irreducible component of I, the equality $\mathrm{supp}(\mathbf{b}) = \mathrm{supp}(\mathbf{a}_I \smallsetminus \mathbf{b})$ follows from the definitions. Thus,

$$\begin{aligned} \sqrt{I^\star} &= \langle \mathbf{x}^{\mathrm{supp}(\mathbf{b})} \mid \mathfrak{m}^{\mathbf{b}} \text{ is an irreducible component of } I\rangle \\ &= \langle \mathbf{x}^F \mid \mathfrak{m}^F \text{ is minimal among primes containing } I\rangle \\ &= \sqrt{I}^\star, \end{aligned}$$

the last equality by reasoning as in the proof of Proposition 1.10. □

Three-dimensions



I $\qquad\qquad\qquad\qquad$ $I^\star$

$$\begin{aligned} I &= \langle z^5, x^2z^2, x^4y^3, x^3y^5, y^4z^3, y^2z^4, xyz\rangle \\ &= \langle x^2, y, z^5\rangle \cap \langle y, z^2\rangle \cap \langle y^3, z\rangle \cap \langle x^4, y^5, z\rangle \cap \langle x^3, z\rangle \cap \langle x, z^3\rangle \cap \langle x, y^4, z^4\rangle \cap \langle x, y^2, z^5\rangle \end{aligned}$$

$$\mathbf{a} := \mathbf{a}_I = (4,5,5)$$

$$\begin{aligned} I^\star &= \langle z\rangle \cap \langle x^3, z^4\rangle \cap \langle x, y^3\rangle \cap \langle x^2, y\rangle \cap \langle y^2, z^3\rangle \cap \langle y^4, z^2\rangle \cap \langle x^4, y^5, z^5\rangle \\ &= \langle x^3y^5z, y^5z^4, y^3z^5, xyz^5, x^2z^5, x^4z^3, x^4y^2z^2, x^4y^4z\rangle. \end{aligned}$$

Figure 1.1: The truncated staircase diagrams of I and $I^\star$ from Example 1.15

Example 1.15 Let $n = 3$, so that $S = k[x,y,z]$. Figure 1.1 lists the minimal generators and irredundant irreducible components of an ideal $I \subseteq S$ and its dual $I^\star$ with respect to $\mathbf{a}_I$. The (truncated) "staircase diagrams" representing the monomials not in these ideals are also rendered in Figure 1.1, where the black lattice points are generators, and the white lattice points indicate irreducible components. The numbers are to be interpreted as vectors, e.g.

205 = (2,0,5). The arrows attached to a white lattice point indicate the directions in which the component continues to infinity; it should be noted that a white point has a zero in some coordinate precisely when it has an arrow pointing in the corresponding direction.

Alexander duality in 3 dimensions comes down to the familiar optical illusion in which isometrically rendered cubes appear alternately to point “in” or “out”. In fact, the staircase diagram for $I^\star$ in Figure 1.1 is gotten by literally turning the staircase diagram for I upside-down (the reader is encouraged to try this). Notice that the support of a minimal generator of I is equal to the support of the corresponding irreducible component of $I^\star$. □

Remark 1.16 Turning the staircase diagram over to get the staircase for the Alexander dual hides a subtlety: different “bounding boxes” give rise to Alexander duals with respect to different vectors $\mathbf{a} \succeq \mathbf{a}_I$. For an illustrated example of this, turn to Figure 5.3 on page 66, in which both of the staircase diagrams yield staircases for I when they're upside-down. One should be particularly careful with the difference between $I^{[\mathbf{a}]}$ and $I^\star$, since $(I^\star)^\star \neq I$, in general; again, see Figure 5.3 for some pictures.

Random vectors in $\mathbb{N}^n$

Later chapters will reveal how the notion of Alexander duality sheds light on the interconnections between some of the developments in [BPS98], [BS98], and [Stu99] concerning certain kinds of ideals, called *generic* and *cogeneric* monomial ideals, whose sets of generators or irreducible components are essentially random. As a consequence, these ideals will serve as a steady source of concrete examples for the theory throughout this dissertation.

Definition 1.17 *A monomial ideal I is called* generic *if the following condition is satisfied: whenever $\mathbf{x^b}$ and $\mathbf{x^c}$ are two distinct minimal generators such that $0 \neq b_i = c_i$ for some index i, then $\mathrm{lcm}(\mathbf{x^b}, \mathbf{x^c})/\mathbf{x}^{\mathrm{supp}(\mathbf{b}+\mathbf{c})} \in I$. A monomial ideal is called* strongly generic *if $0 \neq b_i = c_i$ never occurs for any distinct minimal generators $\mathbf{x^b}$ and $\mathbf{x^c}$.*

For instance, the ideal I from Example 1.15 is strongly generic. The original definition of genericity in [BPS98] included only strongly generic ideals. This definition was revised in [MSY00], in part because of the results of Section 5.3, below.

Definition 1.18 *A monomial ideal with irreducible decomposition $I = \bigcap_{i=1}^r I_i$ is called* cogeneric *if the following condition holds: if distinct irreducible components I_i and I_j have a minimal generator in common, there is an irreducible component $I_l \subset I_i + I_j$ such that I_l and $I_i + I_j$ do not have a minimal generator in common. I is called* strongly cogeneric *if no distinct irreducible components share a minimal generator.*

For example, the ideal $I^\star$ from Example 1.15 is strongly cogeneric. The original definition of cogenericity in [Stu99] included only strongly cogeneric ideals. As was the case for generic ideals, the definition of cogenericity was revised in [MSY00].

Example 1.19 Let Σ_n denote the symmetric group on $\{1, \ldots, n\}$ and $\mathbf{c} = (1, 2, \ldots, n) \in \mathbb{N}^n$. The ideal $I = \langle \mathbf{x}^{\sigma(\mathbf{c})} \mid \sigma \in \Sigma_n \rangle$ is the *permutohedron ideal* determined by $\mathbf{c}$, introduced in [BS98, Example 1.9]. The permutohedron ideal is cogeneric, although this is not obvious from the definition. The results of Example 5.37 below will show that the tight Alexander dual is the *tree ideal*, which is generated by $2^n - 1$ monomials: $I^\star = \langle (\mathbf{x}^F)^{n-|F|+1} \mid \varnothing \neq F \in \Delta \rangle$. For instance, when $n = 3$,

$$\begin{aligned} I &= \langle xy^2z^3, xy^3z^2, x^2yz^3, x^2y^3z, x^3yz^2, x^3y^2z \rangle \\ I^\star &= \langle xyz, x^2y^2, x^2z^2, y^2z^2, x^3, y^3, z^3 \rangle. \end{aligned}$$

It is readily verified from the definition that the tree ideal is generic, but not strongly.

The tree ideal is so named because it has the same number $(n+1)^{n-1}$ of standard monomials (monomials not in the ideal) as there are trees on $n+1$ labelled vertices. However, it might also be called the "parking ideal" since the exponents on the standard monomials are the *parking functions*. These facts are relevant in a study of the algebra (over the complex numbers $\mathbb{C}$) generated by the Chern 2-forms of the tautological hermitian line bundles over the manifold of complete flags in $\mathbb{C}^n$, since the standard monomials of $I^\star$ are a $\mathbb{C}$-basis for this algebra [PSS99]. □

The next proposition is immediate from the definitions.

Proposition 1.20 *I is (strongly) generic if and only if $I^{[\mathbf{a}]}$ is (strongly) cogeneric.*

As is the case with all results about Alexander duals of monomial ideals, the two ideals I and $I^{[\mathbf{a}]}$ may be switched by Corollary 1.4. It is also assumed that $\mathbf{a} \succeq \mathbf{a}_I$ whenever $I^{[\mathbf{a}]}$ is written.

1.3 Matlis duality and order lattices

Definition 1.2 is quite satisfactory for the consequences already obtained from it concerning monomial ideals, especially because of its elementary nature. But there are other $\mathbb{Z}^n$-graded modules that aren't monomial ideals, including those derived from free resolutions, such as Ext and Tor modules. Also, it is unclear how Alexander duality is to deal with maps between ideals. Eventually, the goal is to apply Alexander duality in a homological context.

With a view towards such applications, we set out to find a functorial characterization of Alexander duality (Theorem 1.35); the purpose of the present section is to bridge the conceptual gap between duality for monomial ideals and functoriality. In particular, we review Matlis duality for $\mathbb{Z}^n$-graded modules $M = \bigoplus_{\mathbf{b} \in \mathbb{Z}^n} M_{\mathbf{b}}$, which will be a key ingredient in the functors and categorical equivalences of Section 1.6.

For a $\mathbb{Z}^n$-graded ring R (usually the polynomial ring S or the field k, here), the functor $\underline{\mathrm{Hom}}_R(-,-)$ is defined on $\mathbb{Z}^n$-graded modules by letting its graded piece in degree $\mathbf{b} \in \mathbb{Z}^n$ consist of the homogeneous morphisms of degree $\mathbf{b}$:

$$\underline{\mathrm{Hom}}_R(M,N)_{\mathbf{b}} \quad = \quad \mathrm{Hom}_R(M, N[\mathbf{b}]) \quad = \quad \mathrm{Hom}_R(M[-\mathbf{b}], N).$$

Here, the nonunderlined Hom denotes homogeneous homomorphisms of degree zero, and the shift $N[\mathbf{b}]$ is defined from N by $N[\mathbf{b}]_{\mathbf{c}} = N_{\mathbf{b}+\mathbf{c}}$. The *Matlis dual* $M^\vee$ is the $\mathbb{Z}^n$-graded module defined by the first equality below:

$$M^\vee \quad = \quad \underline{\mathrm{Hom}}_S(M, \underline{E}(k)) \quad \cong \quad \underline{\mathrm{Hom}}_k(M, k),$$

where $E(k) = k[x_1^{-1}, \ldots, x_n^{-1}] \cong \underline{\mathrm{Hom}}_k(S,k)$ is the *injective hull of* k as an S-module. The second isomorphism is a consequence of the canonical isomorphism below, with $J = S$:

$$\underline{\mathrm{Hom}}_k(- \otimes_S J, k) \quad \cong \quad \underline{\mathrm{Hom}}_S(-, \underline{\mathrm{Hom}}_k(J,k))\,. \tag{1.2}$$

The degree $\mathbf{b}$ part of the Matlis dual is

$$(M^\vee)_{\mathbf{b}} \quad = \quad \mathrm{Hom}_k(M_{-\mathbf{b}}, k),$$

so Matlis duality "reverses the grading". This notion of Matlis dual agrees with that in [GW78] and [BH93, Section 3.6], and is applicable to modules M which are not necessarily finitely generated or artinian.

Let $\mathcal{M}$ denote the category of $\mathbb{Z}^n$-graded S-modules, the morphisms in $\mathcal{M}$ being homogeneous of degree $\mathbf{0}$. Matlis duality is an exact, contravariant, S-linear functor from $\mathcal{M}$ to $\mathcal{M}$. If, furthermore, $M_{\mathbf{b}}$ is a finite-dimensional k-vector space for all $\mathbf{b} \in \mathbb{Z}^n$ (such a module M will be called $\mathbb{Z}^n$*-finite*), then $(M^\vee)^\vee \cong M$. Matlis duality interchanges noetherian modules and artinian ones because it turns ascending chains into descending chains. On the category of modules that are both artinian and noetherian—i.e. those with finite length—Matlis duality is therefore a *dualizing functor* by definition.

Matlis duality also interchanges flat objects in $\mathcal{M}$ with injective objects. Before saying more, let us first recall these notions. The $\mathbb{Z}^n$-graded $\underline{\mathrm{Hom}}$ functor is introduced above. A module $J \in \mathcal{M}$ is called *injective* if the functor $\underline{\mathrm{Hom}}_S(-, J)$ is exact (it is automatically left-exact). In contrast, the $\mathbb{Z}^n$-graded tensor product is the usual tensor product $\otimes_S$, which is $\mathbb{Z}^n$-graded because tensor products commute with direct sums. A module $J \in \mathcal{M}$ is called *flat* if the functor $- \otimes_S J$ is exact (it is automatically right-exact).

Lemma 1.21 *$J \in \mathcal{M}$ is flat if and only if $J^\vee$ is injective.*

Proof: The functor $\underline{\mathrm{Hom}}_S(-, J^\vee)$ on the right in (1.2) is exact $\Leftrightarrow$ the functor on the left is exact $\Leftrightarrow$ $- \otimes_S J$ is, because k is a field. □

Remark 1.22 In later chapters, it will be necessary to work with the derived functors $\underline{\mathrm{Ext}}$ and $\underline{\mathrm{Tor}}$ of the functors $\underline{\mathrm{Hom}}$ and $\otimes_S$. In order to compute these derived functors *in the category $\mathcal{M}$ of $\mathbb{Z}^n$-graded S-modules*, we need to know that $\mathcal{M}$ has enough injective, projective, and flat modules, just as in the nongraded case. Of course, there are always free modules, so this takes care of the projective and flat modules; for injectives one can easily modify the proof of [BH93, Theorem 3.6.2] to fit the $\mathbb{Z}^n$-graded case. The abundance of nice modules is used as follows.

By [Wei94, Theorem 2.7.2 and Exercise 2.4.3], the derived functors $\underline{\mathrm{Tor}}^S_\bullet(M,N)$ can be calculated as the homology of the complexes obtained by either tensoring with N a flat resolution of M in $\mathcal{M}$ or by tensoring with M a flat resolution of N in $\mathcal{M}$. Here, a *flat resolution* is defined exactly like a free resolution, except that the resolving modules are required to be flat instead of free. Of course, free modules are flat, so free resolutions would suffice; but the extra generality is useful (e.g. in Theorem 6.7). Recall that for *finitely generated* $\mathbb{Z}^n$-graded S-modules, the adjectives "free", "flat", and "projective" are equivalent; this is a simple consequence of the grading and Nakayama's lemma. However, non-finitely generated flat modules, such as localizations of S, often fail to be free, or even projective.

Similarly, [Wei94, Definition 2.5.1, Example 2.5.3, and Exercise 2.7.4] imply that the derived functors $\underline{\mathrm{Ext}}^{\bullet}_S(M,N)$ can be calculated as the homology of the complexes obtained either by applying $\underline{\mathrm{Hom}}_S(-,N)$ to a projective (or free) resolution of M in $\mathcal{M}$ or by applying $\underline{\mathrm{Hom}}_S(M,-)$ to an injective resolution of N in $\mathcal{M}$. If M is finitely generated then the $\mathbb{Z}^n$-graded functors $\underline{\mathrm{Ext}}^{\bullet}_S(M,-)$ agree with the usual ungraded functor $\mathrm{Ext}^{\bullet}_S(M,-)$ because graded free modules are also ungraded free modules.

The "degree reversal" of Matlis duality can be viewed as duality in the order lattice $\mathbb{Z}^n$. To see this, suppose that $M \subseteq T := S[x_1^{-1},\dots,x_n^{-1}]$ is an S-submodule of the Laurent polynomial ring (i.e. a *monomial module*, in the language of [BS98]). Then $M^\vee$ is the quotient of T characterized by

$$M_{\mathbf{b}} \neq 0 \quad \Longleftrightarrow \quad (M^\vee)_{-\mathbf{b}} \neq 0. \tag{1.3}$$

Thus, while M corresponds to a dual order ideal in $\mathbb{Z}^n$ (consisting of the exponent vectors on monomials in M), the Matlis dual $M^\vee$ corresponds to an order ideal in $\mathbb{Z}^n$ (consisting of the negatives of the exponent vectors on monomials in M).

Example 1.23 For instance, if for some $F \subseteq \{1,\dots,n\}$ the module M is a localization

$$M = S[\mathbf{x}^{-\overline{F}}] := S[x_i^{-1} \mid i \notin F] \quad \text{then} \quad M^\vee = k[x_1^{-1},\dots,x_n^{-1}][x_i \mid i \notin F] =: \underline{E}\,(S/\mathfrak{m}^F),$$

where again $\mathfrak{m}^F = \langle x_i \mid i \in F\rangle$. The module $\underline{E}\,(S/\mathfrak{m}^F)$ is called the *injective hull in $\mathcal{M}$ of $S/\mathfrak{m}^F$*. Note that $S[\mathbf{x}^{-\overline{F}}]$ is a flat S-module (being a localization of S), so that $S[\mathbf{x}^{-\overline{F}}]^\vee$ is indeed an injective object of $\mathcal{M}$ by Lemma 1.21. See [GW78] for more on injectives and injective hulls in $\mathcal{M}$. □

In case $M \in \mathcal{M}$ is arbitrary, the Matlis dual $M^\vee$ can still be described in terms of lattice duality in $\mathbb{Z}^n$ by taking an injective resolution of M, which Matlis duality transforms into a flat resolution of $M^\vee$, by exactness. In fact, this observation underlies many of the developments in Chapter 3, below.

Alexander duality for squarefree monomial ideals can also be thought of as duality in an order lattice, but this time in the much smaller lattice $\{0,1\}^n \subset \mathbb{Z}^n$. Of course, the duality in $\{0,1\}^n$ is $F \mapsto \mathbf{1} - F = \overline{F}$, so this comes from the following reformulation of squarefree Alexander duality: $I^{[\mathbf{1}]}$ is characterized as the squarefree ideal satisfying

$$(S/I)_F \neq 0 \quad \Longleftrightarrow \quad (I^{[\mathbf{1}]})_{\mathbf{1}-F} \neq 0\,. \tag{1.4}$$

Comparing Equations (1.4) and (1.3), we find that Alexander duality looks just like Matlis duality except that for Alexander duality:

1. We only care about degrees F in the interval $[\mathbf{0},\mathbf{1}] = \{0,1\}^n \subset \mathbb{Z}^n$;
2. We assume that $I^{[\mathbf{1}]}$ is a squarefree ideal; and
3. We have to shift $I^{[\mathbf{1}]}$ by $\mathbf{1}$; that is, instead of $F \mapsto -F$, we have $F \mapsto \mathbf{1} - F$.

To sum this up algebraically, define $B_{\mathbf{1}}M = \bigoplus_{\mathbf{0}\preceq\mathbf{b}\preceq\mathbf{1}} M_{\mathbf{b}}$ to be the part of $M \in \mathcal{M}$ which is *bounded in the interval* $[\mathbf{0},\mathbf{1}]$. The precise relation between Matlis duality and Alexander duality for squarefree monomial ideals is then:

> For each squarefree ideal I, the Alexander dual ideal is the unique squarefree monomial ideal $I^{[\mathbf{1}]}$ such that $B_{\mathbf{1}}I^{[\mathbf{1}]}$ is the Matlis dual of $(B_{\mathbf{1}}(S/I))[\mathbf{1}]$.

Remark 1.24 Yanagawa predicted that if there is a natural extension of Alexander duality to modules other than squarefree ideals, then the dual of the Stanley-Reisner ring S/I_Δ for a simplicial complex Δ would be the Stanley-Reisner ideal $I_{\Delta^\star}$, and not $S/I_{\Delta^\star}$ [Yan00a]. The connection with Matlis duality, particularly Equation (1.4), explains why Yanagawa was correct. The unnaturality of the definition $I^{[\mathbf{1}]}$ is why the brackets are there; the more natural $I^{\mathbf{1}} := S/I^{[\mathbf{1}]}$ deserves the better notation.

Yanagawa [Yan00a] asked whether Alexander duality for squarefree ideals extends to his *squarefree modules*, which are called positively $\mathbf{1}$-determined below. Such an extension is provided by Theorem 1.35 with $\mathbf{a} = \mathbf{1}$, which uses Alexander duality as described here, defined through Matlis duality. For instance, "I is a squarefree ideal" is replaced by "M is a positively $\mathbf{1}$-determined module", which means essentially that M is $\mathbb{N}^n$-graded and can be recovered from $B_{\mathbf{1}}M$. The functor which performs the recovery of M from $B_{\mathbf{1}}M$ is called $P_{\mathbf{1}}$, the *positive extension* with respect to $\mathbf{1}$. Then (1.4) becomes

$$I^{[\mathbf{1}]} \quad = \quad P_{\mathbf{1}}\Big(B_{\mathbf{1}}(S/I)[\mathbf{1}]^\vee\Big),$$

which says that $I^{\mathbf{1}}$ is gotten from S/I by (i) restricting S/I to the interval $[\mathbf{0},\mathbf{1}]$; (ii) shifting the result down by $\mathbf{1}$ (which puts it in the interval $[-\mathbf{1},\mathbf{0}]$); (iii) flipping the result by Matlis duality (so that it sits again in $[\mathbf{0},\mathbf{1}]$); and then (iv) extending positively. The vector $\mathbf{1}$ will be replaced below by an arbitrary vector $\mathbf{a} \in \mathbb{N}^n$.

1.4 Finitely determined modules

For the remainder of this chapter, $\mathbf{a} \in \mathbb{N}^n$ *will denote a fixed element satisfying* $\mathbf{a} \succeq \mathbf{1}$.

The Alexander duality functors will be defined on certain full subcategories of the category $\mathcal{M}$ whose objects are completely determined by their homogeneous components in degrees from the interval $[\mathbf{0},\mathbf{a}]$ between $\mathbf{0}$ and $\mathbf{a}$. Given such data, there are 4 relatively obvious ways for it to determine a $\mathbb{Z}^n$-graded module, depending on which degrees outside of $[\mathbf{0},\mathbf{a}]$ are required to be zero. These are the 4 categories of Definition 1.25, and the goal of this section is simply to introduce them. It should be noted that although the language of categories is important for technical reasons, the material in this section and the next is actually quite elementary, dealing mostly with collections of linear maps between finite-dimensional vector spaces; the difficulty lies only in keeping track of the $\mathbb{Z}^n$-graded degrees.

Denote the i^{th} basis vector of $\mathbb{Z}^n$ by $\mathbf{e}_i$, so that multiplication by x_i gives a homomorphism of k-vector spaces $\cdot x_i : M_{\mathbf{b}} \to M_{\mathbf{b}+\mathbf{e}_i}$ for any $M = \bigoplus_{\mathbf{b}\in\mathbb{Z}^n} M_{\mathbf{b}} \in \mathcal{M}$. If M has homogeneous components that are finite-dimensional k-vector spaces, M will be called $\mathbb{Z}^n$-*finite*; this condition prevents Matlis duals from becoming too large. Recall that M is

$\mathbb{N}^n$-graded if $M_{\mathbf{b}} = 0$ for $\mathbf{b} \notin \mathbb{N}^n$. By analogy, M will be called $(\mathbf{a} - \mathbb{N}^n)$-*graded* if $M_{\mathbf{b}} = 0$ unless $\mathbf{b} \preceq \mathbf{a}$.

Definition 1.25 *Each of the following 4 categories is the full subcategory of $\mathcal{M}$ on the $\mathbb{Z}^n$-finite modules $M \in \mathcal{M}$ satisfying the given conditions.*

$\mathcal{M}^{\mathbf{a}}$: *M is* **a**-determined *if $\cdot x_i : M_{\mathbf{b}} \to M_{\mathbf{b}+\mathbf{e}_i}$ is an isomorphism unless $0 \le b_i \le a_i - 1$.*

$\mathcal{M}^{\mathbf{a}}_{+}$: *$M$ is* positively **a**-determined *if it is $\mathbb{N}^n$-graded and $\cdot x_i : M_{\mathbf{b}} \to M_{\mathbf{b}+\mathbf{e}_i}$ is an isomorphism whenever $a_i \le b_i$.*

$\mathcal{M}^{\mathbf{a}}_{-}$: *$M$ is* negatively **a**-determined *if it is $(\mathbf{a} - \mathbb{N}^n)$-graded and $\cdot x_i : M_{\mathbf{b}} \to M_{\mathbf{b}+\mathbf{e}_i}$ is an isomorphism whenever $b_i \le -1$.*

$\overline{\mathcal{M}}^{\mathbf{a}}$: *$M$ is* **a**-bounded *if $M_{\mathbf{b}} = 0$ unless $\mathbf{0} \preceq \mathbf{b} \preceq \mathbf{a}$.*

Example 1.26 The most important modules in this dissertation are shifts of S and of injective hulls $\underline{E}(S/\mathfrak{m}^F)$, the latter having been defined in Example 1.23. We have:

1. A shift $S[-\mathbf{b}]$ is positively **a**-determined if and only if $\mathbf{0} \preceq \mathbf{b} \preceq \mathbf{a}$, and **a**-determined if and only if $\mathbf{1} \preceq \mathbf{b} \preceq \mathbf{a}$. However, $S[-\mathbf{b}]$ is never negatively **a**-determined.
2. A shift $\underline{E}(S/\mathfrak{m}^F)[-\mathbf{b}]$ is **a**-determined if and only if $\mathbf{0} \preceq \mathbf{b} \cdot F \preceq \mathbf{a} - \mathbf{1}$, where $\mathbf{b} \cdot F = \sum_{i \in F} b_i \mathbf{e}_i$. The reason why we can forget about the coordinates b_j for $j \notin F$ is that $\cdot x_j$ is an isomorphism on every graded piece of $\underline{E}(S/\mathfrak{m}^F)$ whenever $j \notin F$. On the other hand, $\cdot x_i$ for $i \in F$ is an isomorphism on $\underline{E}(S/\mathfrak{m}^F)[-\mathbf{b}]_{\mathbf{c}}$ unless $c_i = b_i$, in which case $\cdot x_i$ is the zero map on a 1-dimensional k-vector space.
3. A shift $\underline{E}(S/\mathfrak{m}^F)[-\mathbf{b}]$ is negatively **a**-determined if and only if $F = \{1, \ldots, n\}$ and $\mathbf{0} \preceq \mathbf{b} \preceq \mathbf{a} - \mathbf{1}$. A shift $\underline{E}(S/\mathfrak{m}^F)[-\mathbf{b}]$ is never positively **a**-determined. □

Remark 1.27 A positively **a**-determined module is finitely generated. In particular, the positively **1**-determined modules are precisely the *squarefree modules* as defined in [Yan00a]. It is shown there (though not stated in these terms) that $\mathcal{M}^{\mathbf{1}}_{+}$ is an abelian category. This will follow for all **a** from Theorem 1.34, below.

Example 1.28 Let $J \subseteq S$ be a monomial ideal. Then J and S/J are positively **a**-determined if and only if the generators of J divide $\mathbf{x}^{\mathbf{a}}$. Given that J is in $\mathcal{M}^{\mathbf{a}}_{+}$, the modules $J^{[\mathbf{a}]}$ and $S/J^{[\mathbf{a}]}$ are also in $\mathcal{M}^{\mathbf{a}}_{+}$. □

If $N \in \mathcal{M}$, the tensor product $- \otimes_S N$ is naturally a functor $\mathcal{M} \to \mathcal{M}$. Let $\underline{\mathrm{Tor}}_i^S(-, N)$ be its left derived functors and $\beta_{i,\mathbf{b}}(M) = \dim_k \underline{\mathrm{Tor}}_i^S(M, k)_{\mathbf{b}}$, the i^{th} *Betti number of M in degree* **b**. For finitely generated M, $\beta_{i,\mathbf{b}}$ is the number of summands $S[-\mathbf{b}]$ in homological degree i in any minimal $\mathbb{Z}^n$-graded free resolution of M. The next proposition clarifies the definition of $\mathcal{M}^{\mathbf{a}}_{+}$ and extends Example 1.26.1. It will be used in the proofs of Theorem 1.35 and Corollary 3.24.

Proposition 1.29 *A finitely generated module $M \in \mathcal{M}$ is positively* **a**-*determined if and only if the Betti numbers of M satisfy: $\beta_{0,\mathbf{b}}(M) = \beta_{1,\mathbf{b}}(M) = 0$ unless $\mathbf{0} \preceq \mathbf{b} \preceq \mathbf{a}$.*

Proof: If $\beta_{0,\mathbf{b}} = \beta_{1,\mathbf{b}} = 0$ unless $\mathbf{0} \preceq \mathbf{b} \preceq \mathbf{a}$, then any minimal free presentation $\mathbb{F}$ of M is $\mathbb{N}^n$-graded, so M is, too. Furthermore, $\cdot x_i : \mathbb{F}_{\mathbf{b}} \to \mathbb{F}_{\mathbf{b}+\mathbf{e}_i}$ is an isomorphism if $b_i \ge a_i$, so the same is true with M in place of $\mathbb{F}$.

Now assume that M is positively $\mathbf{a}$-determined and let $\mathbf{b} \not\preceq \mathbf{a}$, say $b_i > a_i$. If M has a minimal generator in degree $\mathbf{b}$ then $\cdot x_i : M_{\mathbf{b}-\mathbf{e}_i} \to M_{\mathbf{b}}$ is not surjective. If $\beta_{1,\mathbf{b}}(M) \neq 0$ then since every minimal generator of M is in a degree with i^{th} coordinate $< b_i$, it follows that $\cdot x_i : M_{\mathbf{b}-\mathbf{e}_i} \to M_{\mathbf{b}}$ is not injective. This is a contradiction. □

1.5 The Čech hull and categorical equivalences

The goal of this section is the equivalences in Theorem 1.34 between the categories of Definition 1.25. The construction of an Alexander duality functor in Section 1.6 will be accomplished by Matlis duality in concert with these equivalences of categories. In order to write down the equivalences, though, we need some intermediate functors. All of the machinery developed in this section will be used extensively also in later chapters.

Recall that the poset $(\mathbb{Z}^n, \preceq)$ is an order lattice with *meet* $\wedge$ and *join* $\vee$ being the componentwise minimum and maximum.

Definition 1.30 *Let $M \in \mathcal{M}$. Define the functors $B_{\mathbf{a}}$, $P_{\mathbf{a}}$, and $\check{C}$ as follows.*

1. *Let $B_{\mathbf{a}}M := \bigoplus_{\mathbf{0}\preceq\mathbf{b}\preceq\mathbf{a}} M_{\mathbf{b}}$ be the subquotient* bounded in the interval $[\mathbf{0}, \mathbf{a}]$.
2. *Let $P_{\mathbf{a}}M = \bigoplus_{\mathbf{b}\in\mathbb{Z}^n} M_{\mathbf{a}\wedge\mathbf{b}}$ (that is, $(P_{\mathbf{a}}M)_{\mathbf{b}} = M_{\mathbf{a}\wedge\mathbf{b}}$) with the S-action*

$$\cdot x_i : (P_{\mathbf{a}}M)_{\mathbf{b}} \to (P_{\mathbf{a}}M)_{\mathbf{b}+\mathbf{e}_i} \quad = \quad \begin{cases} \text{identity} & \text{if } b_i \geq a_i \\ \cdot x_i : M_{\mathbf{a}\wedge\mathbf{b}} \to M_{\mathbf{a}\wedge\mathbf{b}+\mathbf{e}_i} & \text{if } b_i < a_i \end{cases}$$

be the positive extension of M. *$P_{\mathbf{a}}$ is usually applied when M is $(\mathbf{a} - \mathbb{N}^n)$-graded.*

3. *Let $\check{C}M = \bigoplus_{\mathbf{b}\in\mathbb{Z}^n} M_{\mathbf{b}\vee\mathbf{0}}$ (that is, $(\check{C}M)_{\mathbf{b}} = M_{\mathbf{b}\vee\mathbf{0}}$) with the S-action*

$$\cdot x_i : (\check{C}M)_{\mathbf{b}} \to (\check{C}M)_{\mathbf{b}+\mathbf{e}_i} \quad = \quad \begin{cases} \text{identity} & \text{if } b_i < 0 \\ \cdot x_i : M_{\mathbf{b}\vee\mathbf{0}} \to M_{\mathbf{e}_i+\mathbf{b}\vee\mathbf{0}} & \text{if } b_i \geq 0 \end{cases}$$

be the negative extension *or* Čech hull of M. *$\check{C}$ is usually applied when M is $\mathbb{N}^n$-graded.*

Figure 1.2 shows a picture of the Čech hull of an ideal in two variables; note that $\check{C}(I)$ "extends I backwards to infinity" whenever I hits the boundary of the positive orthant. In general, the Čech hull is an essential extension which looks something like a cross between localization and injective hull. In fact, part 1 of the next example shows that $\check{C}$ actually is localization when applied to a free module, while part 3 shows that it takes injective hulls when applied to a quotient of S by a prime ideal. Note that the $\mathbb{Z}^n$-graded prime ideals of S are precisely the monomial primes $\mathfrak{m}^F = \langle x_i \mid i \in F\rangle$ for $F \subseteq \{1, \ldots, n\}$, so that (for instance) the "homogeneous residue class ring" of $\mathfrak{m}^F$ is $(S/\mathfrak{m}^F)[\mathbf{x}^{-\overline{F}}]$.

Example 1.31

1. If $\mathbf{b} \in \mathbb{N}^n$ and $F = \operatorname{supp}(\mathbf{b})$, then $\check{C}(S[-\mathbf{b}]) \cong S[\mathbf{x}^{-\overline{F}}][-\mathbf{b}]$.
2. If $\mathbf{b} \in \mathbb{N}^n$ then $B_{\mathbf{a}}(S[-\mathbf{b}]) = 0$ unless $\mathbf{b} \preceq \mathbf{a}$, in which case we have that

$$B_{\mathbf{a}}(S[-\mathbf{b}]) \cong (S/\mathfrak{m}^{\mathbf{a}+\mathbf{1}-\mathbf{b}})[-\mathbf{b}]$$

is the artinian subquotient of S which is nonzero precisely in degrees from the interval $[\mathbf{b}, \mathbf{a}]$. Applying $P_{\mathbf{a}}$ to this yields back $S[-\mathbf{b}]$, so $P_{\mathbf{a}}B_{\mathbf{a}}(S[-\mathbf{b}]) \cong S[-\mathbf{b}]$ if $\mathbf{b} \preceq \mathbf{a}$.

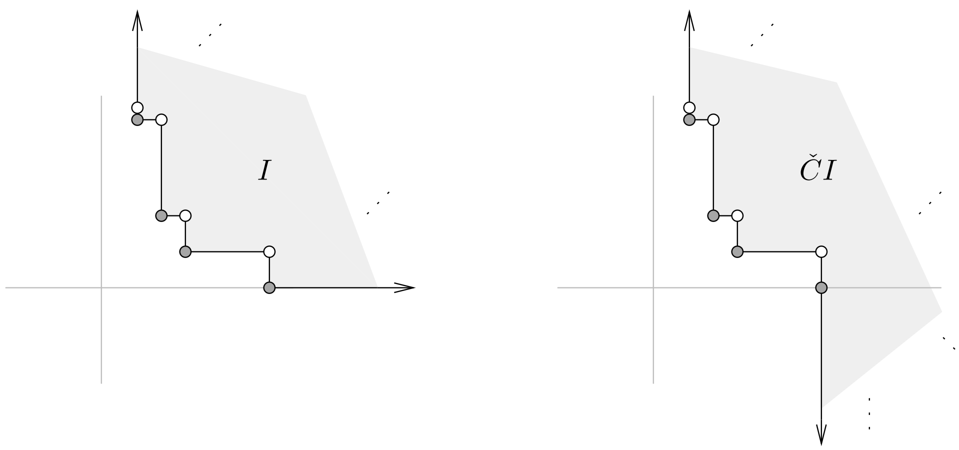


Figure 1.2: The Čech hull

3. If $F \subseteq \{1, \ldots, n\}$, then $\check{C}(S/\mathfrak{m}^F) \cong \underline{E}(S/\mathfrak{m}^F)$, the *injective hull* of $S/\mathfrak{m}^F$ in $\mathcal{M}$; see Example 1.23 and [GW78, Section 1.3].
4. If $\mathbf{b} \preceq \mathbf{a}$ then $B_{\mathbf{a}}(S/\mathfrak{m}^{\mathbf{a} \smallsetminus \mathbf{b}})$ is the artinian quotient $S/\mathfrak{m}^{\mathbf{a}+\mathbf{1}-\mathbf{b}}$ of S which is nonzero precisely in degrees from $[\mathbf{0}, \mathbf{a} - \mathbf{b}]$. Compare this to $B_{\mathbf{a}}(S[-\mathbf{b}])$ from part 2. □

Lemma 1.32 *The functors $B_{\mathbf{a}}$, $P_{\mathbf{a}}$, and $\check{C}$ from $\mathcal{M}$ to itself are exact.*

Proof: Straightforward from the definitions, since a sequence of modules in $\mathcal{M}$ is exact if and only if it is exact in each $\mathbb{Z}^n$-graded degree. □

The functors in Definition 1.30 can be restricted to each of $\mathcal{M}^{\mathbf{a}}$, $\mathcal{M}^{\mathbf{a}}_{+}$, $\mathcal{M}^{\mathbf{a}}_{-}$, and $\overline{\mathcal{M}}^{\mathbf{a}}$. Their effects are summarized in Table 1.1, where the restriction of a functor is assumed to have the indicated target, and is denoted by the same symbol as the functor itself. The $\equiv$ symbol means that the target equals the source, and the restricted functor replaces each object by an isomorphic one. Theorem 1.34 states that all of the restricted functors in the table (with their indicated sources and targets) are actually equivalences of categories.

The following lemma will be used in the proof of Theorem 1.34.

Lemma 1.33 *Morphisms in $\mathcal{M}^{\mathbf{a}}$, $\mathcal{M}^{\mathbf{a}}_{+}$, and $\mathcal{M}^{\mathbf{a}}_{-}$ are uniquely determined by their components in degrees $\mathbf{b}$ with $\mathbf{0} \preceq \mathbf{b} \preceq \mathbf{a}$.*

Table 1.1: The functors of Definition 1.30 on the categories of Definition 1.25 and Theorem 1.34.

	$\mathcal{M}^{\mathbf{a}}$	$\overline{\mathcal{M}}^{\mathbf{a}}$	$\mathcal{M}^{\mathbf{a}}_{+}$	$\mathcal{M}^{\mathbf{a}}_{-}$
$B_{\mathbf{a}}$	$\overline{\mathcal{M}}^{\mathbf{a}}$	$\equiv$	$\overline{\mathcal{M}}^{\mathbf{a}}$	$\overline{\mathcal{M}}^{\mathbf{a}}$
$P_{\mathbf{a}}$	$\equiv$	$\mathcal{M}^{\mathbf{a}}_{+}$	$\equiv$	$\mathcal{M}^{\mathbf{a}}$
$\check{C}$	$\equiv$	$\mathcal{M}^{\mathbf{a}}_{-}$	$\mathcal{M}^{\mathbf{a}}$	$\equiv$

Proof: Only the case $\mathcal{M}^{\mathbf{a}}$ is demonstrated here; the other two cases involve the same arguments. Suppose that $\varphi : M \to M'$ in $\mathcal{M}^{\mathbf{a}}$, and let $\varphi_{\mathbf{b}} : M_{\mathbf{b}} \to M'_{\mathbf{b}}$ be its component in degree $\mathbf{b} \in \mathbb{Z}^n$. Setting $y = \mathbf{x}^{\mathbf{b}\vee\mathbf{0}-\mathbf{b}}$ and $z = \mathbf{x}^{\mathbf{b}-\mathbf{a}\wedge\mathbf{b}} = \mathbf{x}^{\mathbf{b}\vee\mathbf{0}-\mathbf{a}\wedge\mathbf{b}\vee\mathbf{0}}$ (where $\mathbf{a}\wedge\mathbf{b}\vee\mathbf{0} := (\mathbf{a}\wedge\mathbf{b})\vee\mathbf{0} = \mathbf{a}\wedge(\mathbf{b}\vee\mathbf{0})$ since $\mathbf{0} \preceq \mathbf{a}$, so the parentheses can be left off without ambiguity), the multiplication maps

$$\cdot y : M_{\mathbf{b}} \to M_{\mathbf{b}\vee\mathbf{0}} \quad \text{and} \quad \cdot z : M_{\mathbf{a}\wedge\mathbf{b}\vee\mathbf{0}} \to M_{\mathbf{b}\vee\mathbf{0}}$$

are isomorphisms (and also with M replaced by M'). Since φ is a module homomorphism, we have in any case

$$(\varphi_{\mathbf{b}\vee\mathbf{0}})(\cdot y) = (\cdot y)(\varphi_{\mathbf{b}}) \quad \text{and} \quad (\varphi_{\mathbf{b}\vee\mathbf{0}})(\cdot z) = (\cdot z)(\varphi_{\mathbf{a}\wedge\mathbf{b}\vee\mathbf{0}}).$$

Thus $\varphi_{\mathbf{b}} = (\cdot y)^{-1}(\varphi_{\mathbf{b}\vee\mathbf{0}})(\cdot y) = (\cdot y)^{-1}(\cdot z)(\varphi_{\mathbf{a}\wedge\mathbf{b}\vee\mathbf{0}})(\cdot z)^{-1}(\cdot y)$. □

Theorem 1.34 *The abelian categories $\mathcal{M}^{\mathbf{a}}$, $\mathcal{M}^{\mathbf{a}}_{+}$, $\mathcal{M}^{\mathbf{a}}_{-}$, and $\overline{\mathcal{M}}^{\mathbf{a}}$ are all equivalent.*

Proof: By [Mac98, IV.4, Theorem 1] it is enough to show that for each $\mathcal{N} \in \{\mathcal{M}^{\mathbf{a}}, \mathcal{M}^{\mathbf{a}}_{+}, \mathcal{M}^{\mathbf{a}}_{-}\}$,

i. the functor $B_{\mathbf{a}} : \mathcal{N} \to \overline{\mathcal{M}}^{\mathbf{a}}$ is fully faithful, and

ii. every object in $\overline{\mathcal{M}}^{\mathbf{a}}$ is isomorphic to $B_{\mathbf{a}}M$ for some $M \in \mathcal{N}$.

The faithfulness of $B_{\mathbf{a}}$ in (i) is the content of Lemma 1.33. Furthermore, given $\overline{\varphi} : B_{\mathbf{a}}M \to B_{\mathbf{a}}M'$ for $M, M' \in \mathcal{N}$, we can (with y and z as in the proof of Lemma 1.33) *define* $\varphi_{\mathbf{b}} : M_{\mathbf{b}} \to M'_{\mathbf{b}}$ by

$$\varphi_{\mathbf{b}} \quad = \quad \begin{cases} (\cdot y)^{-1}(\cdot z)(\overline{\varphi}_{\mathbf{a}\wedge\mathbf{b}\vee\mathbf{0}})(\cdot z)^{-1}(\cdot y) & \text{if } \mathcal{N} = \mathcal{M}^{\mathbf{a}} \\ (\cdot z)(\overline{\varphi}_{\mathbf{a}\wedge\mathbf{b}})(\cdot z)^{-1} & \text{if } \mathcal{N} = \mathcal{M}^{\mathbf{a}}_{+} \\ (\cdot y)^{-1}(\overline{\varphi}_{\mathbf{b}\vee\mathbf{0}})(\cdot y) & \text{if } \mathcal{N} = \mathcal{M}^{\mathbf{a}}_{-} \end{cases},$$

whence $B_{\mathbf{a}}$ is full when restricted to $\mathcal{N}$. To show (ii) note that, by definition, $B_{\mathbf{a}}\check{C}P_{\mathbf{a}}$, $B_{\mathbf{a}}P_{\mathbf{a}}$, and $B_{\mathbf{a}}\check{C}$ (viewed as functors $\overline{\mathcal{M}}^{\mathbf{a}} \to \overline{\mathcal{M}}^{\mathbf{a}}$) are all isomorphic to the identity of $\overline{\mathcal{M}}^{\mathbf{a}}$. The categories are abelian because $\overline{\mathcal{M}}^{\mathbf{a}}$ obviously is. □

1.6 The Alexander duality functors

The next theorem is the main result of Chapter 1. It says there is only one way to extend Alexander duality for monomial ideals to a functor on $\mathcal{M}^{\mathbf{a}}_{+}$, and that the functor is particularly nice.

Theorem 1.35 *There is a unique (up to isomorphism) exact k-linear contravariant functor $A_{\mathbf{a}} : \mathcal{M}^{\mathbf{a}}_{+} \to \mathcal{M}^{\mathbf{a}}_{+}$ which, for $\mathbf{0} \preceq \mathbf{b} \preceq \mathbf{c} \preceq \mathbf{a}$, takes canonical inclusions $\iota : S[-\mathbf{c}] \to S[-\mathbf{b}]$ to canonical surjections $A_{\mathbf{a}}(\iota) : S/\mathfrak{m}^{\mathbf{a}\smallsetminus\mathbf{b}} \to S/\mathfrak{m}^{\mathbf{a}\smallsetminus\mathbf{c}}$. Any such functor satisfies $A_{\mathbf{a}}A_{\mathbf{a}} \cong \mathrm{id}_{\mathcal{M}^{\mathbf{a}}_{+}}$ as well as $A_{\mathbf{a}}(S/I) \cong I^{[\mathbf{a}]}$ for monomial ideals I.*

Proof: Existence will be provided in Definition 1.36 and Proposition 1.39. Given existence, all of the remaining statements will then follow from uniqueness, which is treated now.

By Proposition 1.29, any module in $\mathcal{M}_+^{\mathbf{a}}$ has a free presentation in $\mathcal{M}_+^{\mathbf{a}}$. Given a map of modules $\varphi : M \to N$ in $\mathcal{M}_+^{\mathbf{a}}$, choose free presentations (with bases) and a lifting of φ as in the left diagram.

$$\begin{array}{ccccccc} F_1 & \xrightarrow{\partial_M} & F_0 & \longrightarrow & M & \longrightarrow & 0 \\ \varphi_1 \downarrow & & \varphi_0 \downarrow & & \varphi \downarrow & & \\ F_1' & \xrightarrow{\partial_N} & F_0' & \longrightarrow & N & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccccccc} A_{\mathbf{a}}(F_1) & \xleftarrow{\partial^M} & A_{\mathbf{a}}(F_0) & \longleftarrow & A_{\mathbf{a}}(M) & \longleftarrow & 0 \\ \varphi^1 \uparrow & & \varphi^0 \uparrow & & A_{\mathbf{a}}(\varphi) \uparrow & & \\ A_{\mathbf{a}}(F_1') & \xleftarrow{\partial^N} & A_{\mathbf{a}}(F_0') & \longleftarrow & A_{\mathbf{a}}(N) & \longleftarrow & 0 \end{array}$$

Choosing an $A_{\mathbf{a}}$ and applying it to the left diagram gives the right diagram, which has exact rows. The maps φ^1, φ^0, ∂^M, and ∂^N are determined by k-linearity and the action of $A_{\mathbf{a}}$ on $S[-\mathbf{b}]$, and are thus independent of which functor $A_{\mathbf{a}}$ is used to obtain the right diagram. By exactness, $A_{\mathbf{a}}(M) \cong \ker(\partial^M)$, whence $A_{\mathbf{a}}$ is uniquely determined on objects. Furthermore, there is at most one map $A_{\mathbf{a}}(\varphi)$ making the diagram commute, namely $\varphi^0 : \ker(\partial^N) \to \ker(\partial^M)$. Thus the effect of $A_{\mathbf{a}}$ on maps is uniquely determined, as well.

The quotient S/I of S by any monomial ideal $I = \langle \mathbf{x}^{\mathbf{b}} \mid \mathbf{b} \in B \rangle$ is the cokernel of the canonical map $\bigoplus_{\mathbf{b} \in B} S[-\mathbf{b}] \to S$. By exactness and the action on principal ideals, the Alexander dual of this map is $S \to \bigoplus_{\mathbf{b} \in B} S/\mathfrak{m}^{\mathbf{a} \smallsetminus \mathbf{b}}$, and has kernel $(S/I)^{\mathbf{a}}$. But the kernel is readily computed to be $I^{[\mathbf{a}]} = \bigcap_{\mathbf{b} \in B} \mathfrak{m}^{\mathbf{a} \smallsetminus \mathbf{b}}$, whence $(S/I)^{\mathbf{a}} \cong I^{[\mathbf{a}]}$. Moreover, this implies that applying $A_{\mathbf{a}}$ to the right diagram above again yields the left diagram, since the modules $A_{\mathbf{a}}(F_0)$ and $A_{\mathbf{a}}(F_1)$ are direct sums of quotients by irreducible ideals. Thus $A_{\mathbf{a}} A_{\mathbf{a}} \cong \mathrm{id}_{\mathcal{M}_+^{\mathbf{a}}}$. □

Existence will come from the functors of Section 1.6. The point of proving Theorem 1.34 is that $\overline{\mathcal{M}}^{\mathbf{a}}$ consists of finite-length modules over S, or even over the ring $B_{\mathbf{a}} S$, and duality for these modules is familiar (this key fact is hidden in the proof of Corollary 1.4). Using the equivalences above, Matlis duality $\underline{\mathrm{Hom}}_k(-, k[-\mathbf{a}]) = (-)^\vee[-\mathbf{a}]$ from $\overline{\mathcal{M}}^{\mathbf{a}}$ to itself becomes Alexander duality on $\mathcal{M}_+^{\mathbf{a}}$. Thus, we get from $M \in \mathcal{M}_+^{\mathbf{a}}$ to its Alexander dual via the following steps, where $\equiv$ is the covariant equivalence of Theorem 1.34:

$$M \quad \overset{\mathcal{M}_+^{\mathbf{a}} \equiv \overline{\mathcal{M}}^{\mathbf{a}}}{\longmapsto} \quad B_{\mathbf{a}} M \quad \overset{\substack{\text{Matlis duality} \\ \overline{\mathcal{M}}^{\mathbf{a}} \to \overline{\mathcal{M}}^{\mathbf{a}}}}{\longmapsto} \quad (B_{\mathbf{a}} M)^\vee[-\mathbf{a}] \quad \overset{\overline{\mathcal{M}}^{\mathbf{a}} \equiv \mathcal{M}_+^{\mathbf{a}}}{\longmapsto} \quad P_{\mathbf{a}}\Big((B_{\mathbf{a}} M)^\vee[-\mathbf{a}]\Big).$$

Alternatively, one can use Matlis duality $\mathcal{M}_-^{\mathbf{a}} \to \mathcal{M}_+^{\mathbf{a}}$:

$$M \quad \overset{\mathcal{M}_+^{\mathbf{a}} \equiv \mathcal{M}_-^{\mathbf{a}}}{\longmapsto} \quad \check{C} B_{\mathbf{a}} M \quad \overset{\substack{\text{Matlis duality} \\ \mathcal{M}_-^{\mathbf{a}} \to \mathcal{M}_+^{\mathbf{a}}}}{\longmapsto} \quad (\check{C} B_{\mathbf{a}} M)^\vee[-\mathbf{a}].$$

The two modules at which we have just arrived are isomorphic by Theorem 1.34 because their restrictions to the interval $[\mathbf{0}, \mathbf{a}]$ (i.e. their images in $\overline{\mathcal{M}}^{\mathbf{a}}$ under $B_{\mathbf{a}}$) are isomorphic.

Definition 1.36 (Alexander duality) *Given $M \in \mathcal{M}_+^{\mathbf{a}}$, define the* Alexander dual

$$M^{\mathbf{a}} \quad = \quad P_{\mathbf{a}}\Big((B_{\mathbf{a}} M)^\vee[-\mathbf{a}]\Big) \quad \cong \quad (\check{C} B_{\mathbf{a}} M)^\vee[-\mathbf{a}]$$

of M with respect to $\mathbf{a}$, where $(-)^\vee$ is the Matlis dual in $\mathcal{M}$, as in Section 1.3.

Equivalently, by applying $B_{\mathbf{a}}$ to the first equality, we find:

Lemma 1.37 *$M^{\mathbf{a}}$ is the positively $\mathbf{a}$-determined module satisfying $B_{\mathbf{a}}(M^{\mathbf{a}}) \cong (B_{\mathbf{a}}M)^{\vee}[-\mathbf{a}]$.*

Remark 1.38 We will never have occasion to take an Alexander dual of the ideal $\mathfrak{m}$, so $\mathfrak{m}^{\mathbf{b}}$ will always denote an irreducible ideal, not the Alexander dual of the module $\mathfrak{m}$ with respect to $\mathbf{b}$.

We can now complete the proof of Theorem 1.35 by verifying that the functor in Definition 1.36 really deserves to be called an Alexander duality functor.

Proposition 1.39 *The functor $(-)^{\mathbf{a}} : \mathcal{M}_+^{\mathbf{a}} \to \mathcal{M}_+^{\mathbf{a}}$ satisfies the conditions of Theorem 1.35.*

Proof: The exactness, k-linearity, and contravariance are obvious. The action on principal ideals follows from Lemma 1.37 along with Examples 1.31.4 and 1.31.2. □

It is likely that the exactness property in Theorem 1.35 follows from the others, since $\mathcal{M}_+^{\mathbf{a}}$ is equivalent to the category $\overline{\mathcal{M}}^{\mathbf{a}}$ consisting of finite-length modules (Theorem 1.34). In any case, Theorem 1.35 essentially says that there is an equivalence of $\mathcal{M}_+^{\mathbf{a}}$ with its opposite category $(\mathcal{M}_+^{\mathbf{a}})^{\mathrm{op}}$ whose square is the identity, and it may seem that specifying $A_{\mathbf{a}}$ on inclusions of principal ideals is unnecessary. This is not the case, however, because there are equivalences $\mathcal{M}_+^{\mathbf{a}}$ to itself which are not isomorphic to the identity functor. For example, any permutation of the indices $\{1, \ldots, n\}$ has this property. Thus, the Alexander duality of Definition 1.36 followed by a permutation of order two of $\{1, \ldots, n\}$ satisfies every part of Theorem 1.35 except for the action on inclusions. (Are there others?) The action on inclusions works as substitute for the S-linearity in the usual definition of dualizing functor. This is necessary because $\underline{\mathrm{Hom}}_S(-,-)$ doesn't make sense in any of the four equivalent categories of Theorem 1.34: it tends to take values outside the desired categories.

It is worth bearing in mind that Alexander duality would be much simpler if one had no preference for finitely generated modules over other kinds (such as artinian ones). Indeed, Alexander duality for all of $\mathcal{M}$ should really be just Matlis duality. However, the interest in finitely generated modules requires an interaction of Matlis duality with the boundary of the positive orthant $\mathbb{N}^n$. In particular, free modules become preferable to arbitrary flat modules. The extension functors of Definition 1.30 are simply the means for dealing with this boundary problem.

Chapter 2

Monomial matrices

A significant theme of this dissertation is that injective resolutions of $\mathbb{Z}^n$-graded modules are not only tractable, but just as easy to write down as free resolutions. This is important because injective resolutions contain much more information than free resolutions (see Corollary 3.29 and Section 4.1, for instance). On the other hand, complexes of flat modules appear naturally in Section 6.1 as generalizations of the Čech complex. Hence this chapter provides the foundations for working with maps and complexes of injective and flat modules in $\mathcal{M}$, including a handy new *monomial matrix* notation (Section 2.1).

One of the recent developments in the study of monomial ideals is the introduction of geometrically defined resolutions. These associate to each of finitely many syzygies a face in some cell complex which may be defined topologically, combinatorially, or through convex geometry. The important classes of *cellular* and *cocellular* monomial matrices are therefore defined in Section 2.3. They provide many of the examples of the more abstract functorial theorems in subsequent chapters, and are themselves the focus of Chapter 5.

2.1 Motivation and definition

The standard notion of matrix for maps of $\mathbb{Z}^n$-graded free modules is a rectangular array whose (p,q)-entry is of the form $\lambda_{pq}\mathbf{x}^{\mathbf{b}_{pq}}$. But the array does not determine the map uniquely; we need also to keep track of the degrees of the generators of the summands in the source and the target. To do this, we use instead a bordermatrix (the TEX command used to produce the arrays below) with each column labelled by the degree of the corresponding source summand, and each row labelled by the degree of the corresponding target summand. Of course, now that we are keeping track of the degrees of summands in the source and target, we can replace the monomial entry $\lambda_{pq}\mathbf{x}^{\mathbf{b}_{pq}}$ by the scalar λ_{pq}, since $\mathbf{b}_{pq}$ is forced to make up the difference between the corresponding column and row degree labels. For instance, the right-hand 1×1 bordermatrix in Equation (2.1) represents the map $S[-(3,2,8)] \to S[-(1,1,8)]$ that is 2 times the canonical inclusion. Recall the partial order $\preceq$ on $\mathbb{Z}^n$, in which $\mathbf{b} \preceq \mathbf{b}'$ if $b_i \leq b'_i$ for all i, and observe that in order for λ_{pq} to be nonzero, it must be that $\mathbf{b}_{p\cdot} \preceq \mathbf{b}_{\cdot q}$. Here, the subscript on $\mathbf{b}_{\cdot q}$ (for instance) indicates that it labels the q^{th} column, whose entries are indexed by replacing the dot with a number.

The goal here is to modify this notation enough to make it work for maps between

$$\begin{array}{c} \\ \mathbf{b}_{1\cdot} \\ \vdots \\ \mathbf{b}_{p\cdot} \\ \vdots \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{b}_{\cdot 1} & \cdots & \mathbf{b}_{\cdot q} & \cdots \end{array} \\ \left(\begin{array}{ccc} & & \\ & & \\ & & \\ & \lambda_{pq} & \\ & & \\ & & \end{array}\right) \end{array} \qquad \begin{array}{cc} & \scriptstyle (3,2,8) \\ \scriptstyle (1,1,8) & \left(\begin{array}{c} 2 \end{array}\right) \end{array} \tag{2.1}$$

$\mathbb{Z}^n$-graded flat or injective modules. It will be shown in Section 2.2 that it is enough to deal with $\mathbb{Z}^n$-shifts of $\mathbb{Z}^n$-graded localizations $S[\mathbf{x}^{-\overline{F}}]$. But what should be the standard way to represent such a localization? The problem is that the multiplication map $\cdot x_i : S[\mathbf{x}^{-\overline{F}}] \to S[\mathbf{x}^{-\overline{F}}][\mathbf{e}_i]$ is an isomorphism *of degree* $\mathbf{0}$ whenever $i \notin F$. The 1×1 bordermatrix for such a map should really have the same label on its row and column, with the single scalar entry being $1 \in k$. This problem is aggravated for maps between shifts of different localizations. To alleviate it, shifts of flat modules are denoted by vectors $\mathbf{b} \in \mathbb{Z}^n_* := (\mathbb{Z} \cup \{*\})^n$, as follows.

The $*$ is meant to represent an arbitrary integer. For purposes of order and addition, the rules governing $*$ treat it like $-\infty$, except that multiplication by -1 leaves $*$ unchanged:

$$* + b \;=\; * \quad \text{and} \quad * \;<\; b \quad \text{for all } b \in \mathbb{Z}\,, \quad \text{while} \quad -1 \cdot * \;=\; *\,. \tag{2.2}$$

Thus $\mathbf{b} \in \mathbb{Z}^n_*$ may be expressed as $\mathbf{b} = \mathbf{b}_{\mathbb{Z}} + \mathbf{b}_*$, where $\mathbf{b}_{\mathbb{Z}} \in \mathbb{Z}^n$ is obtained from $\mathbf{b}$ by setting $*$ to zero, and $\mathbf{b}_*$ is obtained by setting all integers to zero. If the set of indices where $\mathbf{b}_*$ has a $*$ is $F \subseteq \{1, \ldots, n\}$, then we write $\mathbf{b}_* = *F$ and $\overline{\mathbf{b}}_* = *\overline{F}$, and consider them also as subsets of $\{1, \ldots, n\}$. This enables our conventions for shifts of localizations.

Convention 2.1 (Shifts of localizations)

1. *If $S[\mathbf{x}^{-\overline{F}}][\mathbf{b}]$ is written, it is required that $\mathbf{b}_* = *\overline{F}$. In other words, $\mathbf{b}$ has a $*$ in the i^{th} place if and only if $\cdot x_i$ is an automorphism (of degree $\mathbf{e}_i$) of $S[\mathbf{x}^{-\overline{F}}]$. If no shift is explicitly written, then it is assumed $\mathbf{b} = \mathbf{b}_* = *\overline{F}$.*
2. *If $\mathbf{c} \in \mathbb{Z}^n$, then $S[\mathbf{x}^{-\overline{F}}][\mathbf{b}][\mathbf{c}] = S[\mathbf{x}^{-\overline{F}}][\mathbf{b} + \mathbf{c}]$. Observe that $(\mathbf{b} + \mathbf{c})_* = \mathbf{b}_*$ here. In particular, the last sentence of part 1 means that it is allowable to write $S[\mathbf{x}^{-\overline{F}}][\mathbf{c}]$ to mean $S[\mathbf{x}^{-\overline{F}}][\mathbf{c} + *\overline{F}]$, as long as $\mathbf{c} \in \mathbb{Z}^n$.*

The reason for choosing these conventions is because of the next result.

Lemma 2.2 *There exists a nonzero map $\varphi : S[\mathbf{x}^{-\overline{F}'}][-\mathbf{b}'] \to S[\mathbf{x}^{-\overline{F}}][-\mathbf{b}]$ in $\mathcal{M}$ between shifts of localizations of S if and only if $\mathbf{b} \preceq \mathbf{b}'$. If there is a nonzero map, then*

$$\operatorname{Hom}_{\mathcal{M}}(\, S[\mathbf{x}^{-\overline{F}'}][-\mathbf{b}']\,,\, S[\mathbf{x}^{-\overline{F}}][-\mathbf{b}]\,) \cong k\,.$$

Note the relation between $\mathbf{b}$ and F (also $\mathbf{b}'$ and F') implicitly assumed by Convention 2.1.

Proof: Each variable that has been inverted in the source must also be inverted in the target; i.e. $\mathbf{b}_* \supseteq \mathbf{b}'_*$. In addition, since every such nonzero map must be injective, every integer entry of $\mathbf{b}$ must be $\leq$ the corresponding integer entry of $\mathbf{b}'$, or else there will be no place in the target to send the element 1 of degree $\mathbf{b}'_{\mathbb{Z}}$ from the source. On the other hand, there may be an index i such that $b_i = *$ and $b'_i \in \mathbb{Z}$. Using the rules in (2.2) and putting these conditions together amounts to simply $\mathbf{b} \preceq \mathbf{b}'$. The last statement holds because any nonzero map is determined by the image of 1. □

Definition 2.3 *A* monomial matrix *is a bordermatrix* Λ *as in the left-hand side of Equation 2.1 such that* $\mathbf{b}_{p\cdot} \preceq \mathbf{b}_{\cdot q}$ *whenever* $\lambda_{pq} \neq 0$.

The vectors $\mathbf{b}_{p\cdot}$ and $\mathbf{b}_{\cdot q}$ are called the *row* and *column labels of* Λ, respectively, while $(\mathbf{b}_{p\cdot})_*$ and $(\mathbf{b}_{\cdot q})_*$ are the *row* and *column* $*$*-vectors of* Λ. The λ_{pq} are called *scalar entries.* Λ is $\mathbb{Z}^n$*-finite* if for each $\mathbf{c} \in \mathbb{Z}^n$ there are only finitely many labels of Λ that are $\preceq \mathbf{c}$.

Of course, there is the usual notion of *submatrix,* obtained by choosing some of the p and some of the q. The 1×1 monomial submatrix Λ_{pq} is called the (p, q)-component of Λ. A *homologically graded matrix* Λ is a sequence $\{\Lambda^d\}$ with $\mathbf{b}^{d-1}_{\cdot q} = \mathbf{b}^d_{p\cdot}$ if $p = q$. Matrices below may be homologically graded, although the homological indexing is usually suppressed.

Example 2.4 The left bordermatrix in Equation (2.3) is a monomial matrix over $k[x, y, z]$, while the right bordermatrix is not; given the labels on the right bordermatrix, all of the

$$\begin{array}{c} \\ (*,-4,*) \\ (*,*,-3) \\ (-2,*,9) \end{array} \!\!\begin{array}{c} \begin{array}{ccc} (-7,*,-3) & (0,*,17) & (*,3,-1) \end{array} \\ \left(\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 2 & 5 \\ 0 & -1 & 0 \end{array}\right) \end{array} \qquad \begin{array}{c} \\ (*,*,5) \\ (8,*,0) \end{array} \!\!\begin{array}{c} \begin{array}{ccc} (*,*,*) & (-2,*,4) & (9,*,*) \end{array} \\ \left(\begin{array}{ccc} 1 & -1 & 3 \\ -2 & -1 & 1 \end{array}\right) \end{array} \tag{2.3}$$

scalar entries should be zero. Given the labels on the left bordermatrix, all of the zeros are forced by the condition for a monomial matrix. □

Example 2.5 Here is a more substantive example. The Koszul complex on $x, y \in k[x, y]$ is

$$0 \leftarrow S \xleftarrow{\;(0,0)\begin{array}{c}\begin{array}{cc}(1,0) & (0,1)\end{array}\\ \left(\begin{array}{cc}1 & 1\end{array}\right)\end{array}\;} S[(-1,0)] \oplus S[(0,-1)] \xleftarrow{\;\begin{array}{c}(1,0)\\(0,1)\end{array}\begin{array}{c}(1,1)\\ \left(\begin{array}{c}-1\\1\end{array}\right)\end{array}\;} S[(-1,-1)] \leftarrow 0\,.$$

If all of the zeros in the labels of the monomial matrix are changed to $*$ and subsequently $(-1, -1)$ is added to every label, the result is the Čech complex on x, y:

$$0 \leftarrow S[x^{-1}, y^{-1}] \xleftarrow{\;(*,*)\begin{array}{c}\begin{array}{cc}(0,*) & (*,0)\end{array}\\ \left(\begin{array}{cc}1 & 1\end{array}\right)\end{array}\;} S[y^{-1}] \oplus S[x^{-1}] \xleftarrow{\;\begin{array}{c}(0,*)\\(*,0)\end{array}\begin{array}{c}(0,0)\\ \left(\begin{array}{c}-1\\1\end{array}\right)\end{array}\;} S \leftarrow 0\,.$$

It will be seen in Lemma 3.14 that changing a zero to a $*$ is taking the Čech hull. Thus, up to a shift by **1**, *the Čech complex is the Čech hull of the Koszul complex.* This transition is generalized to squarefree monomial ideals other than $\mathfrak{m}$ in Theorem 6.7. □

2.2 Injective modules and complexes

Monomial matrices can be used to represent any map of $\mathbb{Z}^n$-finite injective or flat modules (see the material surrounding Lemma 1.21 for definitions). As the first step toward proving this, here is a description of all injective and $\mathbb{Z}^n$-finite flat modules in the category $\mathcal{M}$ of $\mathbb{Z}^n$-graded S-modules.

Lemma 2.6 *Every injective module in $\mathcal{M}$ is a direct sum of $\mathbb{Z}^n$-graded shifts of* indecomposable injectives $\underline{E}(S/\mathfrak{m}^F)$. *Every $\mathbb{Z}^n$-finite flat module in $\mathcal{M}$ is a direct sum of shifts of* indecomposable flat modules $S[\mathbf{x}^{-\overline{F}}]$.

Proof: The statement about injectives is [GW78, Theorem 1.3.3]. It has already been seen in Section 1.3 that $J \in \mathcal{M}$ is flat if and only if the Matlis dual $J^\vee = \underline{\mathrm{Hom}}_k(J, k)$ of J is injective; therefore the second statement will follow from the first if Matlis duality commutes with direct sums of modules $M_\alpha \in \mathcal{M}$ whenever the direct sum is $\mathbb{Z}^n$-finite. Precisely,

Claim 2.7 *If $M = \bigoplus_\alpha M_\alpha$ is $\mathbb{Z}^n$-finite, then the map $\bigoplus_\alpha (M_\alpha)^\vee \to M^\vee$ is an isomorphism.*

Proof: The map is obviously injective. Furthermore, given $\mathbf{b} \in \mathbb{Z}^n$, the direct sum $M_{[-\mathbf{b}]}$ of all M_α that are nonzero in degree $-\mathbf{b}$ is finite. Taking the Matlis dual of the isomorphism $M \cong M_{[-\mathbf{b}]} \oplus M/M_{[-\mathbf{b}]}$ shows that the summand $(M_{[-\mathbf{b}]})^\vee$ of $\bigoplus_\alpha (M_\alpha)^\vee$ surjects onto $(M^\vee)_\mathbf{b}$. Since $\mathbf{b}$ is arbitrary, the proof of the Claim (and the Lemma) is complete. □

Up until now, the notation in this chapter has been geared only towards flat modules, and not injectives. However, $\cdot x_i$ is an automorphism (of degree $\mathbf{e}_i$) of $\underline{E}(S/\mathfrak{m}^F)$ if and only if it is an automorphism of $S[\mathbf{x}^{-\overline{F}}] \cong \underline{E}(S/\mathfrak{m}^F)^\vee$.

Convention 2.8 (Shifts of injectives) *Convention 2.1 works just as well with $\underline{E}(S/\mathfrak{m}^F)$ in place of $S[\mathbf{x}^{-\overline{F}}]$.*

The next theorem is the main result of the chapter. For notation, $S[\mathbf{x}^{-\overline{F}}][-\mathbf{b}]$ is said to be *generated in degree* $\mathbf{b}$ by the *generator* 1. Similarly, $\underline{E}(S/\mathfrak{m}^F)[\mathbf{b}]$ is *cogenerated in degree* $-\mathbf{b}$ by the *cogenerator* 1. A generator of an irreducible flat S-module is really an honest basis element for the same module over the appropriate localization of S, just as a cogenerator for an irreducible injective module is an honest socle element over the appropriate localization.

Theorem 2.9 *To give a monomial matrix Λ is equivalent to giving either*

1. *a map $\lambda : N' \to N$ of flat modules along with decompositions into irreducibles*

$$N' = \bigoplus_q S[\mathbf{x}^{-\overline{F}_{\cdot q}}][-\mathbf{b}_{\cdot q}] \quad \text{and} \quad N = \bigoplus_p S[\mathbf{x}^{-\overline{F}_{p\cdot}}][-\mathbf{b}_{p\cdot}]$$

as well as generators for the summands; or

2. a map $\lambda : N \to N'$ of injective modules along with decompositions into irreducibles

$$N = \bigoplus_p \underline{E}(S/\mathfrak{m}^{F_{p\cdot}})[\mathbf{b}_{p\cdot}] \quad \text{and} \quad N' = \bigoplus_q \underline{E}(S/\mathfrak{m}^{F_{\cdot q}})[\mathbf{b}_{\cdot q}]$$

as well as cogenerators for the summands.

In either case, Λ is said to represent λ. *Every map of injective modules can be represented by some monomial matrix Λ, as can every map of $\mathbb{Z}^n$-finite flat modules. A homologically graded matrix can be construed as a homological complex of flat modules* or *a cohomological complex of injective modules.*

Proof: Lemma 2.2 and its Matlis dual, along with Lemma 2.6. □

If $\{\Lambda^d\}$ is a matrix representing a homological complex $\mathbb{F}_{\bullet}$ of flat modules with differential $\{\lambda_d\}$ and a cohomological complex $\mathbb{I}^{\bullet}$ of injective modules with differential $\{\lambda^d\}$, the conventions for homologically graded monomial matrices force the conventions

$$\cdots \leftarrow F_{d-1} \stackrel{\lambda_d}{\leftarrow} F_d \leftarrow \cdots \qquad \text{and} \qquad \cdots \to I^{d-1} \stackrel{\lambda^d}{\to} I^d \to \cdots$$

because $\mathbf{b}^d_{q\cdot}$ represents a label in F_d and I^d. It should be emphasized what the relation is between $\mathbb{F}_{\bullet}$ and $\mathbb{I}^{\bullet}$ if they are represented by the same monomial matrix.

Corollary 2.10 *Let λ be any map of $\mathbb{Z}^n$-finite flat or injective modules. Given a matrix Λ for λ, the matrix for $\lambda^\vee$ induced by Matlis duality is the same matrix Λ. [N.B. The homological grading is left unchanged.]*

Proof: Use the isomorphism $\underline{E}(S/\mathfrak{m}^F)[\mathbf{b}]^\vee = S[\mathbf{x}^{-\overline{F}}][-\mathbf{b}]$ and Theorem 2.9. □

Remark 2.11 A map of modules represented by a monomial matrix is also a map of ungraded S-modules. In order to get the resulting matrix for the ungraded map, each scalar entry λ_{pq} needs to be multiplied by $\mathbf{x}^{\mathbf{b}_{pq}}$, where $\mathbf{b}_{pq} = (\mathbf{b}_{\cdot q})_{\mathbb{Z}} - (\mathbf{b}_{p\cdot})_{\mathbb{Z}}$.

2.3 Cellular monomial matrices

This section outlines a class of monomial matrices that generalize the construction in [BS98, Section 1] of *cellular free complexes* from labelled cellular complexes. Specifically, we define the analogous notions of *cellular* and *cocellular flat complexes* as well as *cellular* and *cocellular injective complexes.* Instances of these include the Koszul and Čech complexes in Example 2.5 as well as their more general relatives in Example 6.5, below. Other interesting examples are furnished by the Scarf complexes in Example 2.15, which will be used systematically as test cases for the duality theorems of Section 3.4. More generally, the geometric interpretation of resolutions afforded by cellular and cocellular monomial matrices interacts nicely with the duality theorems in Section 3.4, providing a connection with duality in algebraic topology. A detailed study of these phenomena is postponed until Chapter 5.

We now recall the relevant topological notions, extracting the exposition from [BS98, Section 1]. See also [BH93, Section 6.2] for more details on regular cell complexes.

Let X be a *regular cell complex* with vertex set V; in the examples here, X will be either a simplicial complex or a set of faces of a polytope. X is assumed to come equipped with an *incidence function* $\varepsilon(G, G')$, defined on pairs of cells in X and taking values in $\{-1, 0, 1\}$, whose purpose is to make a complex out the vector space spanned by the faces of X (see [BH93, Section 6.2] for details). The value of $\varepsilon(G, G')$ is nonzero if and only if G' is a facet of G. The empty set $\varnothing$ is always a face of X, and is a facet of each vertex.

The differential ∂ on the *augmented oriented chain complex* $\widetilde{\mathcal{C}}_\bullet(X; k) = \bigoplus_{G \in X} kG$ of X with coefficients in k is defined on $G \in X$ by $\partial\, G = \sum_{G' \in X} \varepsilon(G, G')\, G'$. The complex $\widetilde{\mathcal{C}}_\bullet(X; k)$ is homologically graded by dimension, with the empty set in homological degree -1. The k-dual $\widetilde{\mathcal{C}}^\bullet(X; k) := \mathrm{Hom}_k(\widetilde{\mathcal{C}}_\bullet(X; k), k)$ is called the *augmented oriented cochain complex* of X, and is to be regarded as a homologically graded complex. The homological grading of $\widetilde{\mathcal{C}}^\bullet(X; k)$ has the empty set in homological degree 1, vertices in homological degree 0, edges in homological degree -1, … , and facets in homological degree $-\dim X$.

Definition 2.12 *A monomial matrix Λ whose scalar entries constitute $\widetilde{\mathcal{C}}_\bullet(X; k)$ up to a homological shift for some regular cell complex X is called a* cellular monomial matrix supported on X. *Similarly, if the scalar entries are a homological shift of $\widetilde{\mathcal{C}}^\bullet(X; k)$ then Λ is called* cocellular.

In this section and Chapter 5, the most common (but by no means the only) homological shifts are (-1) for cellular monomial matrices (putting the empty set in homological degree 0), and $(-\dim X)$ for cocellular monomial matrices (putting the empty set in homological degree $1 + \dim X$ and the facets in homological degree 0).

One should think of the labels in the monomial matrix as being (not necessarily distinct) labels on the faces of X (see Figure 2.1). Given such a *labelled cell complex* X, we say that X *determines* a cellular (or cocellular) monomial matrix if the corresponding bordermatrix for $\widetilde{\mathcal{C}}_\bullet(X; k)$ (or $\widetilde{\mathcal{C}}^\bullet(X; k)$) is a monomial matrix. The next proposition says what it means to determine a (co)cellular monomial matrix, in terms of the labelling. For a face G in a labelled regular cell complex X, let $\mathbf{a}_G$ denote the label on G.

Proposition 2.13 *Let X be a labelled regular cell complex. Then X determines a cellular monomial matrix if and only if $\mathbf{a}_{G'} \preceq \mathbf{a}_G$ for all faces $G' \subseteq G \in X$; and X determines a cocellular monomial matrix if and only if $\mathbf{a}_{G'} \succeq \mathbf{a}_G$ for all faces $G' \subseteq G \in X$.*

Proof: By transitivity of $\preceq$ it is enough to check when G' is a facet of G. In this case, the nonvanishing of $\varepsilon(G, G')$ implies the result. □

Example 2.14 In Figure 2.1, the same simplicial complex determines two different cocellular monomial matrices. For clarity, the notation $\bar{1}{*}1$ is used as a shorthand for the vector $(-1, *, 1) \in \mathbb{Z}^3_*$. □

By Theorem 2.9, a cellular or cocellular monomial matrix can represent a complex of either flat or injective modules, and we say that such a complex is a (co)cellular flat or (co)cellular injective complex, as the case may be. With shifts in homological gradings and Matlis duality involved, it can often be confusing to determine when a flat or injective complex is cellular rather than cocellular. The rule of thumb is:

the flat version goes the right way.

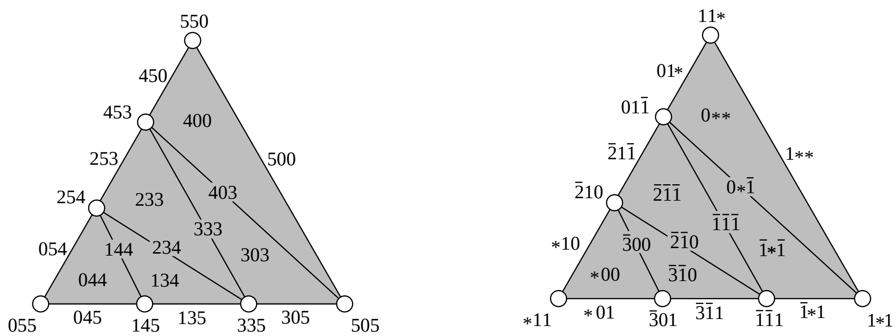
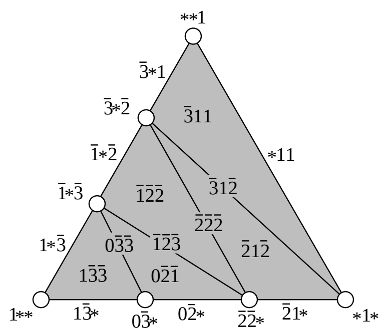


Figure 2.1: Two cocellular monomial matrices supported on the same simplicial complex

* * * * * * * * *

Notation: $\bar{2}1* = (-2, 1, *)$

injective resolution of $M^{444} = J^{[444]}$ ⟶



$$
\begin{array}{c}
 \quad *1*\ \ \bar{2}\bar{2}*\ \ 0\bar{3}*\ \ 1{*}{*}\ \ \bar{1}*\bar{3}\ \ \bar{3}*\bar{2}\ \ {*}{*}1 \\
{\scriptstyle *** \atop 000}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \\
 \quad 050\ \ 220\ \ 410\ \ 500\ \ 301\ \ 102\ \ 005 \\
 \qquad\ \ \mathrm{x}\quad\ \mathrm{x}\qquad\qquad\ \mathrm{x}\quad\ \mathrm{x}
\end{array}
$$

$$
\begin{array}{ccccccccccc}
*11 & \bar{3}*1 & \bar{3}1\bar{2} & \bar{2}1* & \bar{2}\bar{2}\bar{2} & \bar{1}*\bar{2} & \bar{1}\bar{2}\bar{3} & 0\bar{2}* & 0\bar{3}\bar{3} & 1*\bar{3} & 1\bar{3}*
\end{array}
$$
$$
\begin{pmatrix}
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & -1 & 0 \\
0 & 1 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$
$$
\begin{array}{ccccccccccc}
055 & 105 & 152 & 250 & 222 & 302 & 321 & 420 & 411 & 501 & 510 \\
 & & & & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & &
\end{array}
$$

$$
\begin{array}{ccccc}
\bar{3}11 & \bar{2}1\bar{2} & \bar{1}\bar{2}\bar{2} & 0\bar{2}\bar{1} & 1\bar{3}\bar{3}
\end{array}
$$
$$
\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$
$$
\begin{array}{ccccc}
155 & 252 & 322 & 421 & 511 \\
 & & \mathrm{x} & \mathrm{x} &
\end{array}
$$

⟵ free resolution of $B_{444}(M)$

$M = S/J = S/\langle x^2y^2, x^4y, x^3z, xz^2 \rangle$

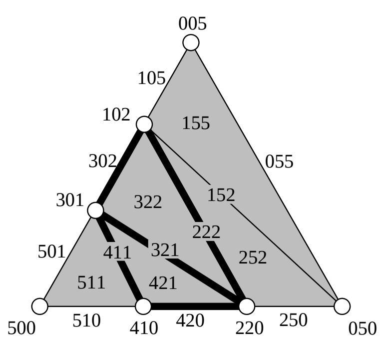


Figure 2.2: Two cellular monomial matrices supported on the complex $X = \Delta_{J+\langle x^5,y^5,z^5 \rangle}$

For instance, the differential of a cellular flat complex goes in the same direction as $\widetilde{\mathcal{C}}_\bullet(X;k)$, while the differential of a cellular injective complex goes in the same direction as $\widetilde{\mathcal{C}}^\bullet(X;k)$. In Figure 2.2, for example, the scalar matrices give the boundary complex of the (unlabelled) simplicial complex depicted there. Both labellings of the simplicial complex determine cellular monomial matrices, with the homological grading increasing to the right: the empty set is labelled by the row label on the leftmost bordermatrix, and the facets are labelled by the column labels of the rightmost bordermatrix. (For clarity, the two sets of column labels are depicted above and below the scalar matrices. The row labels have been left off of all but the first bordermatrix, but the definition of homologically graded monomial matrix says that the row labels equal the column labels on the next bordermatrix to the left.)

The reason why (co)cellular flat and injective complexes are interesting is because sometimes they are resolutions of interesting modules; that is, they are acyclic. (See Example 6.5, though, for an important and generally non-acyclic cocellular flat complex). Historically, the first algebraic complexes to admit descriptions by cellular means were undoubtedly the Koszul and Čech complexes on $x_1, \ldots, x_n$, which are resolutions of k and $\underline{E}(k)$, respectively. Both are supported on a simplex with vertices $x_1, \ldots, x_n$. The *Taylor complex* [Tay61] of a monomial ideal $J = \langle m_1, \ldots, m_r \rangle$ generalizes the Koszul complex for free resolutions, and is supported on a simplex spanned by $m_1, \ldots, m_r$, although still the cellular structure was not explicit at the time of its invention.

Example 2.15 The insight of [BPS98] was to examine a subcomplex of the Taylor complex, called the *Scarf complex* Δ_I, which is defined as follows. Let $m_\sigma = \mathrm{lcm}(m \in \sigma)$ for any subset $\sigma \subseteq \{m_1, \ldots, m_r\}$. Then

$$\Delta_I := \{\sigma \subseteq \{m_1, \ldots, m_r\} \mid \text{if } m_\sigma = m_\tau \text{ for some } \tau \subseteq \{m_1, \ldots, m_r\} \text{ then } \tau = \sigma\}$$

consists of the subsets whose least common multiples are uniquely attained. Each face $\sigma \in \Delta_I$ is labelled by the exponent on m_σ. The point of this definition is that for *generic* monomial ideals (Definition 1.17), the monomial matrix determined by Δ_I represents a free resolution of S/I. This is a theorem originally presented in [BPS98] for strongly generic ideals and later generalized to all generic ideals in [MSY00]. It will be treated more formally and at much greater length in chapter 5.

For a concrete example, the Scarf complex of the generic monomial ideal $J = \langle x^2y^2, x^4y, x^3z, xz^2 \rangle$ is the thickened subdivided quadrilateral (including the two triangles) in the bottom simplicial complex of Figure 2.2. The monomial matrix it determines is the submatrix of the big matrix whose labels are marked with an "x", and it represents a free resolution of S/J. In fact, the entire bottom simplicial complex is the Scarf complex of $J + \langle x^5, y^5, z^5 \rangle$, so it represents a free resolution of $S/(J + \langle x^5, y^5, z^5 \rangle) = B_{(4,4,4)}(S/J)$. □

Remark 2.16 The definition of cellular free complex in [BS98] is a little more restrictive than what Definition 2.12 allows for complexes of free modules. The difference involves the notion of *join* $\vee$ from Section 1.5, which works just as well on the poset $\mathbb{Z}^n_*$. Bayer and Sturmfels require not just that $\mathbf{a}_{G'} \preceq \mathbf{a}_G$ for all faces $G' \subseteq G \in X$, but that the label $\mathbf{a}_G$ on each face G equals the join $\bigvee_{v \in G} \mathbf{a}_v$ of the labels on all vertices $v \in G$. This condition is satisfied in most of the natural examples anyway, and will be revisited in Definition 5.1.

2.4 Minimality

In considering free resolutions of finitely generated graded modules, we are frequently interested in the minimal resolution. In the usual (nonmonomial) notation, the matrices in such a resolution are characterized by having their entries in the maximal ideal. This condition can be restated so as to work for flat modules that aren't free, and for injective modules.

Definition 2.17 *The matrix Λ is called* minimal *if no component Λ_{pq} represents an isomorphism. Equivalently, Λ is minimal if $\lambda_{pq} = 0$ whenever $\mathbf{b}_{p\cdot} = \mathbf{b}_{\cdot q}$. A map λ of flat or injective modules is called* minimal *if there is a minimal matrix for λ.*

Example 2.18 The Scarf complex of any monomial ideal is minimal because its face labels are distinct by definition. In particular, the bottom simplicial complex of Figure 2.2 determines a minimal cellular monomial matrix by Example 2.15. The reader may check that, in fact, all of the face labels in all four of the labelled triangulations of Figures 2.1 and 2.2 have distinct face labels and thus determine minimal injective and flat complexes. This (and much more) will be explained by Theorem 3.23, Corollary 3.27 and Example 3.26. □

Proposition 2.19 *Let λ be a morphism of $\mathbb{Z}^n$-finite flat or injective modules in $\mathcal{M}$. Minimality of λ is independent of the matrix chosen to verify it. Furthermore, λ is minimal if and only if the Matlis dual $\lambda^\vee$ is minimal.*

Proof: It is enough to prove the first statement when the modules are injective and the second statement for any particular matrix Λ representing λ. Assuming, then, that Λ represents the map $\lambda : N \to N'$ of injective modules, we show that Λ is minimal if and only if for every $F \subseteq \{1, \ldots, n\}$ the induced map

$$\lambda^F : \underline{\mathrm{Hom}}_S(S/\mathfrak{m}^F, N) \otimes S[\mathbf{x}^{-\overline{F}}] \quad \to \quad \underline{\mathrm{Hom}}_S(S/\mathfrak{m}^F, N') \otimes S[\mathbf{x}^{-\overline{F}}]$$

is zero. Given F, the localization kills any summands of N and N' that are shifts of $\underline{E}(S/\mathfrak{m}^{F'})$ for $F' \not\subseteq F$; and $\underline{\mathrm{Hom}}_S(S/\mathfrak{m}^F, -)$ depends only on the summands of N and N' that are shifts of $S/\mathfrak{m}^{F'}$ for $F' \supseteq F$. Therefore we are reduced to the case where all summands of both N and N' are shifts of $\underline{E}(S/\mathfrak{m}^F)$—that is, the case where all of the $*$-vectors in Λ are equal to $*\overline{F}$. With this assumption, λ^F is just the map between the socles of $N \otimes S[\mathbf{x}^{-\overline{F}}]$ and $N' \otimes S[\mathbf{x}^{-\overline{F}}]$. Now a homomorphism of indecomposable injectives of the same support is nonzero on the socle if and only if it is an isomorphism (injective hull is an essential extension and every injection splits). Applying this to each component of Λ proves the first statement. The second statement follows from Corollary 2.10. □

Example 2.20 The homologically graded monomial matrix for the Čech complex in Example 2.5 represents a minimal injective resolution of the canonical module $S[-(1,1)]$. That is, using the same monomial matrices, $k[x,y][(-1,-1)]$ is the kernel of the first map in the sequence

$$0 \longrightarrow k[x^{\pm 1}, y^{\pm 1}] \longrightarrow \underline{E}(k[x,y]/\langle x \rangle) \oplus \underline{E}(k[x,y]/\langle y \rangle) \longrightarrow \underline{E}(k) \longrightarrow 0\,. \qquad \square$$

Heuristically, if elements in flat modules act like column vectors, then elements of injective modules act like row vectors. This phenomenon can be seen explicitly in Figure 2.2, where the labels on top of the scalar matrix determine an injective resolution of $J^{[444]}$, while the labels on the bottom determine a free resolution of $B_{444}(S/J)$ going the other way.

Chapter 3

Duality for resolutions

This chapter melds the themes of the previous two chapters, by showing how functorial Alexander duality applies to resolutions, and demonstrating the duality on monomial matrices. The main results here, Theorem 3.23, Corollary 3.27, Corollary 3.29, and Theorem 3.30, form the technical heart of this dissertation. Their proofs involve a connection with local duality (Section 3.2), and provide effective use of monomial matrices (Section 3.3).

The content of the duality theorems is that resolutions of various modules associated to a given module M and its Alexander dual $M^{\mathbf{a}}$ can be transformed functorially into one another. On one hand, this means that the study of both injective and free resolutions of all finitely generated $\mathbb{Z}^n$-graded modules can be reduced to the study of only the free resolutions of finite-length modules. It may seem, therefore, that injective resolutions carry no more information than do free resolutions. On the other hand, this couldn't be farther from the truth: a minimal injective resolution of M contains, in a suitable sense, all minimal free resolutions of all localizations of M, along with maps between these complexes which aren't reflected in any of the localized free resolutions. Reducing to the artinian case only masks this information, although it also shows how explicit computation of injective resolutions can be accomplished using well-known implemented algorithms for free resolutions.

The value of the extra data from injective resolutions will become clearer in subsequent chapters. In this chapter, the examples illustrate how the duality between links and restrictions in simplicial complexes is a combinatorial reflection of localization and restriction of scalars, both on modules and their geometric resolutions (Example 3.32). Also, the connections with duality in algebraic topology (homology versus cohomology) are sketched (Section 3.5), although this is covered in detail in Chapter 5.

3.1 Generalized Alexander duality

In Section 4 we will want to know the consequences of Alexander duality for free and injective resolutions. Since injective resolutions are never positively $\mathbf{a}$-determined, this will necessarily involve Alexander duality between pairs of the equivalent categories of Definition 1.25 and Theorem 1.34 other than $(\mathcal{M}_+^{\mathbf{a}}, \mathcal{M}_+^{\mathbf{a}})$. Of course, composing the duality for this pair (Theorem 1.35) with the equivalences of categories produces a unique Alexander duality functor between each of the 16 pairs. As is the case with $(-)^{\mathbf{a}}$, the functors which define these dualities are essentially Matlis duality with a shift $[-\mathbf{a}]$. In addition, also as with $(-)^{\mathbf{a}}$, these *generalized Alexander duality functors* are defined on all of $\mathcal{M}$, and become anti-equivalences only when the source and target are restricted to the appropriate subcategories. But the fact that these functors can be extended to all of $\mathcal{M}$ is not a point to be dismissed lightly: with the right choice of extension they can, with care, be fruitfully applied to modules—or complexes—which are not $\mathbf{a}$-determined (positively or negatively) for any $\mathbf{a}$.

It is not difficult to write down explicitly all of the 16 generalized Alexander duality functors, but only the ones that are used in the sequel are given notations here.

Definition 3.1 *Define the functors $(-)_{\succeq \mathbf{b}}$, $A_{\mathbf{a}}^{+,0}$, and $A_{\mathbf{a}}^{0,+}$ from $\mathcal{M} \to \mathcal{M}$ by setting, for $M \in \mathcal{M}$ and $\mathbf{b} \in \mathbb{Z}^n$,*

1. $M_{\succeq \mathbf{b}} = \bigoplus_{\mathbf{c} \succeq \mathbf{b}} M_{\mathbf{c}}$.
2. $A_{\mathbf{a}}^{+,0} M = (\check{C} M)^{\vee}[-\mathbf{a}]$.
3. $A_{\mathbf{a}}^{0,+} M = (M^{\vee}[-\mathbf{a}])_{\succeq \mathbf{0}}$.

Lemma 3.2 1. *The functor $(-)_{\succeq \mathbf{0}}$ induces an equivalence $\mathcal{M}^{\mathbf{a}} \to \mathcal{M}_{+}^{\mathbf{a}}$ for every $\mathbf{a}$.*

2. *When restricted to the appropriate subcategories, $A_{\mathbf{a}}^{+,0}$ gives rise to the generalized Alexander duality functors*

$$A_{\mathbf{a}}^{+,0} : \mathcal{M}_{+}^{\mathbf{a}} \to \mathcal{M}^{\mathbf{a}} \qquad \text{and} \qquad A_{\mathbf{a}}^{+,0} : \overline{\mathcal{M}}^{\mathbf{a}} \to \mathcal{M}_{+}^{\mathbf{a}} .$$

3. *When restricted to the appropriate subcategories, $A_{\mathbf{a}}^{0,+}$ gives rise to the generalized Alexander duality functors*

$$A_{\mathbf{a}}^{0,+} : \mathcal{M}^{\mathbf{a}} \to \mathcal{M}_{+}^{\mathbf{a}} \qquad \text{and} \qquad A_{\mathbf{a}}^{0,+} : \mathcal{M}_{+}^{\mathbf{a}} \to \overline{\mathcal{M}}^{\mathbf{a}} .$$

Proof: Use Table 1.1 in Section 1.5 to check that the restrictions do in fact take the source categories to the correct target categories. To see that these functors are the claimed generalized Alexander duality functors, compose on either side with $B_{\mathbf{a}}$ and verify that the composition is Alexander duality on $\overline{\mathcal{M}}^{\mathbf{a}}$. □

Note that part 1 of the lemma says $(-)_{\succeq \mathbf{0}}$ is in some sense a gentle version of $P_{\mathbf{a}} B_{\mathbf{a}}$; it should be thought of as the adjoint to the Čech hull. The next lemma describes how the functors $P_{\mathbf{a}}$, $\check{C}$, and $(-)_{\succeq \mathbf{b}}$ act on free, flat, and injective modules in $\mathcal{M}$. Its proof is immediate from Lemma 2.6, Lemma 1.32, and the definitions.

Lemma 3.3 *The functors in the first column of the following table*

	injective	free	flat
$P_{\mathbf{a}}$	injective	free	flat
$\check{C}$	injective	flat	flat
$(-)_{\succeq \mathbf{b}}$	finite length	free	free

are exact and alter $\mathbb{Z}^n$-finite free, flat, and injective modules in the indicated manner. Furthermore, $P_{\mathbf{a}}$ is the identity on an indecomposable flat module L if and only if $L_{\mathbf{a}} \cong k$; otherwise, $P_{\mathbf{a}} L = 0$. □

Remark 3.4 One could work with projective and injective objects, and even resolutions, in the categories $\mathcal{M}^{\mathbf{a}}$, $\overline{\mathcal{M}}^{\mathbf{a}}$, $\mathcal{M}^{\mathbf{a}}_{+}$, and $\mathcal{M}^{\mathbf{a}}_{-}$. However, this ignores the boundary effects which give the theory its richness (see the end of Chapter 1). Nevertheless, this point of view is exploited to great advantage in [Yan00b] to show that the Bass numbers of $\mathbf{1}$-determined modules (which Yanagawa calls *straight*) are finite.

3.2 Artinian local duality

In Chapter 6 we want to derive local duality as an easy consequence of duality for resolutions (Corollary 6.25). Therefore, it is desirable to prove duality for resolutions without using local duality. Unfortunately, this is not possible; but the good news is that only a very special case (Proposition 3.5) of local duality is needed. Since there is an elementary proof, it is provided here in order to preserve the self-contained nature of this dissertation.

Proposition 3.5 *Let M be an $\mathbb{N}^n$-graded artinian S-module and $\mathbb{F}_{\bullet}$ any free resolution. Then $H^r\underline{\mathrm{Hom}}_S(\mathbb{F}_{\bullet}, S) = 0$ unless $r = n$, in which case $H^n\underline{\mathrm{Hom}}_S(\mathbb{F}_{\bullet}, S) \cong M^\vee[\mathbf{1}]$. Furthermore, if $\mathbb{F}_{\bullet}$ is minimal then it has length $\leq n$.*

Proof: Any minimal free resolution of any finitely generated graded S-module N has length $\leq n$ because the Betti numbers can be calculated by tensoring N with the minimal free resolution of k, the Koszul complex. This proves the last statement. The cohomology of $\underline{\mathrm{Hom}}_S(\mathbb{F}_{\bullet}, S)$ is independent of which free resolution is chosen because the value is always the right derived functor $\underline{\mathrm{Ext}}^{\bullet}_S(M, S)$. Therefore the cohomology is zero unless $r \leq n$, since we can choose $\mathbb{F}_{\bullet}$ to be minimal.

Since M is $\mathbb{N}^n$-graded and artinian, it is $\mathbf{a}$-bounded for some $\mathbf{a} \in \mathbb{N}^n$ (Definition 1.25). Choosing such an $\mathbf{a}$, M is naturally a module over $S/\mathfrak{m}^{\mathbf{a}+\mathbf{1}}$. By Lemma 3.6 with $N = S$ and $\mathbf{b} = (a_1 + 1, \ldots, a_\ell + 1, 0, \ldots, 0)$, the cohomology $H^r\underline{\mathrm{Hom}}_S(\mathbb{F}_{\bullet}, S) \cong \underline{\mathrm{Hom}}_S(M, S/\mathfrak{m}^{\mathbf{b}})[\mathbf{b}]$ is zero if $r < n$, because then $x_n^{a_n+1}$ annihilates M but is regular on $S/\mathfrak{m}^{\mathbf{b}}$. On the other hand, Lemma 3.6 also says that

$$H^n\underline{\mathrm{Hom}}_S(\mathbb{F}_{\bullet}, S) = \underline{\mathrm{Hom}}_S(M, S/\mathfrak{m}^{\mathbf{a}+\mathbf{1}})[\mathbf{a}+\mathbf{1}].$$

Letting $R = S/\mathfrak{m}^{\mathbf{a}+\mathbf{1}}$, this last module is isomorphic to $\underline{\mathrm{Hom}}_R(M, R)[\mathbf{a}+\mathbf{1}]$, and Lemma 3.7 completes the proof. □

Lemma 3.6 *Suppose that M and N are $\mathbb{Z}^n$-graded S-modules such that $x_1^{b_1}, \ldots, x_\ell^{b_\ell}$ is a regular sequence on N which annihilates M. Then $\underline{\mathrm{Hom}}_S(M, N/\mathfrak{m}^{\mathbf{b}}N)[\mathbf{b}] \cong \underline{\mathrm{Ext}}^\ell_S(M, N)$, where $\mathbf{b} = (b_1, \ldots, b_\ell, 0, \ldots, 0)$.*

Proof (adapted from [BH93, Lemma 1.2.4]): The proof is by induction on ℓ, the case $\ell = 0$ being vacuous. Suppose $\ell \geq 1$, and let $\mathbf{b}' = \mathbf{b} - b_\ell\mathbf{e}_\ell$. Induction implies that $\underline{\mathrm{Ext}}^{\ell-1}_S(M, N) \cong \underline{\mathrm{Hom}}_S(M, N/\mathfrak{m}^{\mathbf{b}'}N)[\mathbf{b}']$, but this last module is zero because $x_\ell^{b_\ell}$ is both regular on $N/\mathfrak{m}^{\mathbf{b}'}N$ and zero on M. Therefore, the exact sequence

$$0 \longrightarrow N[-b_1\mathbf{e}_1] \xrightarrow{\cdot x_1^{b_1}} N \longrightarrow N/x_1^{b_1}N \longrightarrow 0$$

yields another exact sequence

$$0 \longrightarrow \underline{\mathrm{Ext}}_S^{\ell-1}(M, N/x_1^{b_1}N) \xrightarrow{\lambda} \underline{\mathrm{Ext}}_S^{\ell}(M, N[-b_1\mathbf{e}_1]) \xrightarrow{\cdot x_1^{b_1}} \underline{\mathrm{Ext}}_S^{\ell}(M, N)\,.$$

By the induction hypothesis applied to $(M,\ N/x_1^{b_1}N$, and $\mathbf{b}'' = \mathbf{b} - b_1\mathbf{e}_1)$, the domain of λ is isomorphic to $\underline{\mathrm{Hom}}_S(M, N/\mathfrak{m}^{\mathbf{b}}N)[\mathbf{b}'']$. But multiplication by $x_1^{b_1}$ is zero on the target of λ, whence λ is an isomorphism. Thus $\underline{\mathrm{Ext}}_S^{\ell}(M,N) \cong \underline{\mathrm{Hom}}_S(M, N/\mathfrak{m}^{\mathbf{b}}N)[\mathbf{b}'' + b_1\mathbf{e}_1] = \underline{\mathrm{Hom}}_S(M, N/\mathfrak{m}^{\mathbf{b}}N)[\mathbf{b}]$. □

Lemma 3.7 *If M is a $\mathbb{Z}^n$-graded module over $R = S/\mathfrak{m}^{\mathbf{a}+\mathbf{1}}$ then $\underline{\mathrm{Hom}}_R(M, R)[\mathbf{a}] \cong M^\vee$.*

Proof: First note that $R[\mathbf{a}] \cong \underline{\mathrm{Hom}}_k(R, k)$, which can be checked directly. Then

$$\begin{aligned} \underline{\mathrm{Hom}}_R(M,R)[\mathbf{a}] &= \underline{\mathrm{Hom}}_R(M, R[\mathbf{a}]) \\ &\cong \underline{\mathrm{Hom}}_R(M, \underline{\mathrm{Hom}}_k(R,k)) \\ &= \underline{\mathrm{Hom}}_k(M \otimes_R R, k) \\ &= M^\vee, \end{aligned}$$

where all of the equalities are canonical. □

Remark 3.8 Lemma 3.7 reflects functorially what was done in the proof of Corollary 1.4 for ideals. Both revolve around the fact that $S/\mathfrak{m}^{\mathbf{a}+\mathbf{1}}$ has a simple socle: in Corollary 1.4 this allowed us to produce an element in the intersection of a collection of irreducible ideals, and in Lemma 3.7 it allowed the isomorphism of $R[\mathbf{a}]$ with the injective hull of k over R. This highlights the importance of being *Gorenstein*.

Remark 3.9 One might wonder why the degree shift $[\mathbf{b}]$ occurs in Lemma 3.6, since many of the modules in its proof are actually zero. However, the key step for this purpose is the isomorphism between $\underline{\mathrm{Ext}}_S^{\ell}(M,N)$ and $\underline{\mathrm{Ext}}_S^{\ell-1}(M, N/x_1^{b_1}N)$, the latter having the correct shift by induction. But a better reason is *naturality*: regardless of M, N, and the sequence of elements $\mathbf{y} = (y_1, \ldots, y_n)$ (and even the ring S, and the grading), there are natural augmentation maps

$$\mathbb{T}^q \longrightarrow \underline{\mathrm{Ext}}^q(M, N/\mathbf{y}N) \qquad \text{and} \qquad \mathbb{T}^q \longrightarrow H_{n-q}\mathbb{K}_{\bullet}(\mathbf{y}; \underline{\mathrm{Ext}}^n(M,N)),$$

where $\mathbb{T} = \underline{\mathbb{T}\mathrm{or}}(\mathbb{F}^{\bullet}, \mathbb{K}_{\bullet}(\mathbf{y}, N))$ is the hypertor functor, $\mathbb{K}_{\bullet}$ is the Koszul complex, and $\mathbb{F}^{\bullet} = \underline{\mathrm{Hom}}_S(\mathbb{F}_{\bullet}, S)$ is the transpose of a free resolution $\mathbb{F}_{\bullet}$ of M. The shift comes from $\mathbb{K}_n(\mathbf{y})$ and is thus a result of the naturality of Koszul homology. The conditions of Lemma 3.6 guarantee that the augmentations are both isomorphisms for $q = 0$. In fact, the two spectral sequences for the double complex $\mathbb{F}^{\bullet} \otimes \mathbb{K}_{\bullet}(\mathbf{y}; N)$, whose total complex has homology $\mathbb{T}$, give a nice conceptual proof of the lemma.

Now that we have artinian local duality at our disposal, we can start making the connection with Alexander duality. The next lemma is used in the proof of Proposition 3.11, which in turn will be applied in the duality theorem for resolutions (Theorem 3.23) as a crucial step in going from an injective resolution of M to a free resolution of M. The proof of the proposition shows how the functors of Section 1.5 can replace limits in the world of finitely determined $\mathbb{Z}^n$-graded modules.

Lemma 3.10 *Every $\mathbb{Z}^n$-finite flat module $L \in \mathcal{M}$ has a unique submodule L' which is minimal among the submodules N such that L/N is free. Such a submodule L' is flat. If the flat module L is, in addition, a complex of S-modules, then L' is a subcomplex.*

Proof: The $\mathbb{Z}^n$-finite injective module $L^\vee$ has a unique maximal submodule which is the Matlis dual of a free module. Indeed, the Matlis duals of $\mathbb{Z}^n$-finite free modules are precisely the $\mathbb{Z}^n$-finite injective modules with support on $\mathfrak{m}$, so the maximal submodule in question is $H^0_\mathfrak{m}(L^\vee)$ by definition, and is injective by Lemma 2.6. Translating back by Matlis duality, $L' = (L^\vee/H^0_\mathfrak{m}(L^\vee))^\vee \subseteq L$ is the desired minimal submodule. L' is flat by Lemmas 2.6 and 1.21, and if L is a complex then L' is, as well, since $H^0_\mathfrak{m}(L^\vee) \subseteq L^\vee$ is a subcomplex. □

Proposition 3.11 *Let M be positively $\mathbf{a}$-determined and $\mathbb{L} = \mathbb{L}_\bullet$ a $\mathbb{Z}^n$-finite flat resolution in $\mathcal{M}$ of $M^\vee$. Then $\mathbb{F} = \underline{\mathrm{Hom}}_S(\mathbb{L}, S[-\mathbf{1}])$ is a complex of free modules whose homology is zero except in homological degree $-n$, where $H_{-n}\mathbb{F} \cong M$. In particular, the homological shift $\mathbb{F}(-n)$ up by n is a free resolution of M whenever $\mathbb{L}_\bullet$ is zero in homological degrees $> n$.*

Proof: Let $\mathbb{L}' \subseteq \mathbb{L}$ be as in Lemma 3.10, so that $\mathbb{F}' = \mathbb{L}/\mathbb{L}'$ is a complex of free modules in $\mathcal{M}$. Then $\underline{\mathrm{Hom}}_S(\mathbb{L}', S) = 0$ by minimality of $\mathbb{L}'$ as a submodule of $\mathbb{L}$. Therefore,

$$\mathbb{F} \;=\; \underline{\mathrm{Hom}}_S(\mathbb{L}, S[-\mathbf{1}]) \;=\; \underline{\mathrm{Hom}}_S(\mathbb{F}', S[-\mathbf{1}]) \tag{3.1}$$

because $\mathbb{L}'$ is a split submodule of $\mathbb{L}$. It remains only to calculate the homology of $\mathbb{F}$.

For $t \in \mathbb{N}$, denote by $\mathbf{t}$ the vector $t \cdot \mathbf{1} = (t, \ldots, t) \in \mathbb{N}^n$. Now choose $t \in \mathbb{N}$ large enough so that $\mathbb{F}'$ is generated in degrees $\succeq -(\mathbf{t} - \mathbf{1})$ and $\mathbf{t} \succeq \mathbf{a}$. Then $\mathbb{L}_{\succeq -\mathbf{t}}$ is a complex of free modules by Lemma 3.3, and is therefore a free resolution of the artinian module $(M^\vee)_{\succeq -\mathbf{t}}$. It follows from the artinian local duality of Proposition 3.5 that $(\mathbb{L}_{\succeq -\mathbf{t}})^* := \underline{\mathrm{Hom}}_S(\mathbb{L}_{\succeq -\mathbf{t}}, S[-\mathbf{1}])$ is a homological free complex which has as its only nonzero homology the module

$$H_{-n}((\mathbb{L}_{\succeq -\mathbf{t}})^*) \;\cong\; ((M^\vee)_{\succeq -\mathbf{t}})^\vee \;\cong\; ((M^\vee)^\vee)_{\preceq \mathbf{t}} \;\cong\; M_{\preceq \mathbf{t}} \;=\; B_\mathbf{t} M\,,$$

where $M_{\preceq \mathbf{t}}$ denotes the quotient module $\bigoplus_{\mathbf{b} \preceq \mathbf{t}} M_\mathbf{b}$ of M.

The complex $\mathbb{L}'$ is a direct sum of flat modules none of which is free, so $\mathbb{L}'_{\succeq -\mathbf{t}}$ is generated in degrees $\not\succeq -(\mathbf{t} - \mathbf{1})$. Therefore, the quotient complex $\underline{\mathrm{Hom}}_S(\mathbb{L}'_{\succeq -\mathbf{t}}, S[-\mathbf{1}])$ of $(\mathbb{L}_{\succeq -\mathbf{t}})^*$ is a free module generated in degrees $\not\preceq \mathbf{t}$. Meanwhile, the assumption on t means that the subcomplex $\mathbb{F} \subseteq (\mathbb{L}_{\succeq -\mathbf{t}})^*$ is generated in degrees $\preceq \mathbf{t}$. By Lemma 3.3, applying the positive extension functor $P_\mathbf{t}$ to the exact sequence of free modules

$$0 \;\to\; \mathbb{F} \;\to\; (\mathbb{L}_{\succeq -\mathbf{t}})^* \;\to\; \underline{\mathrm{Hom}}_S(\mathbb{L}'_{\succeq -\mathbf{t}}, S[-\mathbf{1}]) \;\to\; 0$$

yields an isomorphism $\mathbb{F} \cong P_\mathbf{t}((\mathbb{L}_{\succeq -\mathbf{t}})^*)$. But exactness of $P_\mathbf{t}$ also implies that $H_r\mathbb{F} \cong P_\mathbf{t} H_r((\mathbb{L}_{\succeq -\mathbf{t}})^*)$ is zero except when $r = -n$, in which case $H_{-n}\mathbb{F} \cong P_\mathbf{t}(B_\mathbf{t} M)$. Since M is positively $\mathbf{a}$-determined, it is also positively $\mathbf{t}$-determined, because $\mathbf{t} \succeq \mathbf{a}$. We conclude that $P_\mathbf{t} B_\mathbf{t} M \cong M$ by Theorem 1.34 and Table 1.1 (before Lemma 1.33). □

3.3 Operations on monomial matrices

Monomial matrices separate and keep track of the different aspects of a complex of $\mathbb{Z}^n$-graded flat or injective modules. This makes it easy to take submatrices determined by summands (i.e. labels) with certain properties, or to alter the summands represented by specific rows and columns simply by applying some numerical operation to the corresponding labels. The purpose now is to determine how these numerical operations reflect the functors which arise in Alexander duality.

This section consists of numerous easy but technical points, which have been collected here for ease of organization and reference. The reader is advised to skim this section briefly before proceeding to Section 3.4, returning back as necessary. In particular, nothing after Lemma 3.16 is used in Section 3.4.

The following definition describes the operations that will come up so often in Section 3.4 and in the examples, and gives them a notation making them visible at a glance.

Definition 3.12 (Operations on monomial matrices) *Let Λ be a homologically graded monomial matrix.*

1. *$\Lambda(-n)$ denotes the homological shift of Λ up by n; i.e. $\Lambda(-n)^d = \Lambda^{d-n}$.*
2. *Λ^* denotes the matrix obtained from Λ by switching the rows and columns (taking the transpose) and multiplying the labels as well as the homological degrees by -1. In symbols, $(\lambda^*)^{-d}_{qp} = \lambda^d_{pq}$, $(\mathbf{b}^*)^{-d}_{\cdot p} = -\mathbf{b}^d_{p\cdot}$, and $(\mathbf{b}^*)^{-d}_{q\cdot} = -\mathbf{b}^d_{\cdot q}$.*
3. *A* relabelling function *is a map $\mathbb{Z}^n_* \to \mathbb{Z}^n_*$. Most of the relabelling functions in this paper are products of functions $\mathbb{Z}_* \to \mathbb{Z}_*$ which can be described as follows. Let $t \in \mathbb{Z}$, and $u_1, u_2, u_3 \in \mathbb{Z}_*$. The array*

$$\left[\begin{array}{cccc} b: & < & t & < \\ \mapsto & u_1 & u_2 & b+u_3 \end{array}\right]$$

denotes a function $f : \mathbb{Z}_* \to \mathbb{Z}_*$ which takes the values

$$f(b) \quad = \quad \left\{\begin{array}{lll} u_1 & \text{if} & b < t \\ u_2 & \text{if} & b = t \\ b+u_3 & \text{if} & b > t \end{array}\right. .$$

The first $<$ may be replaced by $*$, meaning that $* < b < t$ doesn't occur in the input.

Example 3.13 Each component of the relabelling function going from the bottom labelling in Figure 2.2 to the top one can be given by the first array below. The same array gives the relabelling to go from left to right in Figure 2.1.

$$\left[\begin{array}{cccc} b: & < & 1 & < \\ \mapsto & * & -3 & b-4 \end{array}\right] \qquad\qquad \left[\begin{array}{cccc} b: & * & -3 & < \\ \mapsto & 0 & 1 & b+4 \end{array}\right]$$

On the other hand, each component of the relabelling function to go from right to left in Figure 2.1 and top to bottom in Figure 2.2 can be given by the right array above. □

The next three lemmas will be crucial to understanding the duality theorem 3.23 for resolutions in terms of matrices.

Lemma 3.14 *Let $\lambda : N' \to N$ be a map of $\mathbb{Z}^n$-finite flat modules.*

1. *The Čech hull replaces the nonpositive coordinates in the labels by $*$.*
2. *The functor $(-)_{\succeq \mathbf{0}}$ of Definition 3.1.1 replaces the nonpositive coordinates (including all coordinates equal to $*$) in the labels by zero.*

Proof: If N is an irreducible flat module generated in degree $\mathbf{b} \in \mathbb{Z}^n_*$, then it follows easily from the definitions that $N_{\succeq \mathbf{0}}$ is a free module generated in degree $\mathbf{b} \vee \mathbf{0}$ (remember that $* < 0$). This proves part 2. Using part 2 and the fact that $\check{C}$ factors through $(-)_{\succeq \mathbf{0}}$, part 1 is a matrix realization of Example 1.31.1, where the labels there on $S[\mathbf{x}^{-\overline{F}}][-\mathbf{b}]$ should be $*\overline{F} + \mathbf{b}$ to agree with Convention 2.1. □

Lemma 3.15 *Let λ be a map $N' \to N$ of flat modules or $N \to N'$ of injective modules. Given a matrix for λ, the matrix for $\lambda[-\mathbf{a}]$ is gotten by replacing each label $\mathbf{b}$ with*

$$\begin{cases} \mathbf{b} + \mathbf{a} & \textit{if the modules are flat, or} \\ \mathbf{b} - \mathbf{a} & \textit{if the modules are injective.} \end{cases}$$

Proof: Follows from conventions for shifts of flat and injective modules. □

Lemma 3.16 *If Λ represents a map λ of finitely generated free modules in $\mathcal{M}$, then Λ induces on $\underline{\mathrm{Hom}}_S(\lambda, S)$ the* free transpose *matrix Λ^*.* □

The remainder of this section will not be used until Section 3.6, where it will be necessary to work with localizations and restrictions of $\mathbb{Z}^n$-graded modules. It will be particularly important for Theorem 3.30, and later for Theorem 4.5 and much of Section 4.2, to have isolated the behavior of maps of injective and flat modules under such operations. This is the purpose of Propositions 3.18, 3.20, and 3.22.

Notationally, the main complication comes from the fact that $N[\mathbf{x}^{-\overline{F}}]$ is a $\mathbb{Z}^n$-graded module over $S[\mathbf{x}^{-\overline{F}}]$, which is not a polynomial ring because of its homogeneous units. To remove these, we define below the $\mathbb{Z}^n$-graded localization $N_{(F)}$, which is obtained from N by setting the variables x_i with $i \in \overline{F}$ equal to 1, or "taking the degree zero part of the homogeneous localization of N at $\mathfrak{m}^F$" as in algebraic geometry.

Definition 3.17 *Let $\mathbb{Z}^F \subseteq \mathbb{Z}^n$ be the coordinate space spanned by $\{\mathbf{e}_i \mid i \in F\}$. Define, for $N \in \mathcal{M}$,*

1. $S_{[F]} := k[x_i \mid i \in F]$ *a $\mathbb{Z}^F$-graded k-subalgebra of S*
2. $N_{[F]} := \bigoplus_{\mathbf{b} \in \mathbb{Z}^F} N_{\mathbf{b}}$ *a $\mathbb{Z}^F$-graded $S_{[F]}$-module*
3. $N_{(F)} := N[\mathbf{x}^{-\overline{F}}]_{[F]}$ *a $\mathbb{Z}^F$-graded $S_{(F)} = S_{[F]}$-module*

and let $\mathcal{M}_{[F]}$ be the category of $\mathbb{Z}^F$-graded $S_{[F]}$-modules. If $\mathbf{b} \in \mathbb{Z}^n_*$, then define the element $\mathbf{b} \cdot F \in \mathbb{Z}^F_*$ by forgetting the coordinates not in F. If $\mathbf{b} \in \mathbb{Z}^n_*$ but the context requires that $\mathbf{b} \in \mathbb{Z}^F_*$, e.g. if $\mathbf{b}$ appears in a monomial matrix over $S_{[F]}$, then it will always be the case that $b_i \in \{0, *\}$ for all $i \notin F$. In this situation, it is assumed that $\mathbf{b}$ is replaced by $\mathbf{b} \cdot F$.

Of course, restriction $N \mapsto N_{[F]}$ is exact, as is homogeneous localization $N \mapsto N_{(F)}$.

Proposition 3.18 *1. Suppose Λ is a matrix representing a map $\lambda : N' \to N$ of free modules. Then the induced matrix $\Lambda_{[F]}$ representing the map $\lambda_{[F]} : N'_{[F]} \to N_{[F]}$ of free $S_{[F]}$-modules is obtained from Λ by taking the rows and columns whose labels $\mathbf{b}$ satisfy $\mathbf{b} \cdot \overline{F} \preceq \mathbf{0} \cdot \overline{F}$, and replacing each such $\mathbf{b}$ by $\mathbf{b} \cdot F$.*

2. If $\mathbb{F} = \mathbb{F}_{\bullet}$ is a free resolution in $\mathcal{M}$ of a module N, then $\mathbb{F}_{[F]}$ is a free resolution in $\mathcal{M}_{[F]}$ of $N_{[F]}$. If $\mathbb{F}$ is minimal then $\mathbb{F}_{[F]}$ is minimal.

Proof: The restriction $S[-\mathbf{b}]_{[F]}$ is $S_{[F]}[-\mathbf{b} \cdot F]$ if $\mathbf{b} \cdot \overline{F} \preceq \mathbf{0} \cdot \overline{F}$ and 0 otherwise. This proves part 1. Exactness of restriction together with the fact that minimality is always preserved under taking submatrices proves part 2 from part 1. □

Lemma 3.19 *Restriction of any module $N \in \mathcal{M}$ commutes with Čech hull. In symbols, $\check{C}(N)_{[F]} \cong \check{C}_{[F]}(N_{[F]})$, where $\check{C}_{[F]}$ is the Čech hull over $S_{[F]}$.*

Proof: Follows from the definitions (1.30.3 and 3.17.2). □

Proposition 3.20 *1. Suppose Λ is a matrix representing a map $\lambda : N \to N'$ of injective modules. Then the induced matrix representing the map $\lambda_{(F)} : N_{(F)} \to N'_{(F)}$ of injectives in $\mathcal{M}_{[F]}$ is the submatrix $\Lambda_{(F)}$ with $*$-vectors containing $*\overline{F}$ (that is, the submatrix where the labels $\mathbf{b}$ satisfy $\mathrm{supp}(\mathbf{b}_{\mathbb{Z}}) \subseteq F$). In particular, $\Lambda_{(F)}$ is minimal if Λ is.*

2. If $\mathbb{I}^{\bullet}$ is an injective resolution in $\mathcal{M}$ of a module N, then $\mathbb{I}^{\bullet}_{(F)}$ is an injective resolution of $N_{(F)}$ in $\mathcal{M}_{[F]}$. If $\mathbb{I}^{\bullet}$ is minimal then so is $\mathbb{I}^{\bullet}_{(F)}$.

Proof: It is an easy fact that $\underline{E}(S/\mathfrak{m}^{F'}) \otimes_S S[\mathbf{x}^{-\overline{F}}]$ is isomorphic to $\underline{E}(S/\mathfrak{m}^{F'})$ if $F' \subseteq F$ and is otherwise zero. Therefore, a component Λ_{pq} survives the localization if and only if both $*$-vectors contain $*\overline{F}$. Furthermore, it is a consequence of Lemma 3.19 and Example 1.31.3 that

$$\Big(\underline{E}(S/\mathfrak{m}^{F'})\Big)_{[F]} \quad \cong \quad \underline{E}_{[F]}\Big((S/\mathfrak{m}^{F'})_{[F]}\Big), \tag{3.2}$$

where $\underline{E}_{[F]}$ is the injective hull in $\mathcal{M}_{[F]}$. For the first statement it remains only to show that (3.2) still holds if the $\underline{E}(S/\mathfrak{m}^{F'})$ on the left is shifted by $\mathbf{b}$ and the right-hand side is shifted by the vector $\mathbf{b} \cdot F \in \mathbb{Z}^F_*$ (Definition 3.17). This follows from the next lemma, below. The exactness of homogeneous localization proves the second statement from the first. □

Lemma 3.21 *Suppose that the variables $\{x_i \mid i \notin F\}$ act as units on $N \in \mathcal{M}$. Then restriction $(-)_{[F]}$ commutes with arbitrary $\mathbb{Z}^n$-shifts of N; i.e. $N[\mathbf{b}]_{[F]} \cong N_{[F]}[\mathbf{b} \cdot F]$ for $\mathbf{b} \in \mathbb{Z}^n$.*

Proof: Any shift of such an N may be accomplished (up to isomorphism in $\mathcal{M}$) by a vector $\mathbf{b} \in \mathbb{Z}^F$, and $(-)_{[F]}$ obviously commutes with such a shift. □

Proposition 3.22 *1. Suppose that Λ is a matrix representing a map $\lambda : N' \to N$ of free modules. Then the induced matrix $\Lambda_{(F)}$ representing $\lambda_{(F)} : N'_{(F)} \to N_{(F)}$ of free $S_{[F]}$-modules is gotten by replacing each label $\mathbf{b}$ with $\mathbf{b} \cdot F \in \mathbb{Z}^F$.*

2. If Λ represents a free resolution of M, then $\Lambda_{(F)}$ represents a free resolution of $M_{(F)}$.

Proof: After tensoring with $S[\mathbf{x}^{-\overline{F}}]$, Lemma 3.21 and the exactness of homogeneous localization imply the result. □

3.4 Duality for resolutions

Now we present the duality theorem for resolutions, the techincal heart of this dissertation. Examples of it are presented in Section 3.5, and its applications occupy much of the remaining chapters. Although the generalized Alexander duality functors (Definition 3.1) appear all over the place in Theorem 3.23, caution is warranted since the resolutions to which they are applied are rarely $\mathbf{a}$-determined (positively or negatively, or neither). Indeed, see Lemma 3.25 and its proof: the free resolution $\mathbb{F}$ in Theorem 3.23 almost always has summands in degrees $\not\preceq \mathbf{a}$ because applying $B_\mathbf{a}$ to a positively $\mathbf{a}$-determined module usually introduces a syzygy (even a first syzygy) in some degree $\not\preceq \mathbf{a}$. An instructive example is the module $B_\mathbf{1} S = S/\langle x_1^2, \ldots, x_n^2\rangle$, which is nonzero only in degrees $\preceq \mathbf{1}$, but whose free resolution has a summand in every degree from the set $\{0, 2\}^n$. Nevertheless, if the resolutions in Theorem 3.23 are minimal, no information is lost in taking the generalized Alexander duals (Corollary 3.27).

Theorem 3.23 says that various kinds of resolutions can be transformed functorially into various other kinds of resolutions. The diagram

$$\begin{array}{ccccccc}
\mathbb{F}(M^\mathbf{a}) & \xleftarrow{\;P_\mathbf{a}\;} & \mathbb{F}(B_\mathbf{a}M^\mathbf{a}) & \overset{A_\mathbf{a}^{0,+}}{\underset{}{\leftrightarrows}} & \mathbb{I}(M) & \xrightarrow{\underline{\mathrm{Hom}}\,(S^\vee,-)} & \mathbb{F}(M) \\
 & & {\scriptstyle \underline{\mathrm{Hom}}\,(-,S[-\mathbf{a}-\mathbf{1}])}\updownarrow & A_\mathbf{a}^{+,0} & & & \\
\mathbb{F}(M) & \xleftarrow[P_\mathbf{a}]{} & \mathbb{F}(B_\mathbf{a}M) & \underset{A_\mathbf{a}^{0,+}}{\rightleftarrows} & \mathbb{I}(M^\mathbf{a}) & \xrightarrow[\underline{\mathrm{Hom}}\,(S^\vee,-)]{} & \mathbb{F}(M^\mathbf{a})
\end{array}$$

serves as a pictorial summary. For example, a free resolution $\mathbb{F}(B_\mathbf{a}M)$ of $B_\mathbf{a}M$ can be made into an injective resolution $\mathbb{I}(M^\mathbf{a})$ of $M^\mathbf{a}$ by applying the functor $A_\mathbf{a}^{+,0}$. Recall from Definition 3.1 that $A_\mathbf{a}^{0,+}(N) = (N^\vee[-\mathbf{a}])_{\succeq \mathbf{0}}$ and $A_\mathbf{a}^{+,0}(N) = (\check{C}N)^\vee[-\mathbf{a}]$ for any $N \in \mathcal{M}$.

Theorem 3.23 (Duality for resolutions) *Let $M \in \mathcal{M}_+^\mathbf{a}$ be positively $\mathbf{a}$-determined, and suppose that $\mathbb{I} = \mathbb{I}^\cdot$ is a $\mathbb{Z}^n$-finite injective resolution of M. Suppose also that $\mathbb{F} := \mathbb{F}_\cdot$ is a $\mathbb{Z}^n$-finite free resolution of $B_\mathbf{a}(M^\mathbf{a})$. Then $\mathbb{I}$ and $\mathbb{F}$ both have length $\geq n$, and whenever either is minimal it has length exactly n. We can construct the following resolutions from $\mathbb{I}$ and $\mathbb{F}$:*

1. *The homological complex $A_\mathbf{a}^{0,+}\,\mathbb{I}$ is a free resolution of $B_\mathbf{a}(M^\mathbf{a})$.*
2. *The cohomological complex $A_\mathbf{a}^{+,0}\,\mathbb{F}$ is an injective resolution of M.*
3. *The homological complex $\mathbb{F}^*[-\mathbf{a}-\mathbf{1}] := \underline{\mathrm{Hom}}_S(\mathbb{F}, S[-\mathbf{a}-\mathbf{1}])(-n)$ is a free resolution of $B_\mathbf{a}M$ whenever $\mathbb{F}$ has length n.*

4. *The homological complex $P_{\mathbf{a}}\mathbb{F}$ is a free resolution of $M^{\mathbf{a}}$.*
5. *The four homological complexes $\underline{\mathrm{Hom}}_S(S^\vee, H^0_{\mathfrak{m}}\mathbb{I})$, $\underline{\mathrm{Hom}}_S(S^\vee, \mathbb{I})$, $\underline{\mathrm{Hom}}_S(\mathbb{I}^\vee, S)$, and $\underline{\mathrm{Hom}}_S((H^0_{\mathfrak{m}}\mathbb{I})^\vee, S)$ are isomorphic. When shifted homologically up by n, each is a free resolution of $M[\mathbf{1}]$ as long as $\mathbb{I}$ has length n.*

In terms of monomial matrices, we have the following (recall the conventions in Definition 3.12). Let Λ be a monomial matrix for $\mathbb{I}$ and Φ a monomial matrix for $\mathbb{F}$.

1′. *Applying $\begin{bmatrix} b_i: & < & -a_i & < \\ \mapsto & 0 & 0 & b_i+a_i \end{bmatrix}$ to Λ yields a matrix for a free resolution of $B_{\mathbf{a}}(M^{\mathbf{a}})$.*

2′. *Applying $\begin{bmatrix} b_i: & < & 0 & < \\ \mapsto & * & * & b_i-a_i \end{bmatrix}$ to Φ yields a matrix for an injective resolution of M.*

3′. *If $\mathbb{F}$ has length n then adding $\mathbf{a}+\mathbf{1}$ to each label of $\Phi^*(-n)$ yields a matrix representing a free resolution of $B_{\mathbf{a}}M$.*

4′. *The submatrix $\Phi^{\preceq\mathbf{a}}$ whose labels are $\preceq \mathbf{a}$ represents a free resolution of $M^{\mathbf{a}}$.*

5′. *Let $\Lambda_{\mathbb{Z}}$ be the submatrix of Λ whose labels are in $\mathbb{Z}^n$. Then $\Lambda^*_{\mathbb{Z}}(-n)$ represents a free resolution of $M[\mathbf{1}]$ whenever $\mathbb{I}$ has length n.*

Of course, M and $M^{\mathbf{a}}$ can be switched throughout. All of the operations preserve minimality.

Proof: The statement about the length of $\mathbb{F}$ follows from Proposition 3.5; the corresponding statement for $\mathbb{I}$ follows from part 2 (whose proof uses nothing about injective resolutions).

(1): The exactness of $A^{0,+}_{\mathbf{a}}$ and the fact that it takes injective modules to free modules both follow from Lemma 3.3. Definition 3.1.3 implies that $A^{0,+}_{\mathbf{a}}\mathbb{I}$ resolves $B_{\mathbf{a}}(M_{\mathbf{a}})$. For part 1′, use Corollary 2.10, Lemma 3.15, and Lemma 3.14: the monomial matrix in question is obtained from Λ by first adding $\mathbf{a}$ to all labels and then replacing all negative coordinates (including $*$) by zero.

(2): The exactness of $A^{+,0}_{\mathbf{a}}$ and the fact that it takes free modules to injective modules both follow from Lemma 3.3, while Lemma 3.2.2 implies that $A^{+,0}_{\mathbf{a}}\mathbb{F}$ resolves M. To get part 2′, use the same lemmas used for part 1′.

(3): By Lemma 1.37 we have an isomorphism $(B_{\mathbf{a}}M^{\mathbf{a}})^\vee[-\mathbf{a}] \cong B_{\mathbf{a}}M$, so this is just local duality for artinian modules, Proposition 3.5. For part 3′ use Lemmas 3.16 and 3.15.

(4): Use Lemma 3.3 and the fact that $P_{\mathbf{a}}B_{\mathbf{a}}(M^{\mathbf{a}}) \cong M^{\mathbf{a}}$ (Theorem 1.34 and Table 1.1). For part 4′, apply Lemma 3.3.

(5): The first complex is isomorphic to the second because $S^\vee$ is artinian, so any homomorphism $S^\vee \to \mathbb{I}$ lands inside $H^0_{\mathfrak{m}}(\mathbb{I})$. The second and third complexes are isomorphic for the same reason that the first and fourth are: because $\underline{\mathrm{Hom}}_S(-^\vee, S) \cong \underline{\mathrm{Hom}}_k(-^\vee \otimes_S S^\vee, k) \cong \underline{\mathrm{Hom}}_S(S^\vee, -)$ by Equation (1.2) in Section 1.3. If the third complex has length n, Proposition 3.11 implies that it is a free resolution of $M[\mathbf{1}]$ because $\mathbb{L} := \mathbb{I}^\vee$ is a flat resolution of $M^\vee$ by Matlis duality. To prove part 5′, first take $H^0_{\mathfrak{m}}\mathbb{I}$, producing the submatrix with labels in $\mathbb{Z}^n$, and then apply Corollary 2.10 and 3.16 to the fourth complex.

The statement about minimality requires an intermediate result, Lemma 3.25. For use in the lemma, the next result [BH95, Theorem 3.1(a)] is derived as a corollary to part 4′.

Corollary 3.24 (Bruns-Herzog) *Suppose M is positively $\mathbf{a}$-determined (equivalently, all nonzero Betti numbers in homological degrees 0 and 1 of the finitely generated $\mathbb{N}^n$-graded module $M \in \mathcal{M}$ occur in degrees $\preceq \mathbf{a}$). Then all nonzero Betti numbers of M occur in nonnegative degrees $\preceq \mathbf{a}$; that is, the minimal free resolution of M is positively $\mathbf{a}$-determined.*

Proof: The condition on the zeroth and first Betti numbers means that M is positively $\mathbf{a}$-determined by Proposition 1.29. The minimal free resolution of M is $\mathbb{N}^n$-graded, so the result follows from part $4'$ above. □

Lemma 3.25 *If $\mathbb{F}$ is minimal then $\mathbb{F}$ is positively $(\mathbf{a}+\mathbf{1})$-determined (equivalently, all labels $\mathbf{b}$ in Φ satisfy $\mathbf{0} \preceq \mathbf{b} \preceq \mathbf{a}+\mathbf{1}$). If $\mathbb{I}$ is minimal, then $\mathbb{I}[-\mathbf{1}]$ is $(\mathbf{a}+\mathbf{1})$-determined (equivalently, the labels $\mathbf{b}$ in Λ satisfy $\mathbf{1}-\mathbf{a} \preceq \mathbf{b}_{\mathbb{Z}} \preceq \mathbf{1}$).*

Proof: The parenthesized statements come from Example 1.26.2 (and Lemma 3.15 for $\mathbb{I}[-\mathbf{1}]$). The module $B_{\mathbf{a}}(M^{\mathbf{a}}) = P_{\mathbf{a}+\mathbf{1}}(B_{\mathbf{a}}(M^{\mathbf{a}}))$ is positively $(\mathbf{a}+\mathbf{1})$-determined by definition, so $\mathbb{F}$ is positively $(\mathbf{a}+\mathbf{1})$-determined by Corollary 3.24. It is now enough to show that there exists *some* injective resolution of $M \in \mathcal{M}_+^{\mathbf{a}}$ whose shift by $[-\mathbf{1}]$ is $(\mathbf{a}+\mathbf{1})$-determined, since the minimal injective resolution will then be forced to satisfy the condition, as well. But the complex $A_{\mathbf{a}}^{+,0}\mathbb{F}$ is an injective resolution of M by part 2 above, so the equality $(A_{\mathbf{a}}^{+,0}\mathbb{F})[-\mathbf{1}] = A_{\mathbf{a}+\mathbf{1}}^{+,0}\mathbb{F}$ implies the proposition because $\mathbb{F}$ started out in $\mathcal{M}_+^{\mathbf{a}+\mathbf{1}}$. □

To finish the proof of minimality preservation, check that inequalities $\mathbf{b}_{p\cdot} \neq \mathbf{b}_{\cdot q}$ persist under the operations in parts $1'$–$5'$ whenever the vectors satisfy the conditions of the Lemma. Note that taking submatrices always has this property. The proof of the theorem is complete. □

3.5 Examples of duality for resolutions

Example 3.26 Each part of Theorem 3.23 can be illustrated by the cocellular and cellular monomial matrices represented in Figures 2.1 and 2.2 using the module $M = S/J$ there, where $J = \langle x^2y^2, x^4y, x^3z, xz^2\rangle$ and $\mathbf{a} = (4,4,4) =: 444$. The numbering of the following list corresponds to the numbering in Theorem 3.23.

1. The top labelling in Figure 2.2 determines a cellular minimal injective resolution of $M^{444} = J^{[444]}$. The bottom labelling determines a cellular minimal free resolution of $B_{444}(M)$. See the relabelling functions in Example 3.13. Similarly, the right labelling in Figure 2.1 determines a cocellular minimal injective resolution of $M = S/J$, while the left labelling determines a cocellular free resolution of $B_{444}(M^{444})$.
2. This is just part 1 in reverse. Again, see the arrays in Example 3.13.
3. This part has not been mentioned yet in the examples; it connects the left diagram in Figure 2.1 with the bottom diagram in Figure 2.2. Indeed, the bottom labels of Figure 2.2 give a minimal free resolution of $B_{444}(S/J)$, and are obtained by subtracting from $555 = \mathbf{a}+\mathbf{1}$ the labels on the left of Figure 2.1, which give a minimal free resolution of

$$B_{444}((S/J)^{444}) \quad = \quad (J^{[444]} + \langle x^5, y^5, z^5\rangle)/\langle x^5, y^5, z^5\rangle\,.$$

   Note that this transition is the one that goes from cellular to cocellular (and back).

4. This is why the thickened subdivided quadrilateral in Figure 2.2 gives the minimal free resolution of $M = S/J$: the labels marked with an "x" are precisely those that are $\preceq 444$ in the free resolution of $B_{444}(M)$. (Switch M and $M^{\mathbf{a}}$ in Theorem 3.23.4 to get the notation to agree here.)

5. This says that the *interior* faces in the top picture of Figure 2.2 give a *free* resolution of $J^{[444]}$, since the interior faces are precisely those with no $*$. More precisely, if one takes the negatives of the interior face labels and adds $\mathbf{1} = 111$, then the coboundary complex $\widetilde{\mathcal{C}}^{\bullet}(X;k)$ induces a free resolution of $J^{[444]}$.

This example is a special case of the material in Chapter 5, where the various resolutions of generic and cogeneric ideals presented in the context of duality for cellular resolutions. □

There is a fundamental difference between parts 1, 2, and 3 of Theorem 3.23 and parts 4 and 5: the latter parts involve a certain loss of information (by taking submatrices), whereas the former do not. This is evident from the examples above: one can get all the way from the right of Figure 2.1 to the top of Figure 2.2 by going through the left of Figure 2.1 and the bottom of Figure 2.2. Corollary 3.27 says that this is phenomenon is completely general.

Corollary 3.27 (Duality for minimal resolutions) *The following resolutions are equivalent, in the sense that the functors in Theorem 3.23 (and their compositions) translate between them, and there are matrices for them which are the same up to relabelling the rows and columns and taking a transpose:*

1. *A minimal injective resolution of M.*
2. *A minimal free resolution of $B_{\mathbf{a}}(M^{\mathbf{a}})$.*
3. *A minimal free resolution of $B_{\mathbf{a}}M$.*
4. *A minimal injective resolution of $M^{\mathbf{a}}$.*

A matrix for the minimal injective resolution of $M^{\mathbf{a}}$ is obtained by applying the relabelling function

$$\begin{bmatrix} b_i: & * & -1 & < \\ \mapsto & 1 & * & b_i + 1 - a_i \end{bmatrix}$$

to the matrix $\Lambda^*(-n)$, for any matrix Λ representing a minimal injective resolution of M.

Proof: Straightforward from Theorem 3.23. (Notice that the labels satisfy the conditions of Lemma 3.25, so the relabelling function need not be specified outside those ranges.) □

Example 3.28 Here is a synopsis of what duality for resolutions says about cellular and cocellular resolutions in general. Suppose the free resolution $\mathbb{F}$ in Theorem 3.23 is a cellular resolution $\mathbb{F}_X$ of $B_{\mathbf{a}}(S/J)$ for some monomial ideal J, supported on a labelled cell complex X of dimension $n-1$. Heuristically, the resolutions obtained from $\mathbb{F}_X$ in Theorem 3.23 and

Corollary 3.27 are described as follows:

free resolution of S/J	$\leftrightarrow$	boundary complex of $X\vert_{\text{vertices} \neq x_i^{a_i+1}}$
free resolution of $B_{\mathbf{a}}(S/J)$	$\leftrightarrow$	boundary complex of X
injective resolution of $J^{[\mathbf{a}]}$	$\leftrightarrow$	coboundary complex of X
injective resolution of S/J	$\leftrightarrow$	boundary complex of X
free resolution of $B_{\mathbf{a}}(J^{[\mathbf{a}]})$	$\leftrightarrow$	coboundary complex of X
free resolution of $J^{[\mathbf{a}]}$	$\leftrightarrow$	coboundary complex on interior faces of X

where, of course, the labellings vary. The first three are cellular, the fourth and fifth are cocellular, and the last is *relative cocellular*. These can all be matched with Examples 2.14, 2.15, and 3.26, along with Figures 2.1 and 2.2. Many details of this sketch are stated precisely in Chapter 5, where they are used to study generic and cogeneric ideals. □

3.6 Localization, restriction, and duality

We turn our attention to the interaction of localization with duality for resolutions. The first result (Corollary 3.29) describes the structure of a minimal injective resolution of a finitely generated module M by determining where all of the summands come from: the shifts of $\underline{E}(S/\mathfrak{m}^F)$ correspond to the summands in a minimal free resolution of $M_{(F)}$. This observation lends credence to claim that injective resolutions contain much more complete homological information than free resolutions.

Given an injective module $I \in \mathcal{M}$ and $F \subseteq \{1, \ldots, n\}$, the localization $I_{(F)}$ kills all of the summands in I whose support is not contained in $\mathfrak{m}^F$ (by Proposition 3.20). On the other hand, the functor $H^0_{\mathfrak{m}^F}$ picks out those summands whose supports contain $\mathfrak{m}^F$ (by definition). Combining these two functors, we find that $(H^0_{\mathfrak{m}^F} I)_{(F)} = H^0_{\mathfrak{m}^F}(I_{(F)})$ is obtained from I by first taking the subquotient consisting of all summands which are shifts of $\underline{E}(S/\mathfrak{m}^F)$, and then restricting to $S_{[F]}$. If $I = \mathbb{I}$ is a minimal injective resolution of $M \in \mathcal{M}^{\mathbf{a}}_{+}$, then Theorem 3.23.5 implies the next corollary; if M is just finitely generated, we can shift everything back by $[\mathbf{b} \cdot F]$ after applying the argument to $M[-\mathbf{b}]$.

Corollary 3.29 *Let $M \in \mathcal{M}$ be finitely generated, with minimal injective resolution $\mathbb{I}$. If $|F| = i$ and $(-i)$ denotes homological shift by i (Definition 3.12), then the complex*

$$\underline{\mathrm{Hom}}_{S_{[F]}}\Big(S^{\vee}_{[F]}\,,\,(H^0_{\mathfrak{m}^F}\mathbb{I})_{(F)}\Big)(-i)$$

of $S_{[F]}$-modules is a minimal free resolution of $M_{(F)}[F]$. In terms of matrices, if Λ is a monomial matrix representing $\mathbb{I}$ and $\Lambda_{(F)}$ is the submatrix with $*$-vectors equal to $*\overline{F}$, then $\Lambda^*_{(F)}(-i)$ represents a minimal free resolution of $M_{(F)}[F]$. □

For more comments on the meaning of this corollary, see Question 4.14.

The next theorem says that localization is dual to restriction, in a functorial sense. The subscript $\mathbf{a} \cdot F$ in $A^{+,0}_{\mathbf{a}\cdot F}$ and $A^{0,+}_{\mathbf{a}\cdot F}$ indicates that the duality is taking place over $S_{[F]}$, that is, in $\mathcal{M}_{[F]}$. For the matrices over $S_{[F]}$, recall the convention that all coordinates of any vector $\overline{F}$ coming from a row or column label (see Theorem 2.9) will either be 0 or $*$, and these are to be ignored (Definition 3.17).

Theorem 3.30 (Restriction-localization) *Let $M \in \mathcal{M}_+^{\mathbf{a}}$. Suppose that $\mathbb{I}$ is a $\mathbb{Z}^n$-finite injective resolution of M, $\mathbb{F}$ is a $\mathbb{Z}^n$-finite free resolution of $B_{\mathbf{a}}(M^{\mathbf{a}})$, and $F \subseteq \{1, \dots, n\}$.*

1. *$A^{0,+}_{\mathbf{a}\cdot F}(\mathbb{I}_{(F)})$ and $(A^{0,+}_{\mathbf{a}}\mathbb{I})_{[F]}$ are both free resolutions in $\mathcal{M}_{[F]}$ of $B_{\mathbf{a}}(M^{\mathbf{a}})_{[F]}$. If $\mathbb{I}$ is minimal then $A^{0,+}_{\mathbf{a}\cdot F}(\mathbb{I}_{(F)}) \cong (A^{0,+}_{\mathbf{a}}\mathbb{I})_{[F]}$, and this free resolution is minimal.*
2. *$A^{+,0}_{\mathbf{a}\cdot F}(\mathbb{F}_{[F]}) \cong (A^{+,0}_{\mathbf{a}}\mathbb{F})_{(F)}$ is an injective resolution in $\mathcal{M}_{[F]}$ of $M_{(F)}$ which is minimal if $\mathbb{F}$ is minimal.*

The functors in parts 1 and 2 preserve minimality. In terms of monomial matrices, we have the following. Suppose Λ is a monomial matrix for $\mathbb{I}$ and Φ is a monomial matrix for $\mathbb{F}$.

1′. Let $\Lambda_{(F)}$ be the submatrix of Λ whose $*$-vectors contain $*\overline{F}$. Applying the relabelling function $\left[\begin{array}{cccc} b_i: & < & -a_i & < \\ \mapsto & 0 & 0 & b_i + a_i \end{array}\right]$ to $\Lambda_{(F)}$ yields a matrix for $A^{0,+}_{\mathbf{a}\cdot F}(\mathbb{I}_{(F)})$ of part 1.

2′. Let $\Phi_{[F]}$ be the submatrix of Φ whose labels $\mathbf{b}$ satisfy $\mathbf{b} \cdot \overline{F} \preceq \mathbf{0} \cdot \overline{F}$. Applying the relabelling function $\left[\begin{array}{cccc} b_i: & < & 0 & < \\ \mapsto & * & * & b_i - a_i \end{array}\right]$ to $\Phi_{[F]}$ yields a matrix for part 2.

If Λ is minimal and the matrix output by part 1′ is $\Phi_{[F]}$, then the matrix output by part 2′ is $\Lambda_{(F)}$. Similarly, if Φ is minimal and the matrix output by part 2′ is $\Lambda_{(F)}$, then the matrix output by part 1′ is $\Phi_{[F]}$.

Proof: (1): That $(A^{0,+}_{\mathbf{a}}\mathbb{I})_{[F]}$ is a free resolution in $\mathcal{M}_{[F]}$ of $B_{\mathbf{a}}(M^{\mathbf{a}})_{[F]}$ and preserves minimality is Proposition 3.18 applied to Theorem 3.23.1. That $A^{0,+}_{\mathbf{a}\cdot F}(\mathbb{I}_{(F)})$ is a free resolution of some module N and preserves minimality is Theorem 3.23.1 applied (over $S_{[F]}$) to Proposition 3.20. Furthermore, the module $N \cong A^{0,+}_{\mathbf{a}\cdot F}(M_{(F)})$ is independent of whether $\mathbb{I}$ is minimal. Thus we need only verify the isomorphism of complexes when $\mathbb{I}$ is minimal.

In fact, the proof only uses $\mathbb{I}[-\mathbf{1}] \in \mathcal{M}^{\mathbf{a}+\mathbf{1}}$ (see Lemma 3.25). Indeed, with this hypothesis we show that the matrix produced in part 1′ represents both complexes of part 1. For $A^{0,+}_{\mathbf{a}\cdot F}(\mathbb{I}_{(F)})$ this is Theorem 3.23.2′ applied (over $S_{[F]}$) after Proposition 3.20. For $(A^{0,+}_{\mathbf{a}}\mathbb{I})_{[F]}$, first apply Theorem 3.23.1′: the hypothesis on $\mathbb{I}$ means that that $b_i \mapsto 0$ if and only if $b_i = *$, and all of the other coordinates are strictly positive. The result now follows from Proposition 3.18.

(2): That $(A^{+,0}_{\mathbf{a}}\mathbb{F})_{(F)}$ is an injective resolution in $\mathcal{M}_{[F]}$ of $M_{(F)}$ and preserves minimality is Proposition 3.20 applied to Theorem 3.23.2. Therefore, we need only show that the matrix produced by part 2′ represents both complexes of part 2. That it represents $A^{+,0}_{\mathbf{a}\cdot F}(\mathbb{F}_{[F]})$ is Theorem 3.23 applied (over $S_{[F]}$) to Proposition 3.18. For $(A^{+,0}_{\mathbf{a}}\mathbb{F})_{(F)}$, first apply Theorem 3.23.2′, whose relabelling array is the same. The $*$-vector of a relabelled row or column ends up containing $*\overline{F}$ if and only if the original label $\mathbf{b}$ satisfied $\mathbf{b} \cdot \overline{F} \preceq \mathbf{0} \cdot \overline{F}$. Therefore, the result follows from Proposition 3.20.

The final claims are in fact true under conditions weaker than minimality; it is enough to satisfy the conclusions of Lemma 3.25 (rather than its hypotheses). □

Remark 3.31 The two free resolutions $A^{0,+}_{\mathbf{a}\cdot F}(\mathbb{I}_{(F)})$ and $(A^{0,+}_{\mathbf{a}}\mathbb{I})_{[F]}$ of from Theorem 3.30.1 may very well be nonisomorphic if $\mathbb{I}$ is not minimal. In general, $A^{0,+}_{\mathbf{a}\cdot F}(\mathbb{I}_{(F)})$ has fewer summands than $(A^{0,+}_{\mathbf{a}}\mathbb{I})_{[F]}$. For instance, suppose $\underline{E}(k)[-\mathbf{b}]$ is an artinian injective summand of $\mathbb{I}$ with $\mathbf{b}\cdot\overline{F} \not\preceq (\mathbf{a}-\mathbf{1})\cdot\overline{F}$ (see Lemma 3.25). Then $\underline{E}(k)[-\mathbf{b}]$ becomes a free summand $S[(\mathbf{b}-\mathbf{a})\cdot F]$ of $(A^{0,+}_{\mathbf{a}}\mathbb{I})_{[F]}$, but is erased by the localization in $A^{0,+}_{\mathbf{a}\cdot F}(\mathbb{I}_{(F)})$ whenever $\overline{F}\neq\varnothing$.

Example 3.32 The restriction-localization theorem is easiest to see in Figure 2.2. If we invert y in the injective resolution of the cogeneric ideal $J^{[444]}$ represented by the top labelling, we simply restrict to the faces which have a $*$ in their second slot. Applying Theorem 3.30, this gives the same subcomplex as picking the faces in the bottom labelling whose second entries are 0. In general, localization of cogeneric monomial ideals acts on this triangulation of a simplex by restriction to a face of the simplex.

In Figure 2.1, on the other hand, inverting the variable y in the injective resolution of S/J in the right of Figure 2.1 yields the set of faces Y containing the vertex v labelled $1{*}1$. These 2 triangles, 3 edges, and vertex are, by Theorem 3.30, the same faces which in the left labelling have second coordinate 0. The ideal J is generic and therefore the localization $J_{(F)}$ is also generic, so its minimal injective resolution should still be a triangulation of a simplex (see Section 5.3 for details). To see how Y is a triangulation of a simplex, define the *contrastar* and *link* of v in the simplicial complex X to be

$$\mathrm{cost}_v X := \{G\in X \mid v\notin G\} \quad \text{and} \quad \mathrm{link}_v X := \{G\in X \mid v\notin G \text{ and } v\cup G\in X\}. \tag{3.3}$$

Then the *relative chain complex* $\widetilde{\mathcal{C}}_{\bullet}(X,\mathrm{cost}_v X;k) := \widetilde{\mathcal{C}}_{\bullet}(X;k)/\widetilde{\mathcal{C}}_{\bullet}(\mathrm{cost}_v X;k)$ is isomorphic to $\widetilde{\mathcal{C}}_{\bullet}(\mathrm{link}_v X)$, essentially because adding v to every face in $\mathrm{link}_v X$ gives $Y = X\setminus\mathrm{cost}_v X$ [Grä84, Lemma 1.3]. In general, localization of generic ideals takes links in the Scarf triangulation.

The reader familiar with Alexander duality for squarefree monomial ideals will notice that the Alexander dual to restriction (which is what we got in the earlier part of this example) is taking links (which is what we got in the latter part). Only this time, it happens on Scarf complexes instead of on Stanley-Reisner complexes—that is, on resolutions instead of on modules. □

Corollary 3.33 $(M_{(F)})^{\mathbf{a}\cdot F}\cong(M^{\mathbf{a}})_{[F]}$ *and* $(M^{\mathbf{a}})_{(F)}\cong(M_{[F]})^{\mathbf{a}\cdot F}$ *for* $M\in\mathcal{M}^{\mathbf{a}}_{+}$. *In words, localizing and then dualizing is the same as dualizing and then restricting; while dualizing and then localizing is the same as restricting and then dualizing.*

Proof: It is enough to show the first isomorphism, switching M and $M^{\mathbf{a}}$ for the second. In Theorem 3.30.2, $\mathbb{F}_{[F]}$ resolves $B_{\mathbf{a}}(M^{\mathbf{a}})_{[F]} = B_{\mathbf{a}\cdot F}({M^{\mathbf{a}}}_{[F]})$, so applying Theorem 3.23.2 (over $S_{[F]}$) we find that $M_{(F)}$ is the Alexander dual of $(M^{\mathbf{a}})_{[F]}$ with respect to $\mathbf{a}\cdot F$. □

Chapter 4

Applications of duality

The contents of this chapter are a smorgasbord of theorems illustrating the flexibility of duality for resolutions. The results range from the algebraic in Section 4.1, to the numerical in Section 4.2, to the combinatorial in Section 4.3, to the topological in Section 4.4. This last one, which interprets Alexander duality as a change of basis between between equivariant K-homology and equivariant K-cohomology, in fact uses only Chapter 1 to prove the *Alexander inversion formula*. This formula is an identity involving generating functions, and can also be interpreted as a combinatorial statement about h-vectors of dual simplicial complexes. Each of the other sections reflects the wealth of data from injective resolutions in its own way.

Section 4.1 contains the generalization to $\mathbb{N}^n$-graded modules of the theorems of Eagon-Reiner [ER98] and Terai [Ter97] relating the projective dimension of a squarefree monomial ideal to the row and column labels in the minimal free resolution of its Alexander dual. While the free resolution of a module M corresponds to the artinian summands in its injective resolution, the free resolution of $M^{\mathbf{a}}$ corresponds to the summands that are shifted by nonpositive vectors. As it turns out, it is possible to read the projective dimension of M simply by looking at how the codimensions of these nonpositive-shift summands increase, and by a generalized Alexander duality these codimensions equal the support sizes of row and column labels in the free resolution of $M^{\mathbf{a}}$. This essential use of injective resolutions casts new light on the principles underlying the results in [ER98, Ter97].

Section 4.2 contains the menagerie of equalities and inequalities between Betti and Bass numbers of a positively $\mathbf{a}$-determined module and its Alexander dual. These are seen to be immediate corollaries of Theorem 3.23. The moral of the story here is that if you're going to write down numerical statements to express Alexander duality, they ought to involve Bass numbers. In particular, a Betti-Betti inequality of Bayer, Charalambous, and Popescu [BCP99] is generalized to $\mathbb{N}^n$-graded modules. They proved their result using simplicial methods for squarefree monomial ideals; here, we instead subject Bass number *equalities* to localization.

Section 4.3 interprets duality for resolutions in terms of lcm-lattices of monomial ideals [GPW00], showing how they determine minimal injective resolutions. The order-theoretic translation of the rigidity of injective resolutions is: given a finite order-sublattice of $\mathbb{N}^n$, it may be possible to move the atoms a little and not change anything significant.

However, adding elements of $\mathbb{N}^n$ along the axes is like placing tent pegs: having them in makes it much harder to move things around without significant change.

4.1 Projective dimension and support-regularity

A great deal of recent research on Alexander duality concerns the fundamental result of Eagon and Reiner [ER98] relating the degrees in the maps in the minimal free resolution of a squarefree monomial ideal to the length of the minimal free resolution of its squarefree Alexander dual, and the subsequent generalization of their result by Terai [Ter97]. To be precise, define the *regularity* of an $\mathbb{N}^n$-graded module M by

$$\mathrm{reg}(M) := \max_i \{\, |\mathbf{b}| - i \mid \beta_{i,\mathbf{b}}(M) \neq 0\},$$

where $|\mathbf{b}| = \sum_{i=1}^n b_i$, and the projective dimension of M by

$$\mathrm{proj.\,dim}(M) := \max\{i \mid \beta_{i,\mathbf{b}}(M) \neq 0 \text{ for some } \mathbf{b} \in \mathbb{N}^n\}.$$

The module M has *linear free resolution* if every homogeneous minimal generator of M has $\mathbb{Z}$-graded degree $\mathrm{reg}(M)$. In terms of monomial matrices, a free resolution is linear if all of the row labels $\mathbf{b}$ in homological degree 0 have the same size $d := |\mathbf{b}|$, and all of the row labels in a higher homological degree i have size $d + i$.

Theorem 4.1 *Let I be a squarefree monomial ideal and $I^{\mathbf{1}} = S/I^{[\mathbf{1}]}$ the Alexander dual.*
[ER98, Theorem 3]: *I has linear free resolution if and only if $I^{\mathbf{1}}$ is Cohen-Macaulay.*
[Ter97, Corollary 0.3]: $\mathrm{reg}(I) = \mathrm{proj.\,dim}(I^{\mathbf{1}})$.

Note that Terai's result implies the Eagon-Reiner result, since $S/I^{[\mathbf{1}]}$ is Cohen-Macaulay if and only if $\mathrm{proj.\,dim}(S/I^{[\mathbf{1}]}) = \mathrm{codim}(S/I^{[\mathbf{1}]})$, and the codimension is the smallest degree of a generator of I by the definition of Alexander dual ideal $I^{[\mathbf{1}]}$ (Definition 1.2).

Herzog, Reiner, and Welker [HRW99] generalized the Eagon-Reiner part of Theorem 4.1 in another direction, showing that I is *componentwise linear* if and only if $S/I^{[\mathbf{1}]}$ is *sequentially Cohen-Macaulay.* They then posed question of whether there is a duality for possibly nonradical monomial ideals, or some other class of modules, such that the "amazing properties" of Theorem 4.1 still hold [HRW99, Question 10]. Having extended Alexander duality in the unique way to a functor on $\mathbb{N}^n$-graded modules in Theorem 1.35, it is natural to think that $(-)^{\mathbf{1}}$, and maybe even $(-)^{\mathbf{a}}$, will satisfy the conclusions of Theorem 4.1.

It is in fact true that Theorem 4.1 holds for arbitrary squarefree modules $M \in \mathcal{M}_+^{\mathbf{1}}$ (Corollary 4.6). But when $\mathbf{a} \neq \mathbf{1}$, Terai's contribution to Theorem 4.1 cannot be expected to hold verbatim for Alexander duality with respect to arbitrary $\mathbf{a}$, because the projective dimension is universally bounded by n, while the regularity is not:

Example 4.2 If $d \in \mathbb{N}$, then $\mathrm{reg}(\langle x_1^d, \ldots, x_n^d\rangle) = n(d-1)$, while the functorial Alexander dual $\langle x_1^d, \ldots, x_n^d\rangle^{d\cdot\mathbf{1}} = S/\mathfrak{m}$ with respect to $(d, \ldots, d)$ has projective dimension n. □

Furthermore, even Eagon-Reiner's contribution fails for general $\mathbf{a}$, for a seemingly silly reason: almost every positively $\mathbf{a}$-determined module has an artinian Alexander dual. Specifically, if $M \in \mathcal{M}_+^{\mathbf{a}-\mathbf{1}}$ is arbitrary, then $(M[-\mathbf{1}])^{\mathbf{a}}$ is artinian, and hence Cohen-Macaulay.

But the minimal free resolution of $M[-\mathbf{1}]$ is just the shift by $\mathbf{1}$ of the minimal resolution of M. Thus every minimal resolution, be it linear or not, appears as the resolution of a module whose dual is Cohen-Macaulay; thus M Cohen-Macaulay $\not\Rightarrow M^{\mathbf{a}}$ has linear free resolution.

One might still hope that the implication "M has a linear resolution $\Rightarrow M^{\mathbf{a}}$ is Cohen-Macaulay" would hold, but this fails, too, even if we allow $M^{\mathbf{a}}$ to be merely sequentially Cohen-Macaulay (defined in the next example).

Example 4.3 Let $I' = \langle ab, bc, cd\rangle \subseteq S = k[a,b,c,d]$ be the ideal of the "stick twisted cubic" simplicial complex spanned by the edges $\{b,d\}, \{b,c\}$, and $\{a,c\}$. It is readily checked that I' has a linear resolution by Theorem 4.1: $(I')^{\mathbf{1}} = S/(I')^{[\mathbf{1}]}$ is the coordinate ring of another stick twisted cubic, and is thus Cohen-Macaulay because the stick twisted cubic is connected and has dimension 1. Now define

$$I = \mathfrak{m}I' = \langle a^2b, abc, acd, ab^2, b^2c, bcd, abc, bc^2, c^2d, abd, bcd, cd^2\rangle$$
$$I^{[\mathbf{a}]} = \langle b^2d^2, b^2c^2, a^2c^2, abc^2d^2, a^2bcd^2, a^2b^2cd\rangle$$

with $\mathbf{a} = (2,2,2,2)$. Then I also has a linear free resolution by [HRW99, Lemma 1], and we show that $S/I^{[\mathbf{a}]}$ is not sequentially Cohen-Macaulay.

For a module N to be sequentially Cohen-Macaulay, we require that there exist a filtration $0 = N_0 \subset N_1 \subset \cdots \subset N_r = N$ such that N_i/N_{i-1} is Cohen-Macaulay for all $i \le r$ and $\dim(N_{i+1}/N_i) > \dim(N_i/N_{i-1})$ for all $i < r$. It follows from the equidimensionality of N/N_{r-1} and the strict reduction of dimension in successive quotients that N_{r-1} is the top dimensional piece of N; i.e. N_{r-1} is the intersection of all primary components (of 0 in N) which have dimension $\dim(N)$. Thus it suffices to check that $S/I^{[\mathbf{a}]}_{\text{top}}$ is not Cohen-Macaulay, where $I^{[\mathbf{a}]}_{\text{top}} = \langle b^2d^2, b^2cd, abcd, b^2c^2, abc^2, a^2c^2\rangle$ is the intersection of all primary components of $I^{[\mathbf{a}]}$ which have dimension $2 = \dim(S/I^{[\mathbf{a}]})$. This can be done directly, for instance using the symbolic algebra program CoCoA [CNR99]. □

The fundamental problem with the case $\mathbf{a} \neq \mathbf{1}$ is that the $\mathbb{Z}$-degree of an element of $\mathbb{N}^n$ which is $\preceq \mathbf{a}$ is not determined by the support of its $\mathbb{Z}^n$-graded degree, as it is with an element $\preceq \mathbf{1}$. Thus a module might have a linear resolution while its generators have support sets of varying sizes, wreaking havoc with the equidimensionality required for the Cohen-Macaulayness of the dual. Nonetheless, it will be shown in this section that Theorem 4.1 has an explanation in terms of injective resolutions, and can therefore be directly generalized to the $\mathbb{N}^n$-graded case.

Heuristically, the projective dimension of a finitely generated module M depends only on the codimensions of the irreducible summands in a minimal injective resolution of M, and in which cohomological degrees they occur. If M is positively $\mathbf{a}$-determined, Alexander duality changes these codimensions into the sizes of the supports of the Betti degrees of $M^{\mathbf{a}}$. This motivates the following definition, which is an analog of regularity that more appropriately reflects the $\mathbb{Z}^n$-grading.

Definition 4.4 *The* support-regularity *of an* $\mathbb{N}^n$*-graded module is*

$$\operatorname{supp.reg}(M) := \max\{|\operatorname{supp}(\mathbf{b})| - i \mid \beta_{i,\mathbf{b}}(M) \neq 0\},$$

and M is said to have a support-linear *free resolution if there is a nonnegative integer d such that $|\operatorname{supp}(\deg m)| = d$ for all minimal generators m of M and* $\operatorname{supp.reg}(M) = d$.

Observe that in case M is a squarefree module (i.e. $M \in \mathcal{M}_+^{\mathbf{1}}$), all of the nonzero Betti numbers of M occur in degrees $\mathbf{b} \in \{0,1\}^n$, so that $\operatorname{supp.reg}(M)$ agrees with the usual regularity $\operatorname{reg}(M)$. Moreover, support-linearity is the same thing as linearity for a minimal free resolution of such an M.

The next theorem answers [HRW99, Question 10] by giving the most general form of Theorem 4.1. It is the main result of this section, and the search for its statement as well as its proof was the most important motivation for developing the machinery of Chapter 3.

Theorem 4.5 *Let M be positively $\mathbf{a}$-determined. Then* $\operatorname{proj.dim}(M) = \operatorname{supp.reg}(M^{\mathbf{a}})$. *In particular, M is Cohen-Macaulay if and only if $M^{\mathbf{a}}$ has a support-linear free resolution.*

When $\mathbf{a} = \mathbf{1}$, the word "support" can be excised from the theorem. In particular, Theorem 4.1 is recovered as the special case of the next corollary in which M is the quotient of S by a squarefree monomial ideal.

Corollary 4.6 *Let $M \in \mathcal{M}_+^{\mathbf{1}}$ be squarefree. Then* $\operatorname{proj.dim}(M) = \operatorname{reg}(M^{\mathbf{1}})$. *In particular, M is Cohen-Macaulay if and only if $M^{\mathbf{1}}$ has a linear free resolution.*

The proof of the theorem will follow after some analysis of the codimensions of indecomposable summands in a minimal injective resolution.

Definition 4.7 *The* elevation *of an injective module I is* $\operatorname{elev}(I) := \max\{c \mid I \text{ has a summand of codimension } c\}$. *The* rise *of a complex $\mathbb{I} : 0 \to I^0 \to I^1 \to \cdots$ of injectives is* $\operatorname{rise}(\mathbb{I}) := \max_i\{\operatorname{elev}(I^i) - i\}$.

Proposition 4.8 *Let $N \in \mathcal{M}$ be finitely generated and $\mathbb{I}$ its minimal injective resolution. Then* $\operatorname{proj.dim}(N) = \operatorname{rise}(\mathbb{I})$, *and N is Cohen-Macaulay if and only if* $\operatorname{rise}(\mathbb{I}) = \operatorname{codim}(I^0)$.

Proof: Taking a $\mathbb{Z}^n$-shift of N if necessary (and choosing a large $\mathbf{a}$), we may assume $N \in \mathcal{M}_+^{\mathbf{a}}$. $\mathbb{I}$ has at least one summand isomorphic to $\underline{E}(k)$, and the lowest cohomological degree in which this occurs is $n - \operatorname{proj.dim}(N)$ by Theorem 3.23.5. Since $\underline{E}(k)$ has codimension n, we have $\operatorname{rise}(\mathbb{I}) \geq n - (n - \operatorname{proj.dim}(N)) = \operatorname{proj.dim}(N)$. Choose now a summand $\underline{E}(S/\mathfrak{m}^F)[\mathbf{b}]$ of $\mathbb{I}$ in cohomological degree i such that $\operatorname{rise}(\mathbb{I}) = |F| - i$. Applying Proposition 3.20 and subsequently Theorem 3.23.5 over $S_{[F]}$, we find that $|F| - i \leq \operatorname{proj.dim}_{S_{[F]}}(N_{(F)})$, and this is obviously $\leq \operatorname{proj.dim}(N)$. Thus $\operatorname{rise}(\mathbb{I}) = \operatorname{proj.dim}(N)$. The statement about Cohen-Macaulayness follows because $\operatorname{codim}(N) = \operatorname{codim}(I^0)$. □

Example 4.9 If J is a generic monomial ideal, the injective resolution $\mathbb{I}$ of S/J is given by the boundary map on a triangulation Γ of a simplex on $x_1, \ldots, x_n$ (Figure 2.1, right). The dimension of the indecomposable summand corresponding to a face $G \in \Gamma$ is the number of $x_i \in G$. Since the boundary map on Γ removes one vertex at a time, $\operatorname{elev}(I^{j+1}) = \min\{n, \operatorname{elev}(I^j) + 1\}$. Thus $\operatorname{rise}(\mathbb{I}) = \operatorname{elev}(I^0)$, and $\operatorname{proj.dim}(S/J)$ is the largest codimension of an associated prime of S/J; see [Yan99] or [MSY00] for stronger results of this type. In particular, S/J is Cohen-Macaulay if and only if it is equidimensional [BPS98]. □

In view of Proposition 4.8, why introduce the notion of rise? The reason is because the notion of rise is what translates via duality for resolutions into support-regularity of the Alexander dual. Unfortunately, if $\mathbb{I}$ is a minimal injective resolution of $M \in \mathcal{M}_+^{\mathbf{a}}$, then $\operatorname{supp.reg}(M^{\mathbf{a}})$ comes not directly from the rise of $\mathbb{I}$, but from that of $\check{C}\,\mathbb{I}$. To make sense of $\operatorname{rise}(\check{C}\,\mathbb{I})$, we need to know that $\check{C}\,\mathbb{I}$ is still a complex of injective modules.

Lemma 4.10 *If N is an indecomposable injective, then $\check{C}N \cong N$ if the $\mathbb{Z}^n$-graded component $N_{\mathbf{0}}$ is nonzero; otherwise, $\check{C}N = 0$. Taking Čech hull of an injective module commutes with localization: $(\check{C}N)_{(F)} \cong \check{C}(N_{(F)})$, where the second Čech hull is over $S_{[F]}$.*

Proof: The first statement is immediate from the definitions (and is Matlis dual to the last statement of Lemma 3.3). The second is by Lemma 3.19 and Proposition 3.20. □

Proposition 4.11 $\operatorname{rise}(\mathbb{I}) = \operatorname{rise}(\check{C}\,\mathbb{I})$ *if $\mathbb{I}$ is a minimal injective resolution of $M \in \mathcal{M}_+^{\mathbf{a}}$.*

Proof: Choose a summand $\underline{E}\,(S/\mathfrak{m}^F)[-\mathbf{b}]$ of $\mathbb{I}$ in minimal cohomological degree i (and therefore with minimal F) such that $\operatorname{rise}(\mathbb{I}) = |F| - i$. By Lemma 4.10 and Proposition 3.20 $\operatorname{rise}(\mathbb{I}_{(F)}) = \operatorname{rise}(\mathbb{I}) \geq \operatorname{rise}(\check{C}\,\mathbb{I}) \geq \operatorname{rise}(\check{C}\,\mathbb{I}_{(F)})$. Therefore, by localizing at $\mathfrak{m}^F$ if necessary, we assume $F = \mathbf{1}$ and $\mathbf{b} \in \mathbb{Z}^n$. It is enough to show that $\mathbf{b}$ can be chosen $\succeq \mathbf{0}$, for then $\underline{E}\,(k)[-\mathbf{b}]$ survives the Čech hull. This is, by Theorem 3.23.5′ and the minimality of i, equivalent to showing that the minimal free resolution of M has a summand $S[-\mathbf{c}]$ in homological degree $\operatorname{proj.dim}(M)$ with $\mathbf{1} \preceq \mathbf{c} := \mathbf{b} + \mathbf{1}$.

Let $d = \operatorname{proj.dim}(M)$. The assumptions above have reduced to the case where $\operatorname{elev}(I^j) < j + d$ whenever $\operatorname{elev}(I^j) < n$. Using Propositions 3.20 and 4.8, this implies that $d > \operatorname{proj.dim}_{S_{[F]}}(M_{[F]})$ whenever $\mathbf{0} \preceq F \prec \mathbf{1}$. It follows that $\underline{\operatorname{Ext}}_S^d(M, S)$ has support on $\mathfrak{m}$, because its localization at every nonmaximal associated prime is zero (every associated prime is monomial). Suppose e is a generator for a summand $S[-\mathbf{c}]$ in homological degree d of the minimal free resolution $\mathbb{F}$ of M. Then the image in $\underline{\operatorname{Ext}}_S^d(M, S)$ of the dual basis vector e^* (for any basis) is nonzero and annihilated by some power of $\mathfrak{m}$. If $(m_1, \ldots, m_r)$ is the image of e (in some basis) under the differential of $\mathbb{F}$, then the annihilator of e^* is contained in the ideal generated by the m_i. In particular, every variable appears in at least one of the monomials $m_1, \ldots, m_r$. But M is $\mathbb{N}^n$-graded, so $\mathbb{F}$ is, as well, whence $\mathbf{x}^{\mathbf{c}}$ must be divisible by the least common multiple of the m_i. □

Proof of Theorem 4.5: Let $\mathbb{I}$ be a minimal injective resolution of $M \in \mathcal{M}_+^{\mathbf{a}}$ and $\mathbb{F} = P_{\mathbf{a}}(A_{\mathbf{a}}^{0,+}\mathbb{I})$ the minimal free resolution of $M^{\mathbf{a}}$ induced by Alexander duality (Theorem 3.23.1 and 3.23.4). In addition, suppose Λ is a monomial matrix for $\mathbb{I}$ and let $\Phi^{\preceq \mathbf{a}}$ be the induced matrix for $\mathbb{F}$ (by applying Theorem 3.23.1′ and 3.23.4′). The submatrix $\check{\Lambda}$ of Λ whose labels do not have any coordinates equal to 1 is the induced matrix for $\check{C}\,\mathbb{I}$ by Lemma 4.10. Therefore, applying the relabelling function in Theorem 3.23.1′ to $\check{\Lambda}$ yields the same monomial matrix $\Phi^{\preceq \mathbf{a}}$. In other words,

Lemma 4.12 $\mathbb{F} = A_{\mathbf{a}}^{0,+}(\check{C}\,\mathbb{I})$ *is the minimal free resolution of $M^{\mathbf{a}}$.*

It follows that the support of the degree label on a summand in $\mathbb{F}$ is the support of the integer part of the label of the corresponding summand in $\check{C}\mathbb{I}$. Since the support size of the integer part is the codimension, the theorem follows from Propositions 4.8 and 4.11. □

Proposition 4.11 says that the elevations in a minimal injective resolution $\mathbb{I}$ of a well-behaved (e.g. Cohen-Macaulay) module do not rise too quickly. However, it does *not* rule out the possibility of some nonzero component λ_{pq} in $\mathbb{I}$ mapping a low-codimension summand to a high-codimension summand. Let's make this precise.

Definition 4.13 *The* jump *of a homomorphism* $\lambda : I \to I'$ *of indecomposable injectives is* $\mathrm{jump}(\lambda) = \mathrm{codim}(I') - \mathrm{codim}(I)$ *if* $\lambda \neq 0$. *If* $\lambda = 0$ *then set* $\mathrm{jump}(\lambda) = -1$.

For instance, Corollary 3.29 says that the components of $\mathbb{I}$ having jump zero are reasonably well-understood, inasmuch as free resolutions have been studied thoroughly in the literature. But what about the components of jump > 0?

Question 4.14 What determines the jumps in components of a minimal injective resolution of a finitely generated $\mathbb{Z}^n$-graded module?

The question makes sense also for $\mathbb{Z}$-graded or non-graded modules. But for $\mathbb{Z}^n$-graded modules we already have ways of analyzing some of the jumps in the minimal injective resolution $\mathbb{I}$. For example, suppose that $M \in \mathcal{M}_+^{\mathbf{1}}$ is squarefree with matrix Λ representing $\mathbb{I}$. Lemma 3.25 says that all of the labels in Λ are in $\{1, 0, *\}^n$. Furthermore, Lemma 4.10 says that $\check{C}\,\mathbb{I}$ is the $\mathbf{1}$-determined part of $\mathbb{I}$, taking the submatrix with labels in $\{0, *\}^n$. By Lemma 4.12, the jump between any two such summands is the ($\mathbb{Z}$-graded) degree of the corresponding map in the dual free resolution of $M^{\mathbf{1}}$ from Theorem 3.23.2. Therefore, if M is Cohen-Macaulay, the jumps between $\mathbf{1}$-determined summands in Λ are all ≤ 1 by Corollary 4.6. It is not true, however, that all of the jumps in Λ are ≤ 1, even if M is Cohen-Macaulay.

Example 4.15 A counterexample is given by $M = k[t, x, y, z]/\langle xyz, tx, ty\rangle$, which is Cohen-Macaulay, but whose minimal injective resolution has a component Λ_{pq} in a matrix for $I^1 \to I^2$ with $\mathbf{b}_{p\cdot} = (0, *, *, 1)$ and $\mathbf{b}_{\cdot q} = (0, 0, 0, 1)$. We have $\mathrm{jump}(\Lambda_{pq}) = 4 - 2 = 2$. This calculation can be (and was) done by computer [CNR99], using Theorem 3.23. □

4.2 Duality for Bass and Betti numbers

One of the features in the recent literature on Alexander duality for Stanley-Reisner rings is a collection of equalities and inequalities between Betti numbers of an ideal its dual; e.g. see [BCP99]. The proofs always involved calculating the Betti numbers as Betti numbers of certain simplicial complexes and getting relations among them using topological methods. Thus these proofs worked by comparing some homology or cohomology. The aim here is twofold: first, to demonstrate that Alexander duality is better expressed using *Bass numbers* (Definition 4.16) in addition to Betti numbers; and second, to show how all of the relations between Betti and Bass numbers are really residual data from duality for resolutions. In particular, each of the equalities below follows from the equivalence of some pair of *complexes*, so that, of course, the numerical information must be the same.

Definition 4.16 *Suppose* $0 \to I^0 \to I^1 \to \cdots$ *is a minimal injective resolution of* $M \in \mathcal{M}$. *The* i^{th} Bass number at F (or $\mathfrak{m}^F$) in degree $\mathbf{b} \cdot F \in \mathbb{Z}^F$ *is the number* $\mu_{i,\mathbf{b}}(F, M)$ *of summands of* I^i *that are isomorphic to* $\underline{E}(S/\mathfrak{m}^F)[-\mathbf{b}]$. *Set* $\mu_{i,\mathbf{b}}(M) = \mu_{i,\mathbf{b}}(\mathbf{1}, M)$, *the Bass number of* M *at* $\mathfrak{m}$ *in degree* $\mathbf{b}$ *if* $\mathbf{b} \in \mathbb{Z}^n$.

Note that $\underline{E}(S/\mathfrak{m}^F)[-\mathbf{b}]$ corresponds to a label $-\mathbf{b}$ (not $\mathbf{b}$) in a monomial matrix. The above definition of Bass numbers at monomial primes agrees with the usual definition in terms of certain $\underline{\mathrm{Ext}}$ functors.

Proposition 4.17 ([GW78, Theorem 1.3.4]) $\mu_{i,\mathbf{b}}(F,N) = \dim_k(\underline{\mathrm{Ext}}^i_{S(F)}(k, N_{(F)})_\mathbf{b})$.

When summed over $\mathbf{b}$, these Bass numbers also agree with the usual (ungraded) Bass numbers at $\mathfrak{m}^F$. Furthermore, they determine the ungraded Bass numbers at all primes [GW78, Theorem 1.2.3].

The next corollary is a first indication of how duality for resolutions "explains" equalities between Bass and Betti numbers. Though it is not new, it will be needed in the proof of Theorem 4.19.

Corollary 4.18 *For any finitely generated $M \in \mathcal{M}$, $\beta_{n-i,\mathbf{b}}(M) = \mu_{i,\mathbf{b}-\mathbf{1}}(M)$ for all $i \in \mathbb{Z}$ and $\mathbf{b} \in \mathbb{Z}^n$.*

Proof: Shifting degrees and choosing $\mathbf{a}$ large enough, we can assume that M is positively $\mathbf{a}$-determined. The result now follows from Theorem 3.23.5 and 3.23.5′. □

The next theorem is the main result of the section, and should be thought of as the quantitative translation of duality for resolutions. Recall from Definition 1.1 that $\mathbf{a} \smallsetminus \mathbf{b}$ is chosen to make $\langle \mathbf{x}^\mathbf{b} \rangle^{[\mathbf{a}]} = \mathfrak{m}^{\mathbf{a} \smallsetminus \mathbf{b}}$.

Theorem 4.19 *Suppose that $\mathbf{0} \preceq F \preceq \mathbf{b} \preceq \mathbf{a} \cdot F$, i.e. $F = \mathrm{supp}(\mathbf{b})$ and $\mathbf{0} \preceq \mathbf{b} \preceq \mathbf{a}$. Then:*

1. $\beta_{i,\mathbf{b}}(M^\mathbf{a}) = \mu_{i,(\mathbf{a} \smallsetminus \mathbf{b})-F}(F,M)$,
2. $\mu_{n-i,\mathbf{b}-\mathbf{1}}(M^\mathbf{a}) = \mu_{i,(\mathbf{a} \smallsetminus \mathbf{b})-F}(F,M)$, and
3. $\beta_{i,\mathbf{b}}(M^\mathbf{a}) = \beta_{|F|-i,\,\mathbf{a} \smallsetminus \mathbf{b}}(M_{(F)})$.

Of course, the pair $(M, M^\mathbf{a})$ can be switched, as can $(\mathbf{b}, \mathbf{a} \smallsetminus \mathbf{b})$, $(i, n-i)$, and $(i, |F|-i)$.

Proof: Suppose Φ represents a minimal free resolution of $B_\mathbf{a} M^\mathbf{a}$, and let Λ be the matrix output from Theorem 3.23.2′. Each label $\mathbf{b}$ satisfying $F \preceq \mathbf{b} \preceq \mathbf{a} \cdot F$ in the matrix $\Phi^{\preceq \mathbf{a}}$ from Theorem 3.23.4′ becomes a label $*\overline{F} + \mathbf{b} - (\mathbf{a} \cdot F)$ in Λ. Such a label in Λ contributes to the Bass number $\mu_{i,(\mathbf{a}\cdot F)-\mathbf{b}}(F,M)$, so the equality $(\mathbf{a} \cdot F) - \mathbf{b} = (\mathbf{a} \smallsetminus \mathbf{b}) - F$ proves part 1. Part 2 follows from part 1 by Corollary 4.18. Finally, the right-hand sides of parts 1 and 3 are equal by the matrix part of Corollary 3.29, so part 3 follows from part 1. □

Remark 4.20 There is an even more comprehensive equality like Theorem 4.19.2 involving *every* Bass number of both M and $M^\mathbf{a}$. It comes from the equivalence of Corollary 3.27, and is best read off the relabelling function given there.

Example 4.21 The previous theorem and remark are so transparent for generic and cogeneric monomial ideals that almost nothing is going on. The only thing we're doing is taking the same faces of a simplicial complex (Figures 2.1 and 2.2) and putting different $\mathbb{Z}^n_*$ labels on them with different homological degrees, sometimes with β, and sometimes with μ. A similar observation holds for any minimal cellular or cocellular injective resolution. □

Remark 4.22 When $M = S/I$ is the quotient by an arbitrary monomial ideal, Theorem 4.19.3 is the generalization of the formulas in [ER98, Proposition 1] and [BCP99, Theorem 2.4] for squarefree ideals. Indeed, the links which appear in their formulas are the result of localization [Hoc77, Proposition 5.6], which occurs in Theorem 4.19 when the Bass numbers are computed at nonmaximal primes (Proposition 4.17).

No matter which sufficiently large $\mathbf{a}$ is chosen, the list of Betti numbers of $M^{\mathbf{a}}$ is just the list of (localized) Bass numbers of M by Theorem 4.19.1. Thus the collection of modules that are Alexander dual to M are very closely related homologically. Of course, one could see this already from Theorem 3.23.

Theorem 4.19, particularly part (1), can be thought of as a broad generalization of the fact that the generators of a monomial ideal are in bijection with the irreducible components of the Alexander dual, as in Definition 1.2. Having gone through a great deal of functorial rigamarole to get here, let's take a moment to see the connection. It is well-known that the generators of a monomial ideal I correspond to the zeroth Betti numbers of I. On the other hand, it is less well-known that the irreducible components of I correspond to the zeroth Bass numbers of S/I.

Proposition 4.23 *Given an ideal $I \subseteq S$ the following are equivalent for $\mathbf{b} \in \mathbb{Z}^F$:*

- (i) $\mathfrak{m}^{\mathbf{b}}$ is an irredundant irreducible component of I.
- (ii) $\mu_{0,\mathbf{b}-F}(F, S/I) = 1$.
- (iii) $\mu_{0,\mathbf{b}-F}(F, S/I) \neq 0$.

Proof: (i) $\Rightarrow$ (ii) because $\mathfrak{m}^{\mathbf{b}}$ corresponds to a minimal generator $\mathbf{x}^{\mathbf{a}\smallsetminus\mathbf{b}}$ of $I^{[\mathbf{a}]}$ (Theorem 1.7), and this corresponds to a Betti number $\beta_{i,\mathbf{a}\smallsetminus\mathbf{b}}(I^{[\mathbf{a}]}) = \mu_{i,\mathbf{b}-F}(F, S/I)$ by Theorem 4.19. (ii) $\Rightarrow$ (iii) is trivial. (iii) $\Rightarrow$ (i) can be argued as in (i) $\Rightarrow$ (ii) but backwards, as long as $\mu_{i,\mathbf{b}-F}(F, S/I)$ is zero unless $F \preceq \mathbf{b} \preceq \mathbf{a} \cdot F$. To show this, use the injection $S/I \hookrightarrow \bigoplus_{\mathbf{c}\in C} S/\mathfrak{m}^{\mathbf{c}}$, where $I = \bigcap_{\mathbf{c}\in C} \mathfrak{m}^{\mathbf{c}}$ is the irredundant decomposition of I. The initial terms of the long exact sequences for $\underline{\mathrm{Ext}}$ coming from Proposition 4.17 (with various F) imply that $\mu_{0,\mathbf{b}-F}(F, S/I) \leq \sum_{\mathbf{c}\in C} \mu_{0,\mathbf{b}-F}(S/\mathfrak{m}^{\mathbf{c}})$. But $S/\mathfrak{m}^{\mathbf{c}}$ has only one zeroth Bass number that is nonzero: $\mu_{0,\mathbf{c}-G}(G, S/\mathfrak{m}^{\mathbf{c}}) = 1$, where $G = \mathrm{supp}(\mathbf{c})$. □

Remark 4.24 Proposition 4.23 and its proof imply that the map $S/I \hookrightarrow \bigoplus_{\mathbf{c}\in C} S/\mathfrak{m}^{\mathbf{c}}$ induces an isomorphism on their injective hulls. This can be shown directly, without resorting to Alexander duality, if one is willing to rely more heavily on localization and Proposition 4.17; see [Mil98, Proposition 4.12].

Theorem 4.19 can be used to prove the following Betti-Betti inequality theorem, which generalizes to positively $\mathbf{a}$-determined modules an inequality [BCP99, Corollary 2.6] for squarefree ideals. The topological argument involving links employed there is preempted here by a simple algebraic observation involving localization (which gives links in the squarefree case [Hoc77, Proposition 5.6]). The result will be applied in the proof of Theorem 5.55.

Theorem 4.25 *If $\mathbf{0} \preceq F \preceq \mathbf{b} \preceq \mathbf{a} \cdot F$ and M is positively $\mathbf{a}$-determined, then*

$$\beta_{i,\mathbf{b}}(M^{\mathbf{a}}) \quad \leq \quad \sum_{\mathbf{c}\cdot F = \mathbf{a}\smallsetminus\mathbf{b}} \beta_{|F|-i,\mathbf{c}}(M).$$

Of course, M and $M^{\mathbf{a}}$ can be switched, as can $\mathbf{b}$ and $\mathbf{a} \smallsetminus \mathbf{b}$, or i and $|F| - i$.

Proof: Note that imposing the given condition on $\mathbf{b}$ is equivalent to imposing it on $\mathbf{a} \setminus \mathbf{b}$. Since $(\mathbf{a} \setminus \mathbf{b}) \cdot F = \mathbf{a} \setminus \mathbf{b}$, the sum on the right is $\geq \beta_{|F|-i,\mathbf{a}\setminus\mathbf{b}}(M_{(F)})$ by Proposition 3.22, with equality if and only if the matrix $\Lambda_{(F)}$ there is minimal. Now use Theorem 4.19.3. $\square$

The aforementioned inequality of [BCP99] is the next corollary when $M = I$ is a squarefree monomial ideal.

Corollary 4.26 *Let $M \in \mathcal{M}_+^{\mathbf{1}}$ be a squarefree module. Then*

$$\beta_{i,\mathbf{b}}(M^{\mathbf{1}}) \quad \leq \quad \sum_{\mathbf{b} \preceq \mathbf{c} \preceq \mathbf{1}} \beta_{|\mathbf{b}|-i,\mathbf{c}}(M).$$

Of course, M and $M^{\mathbf{1}}$ can be switched, as can i and $|\mathbf{b}| - i$.

Proof: $\mathbf{1} \setminus \mathbf{b} = \mathbf{b} = \operatorname{supp}(\mathbf{b})$ whenever $\mathbf{0} \preceq \mathbf{b} \preceq \mathbf{1}$. Furthermore, $\mathbf{c} \cdot \mathbf{b} = \mathbf{b}$ implies that $\mathbf{c} \succeq \mathbf{b}$. Finally, $\beta_{|\mathbf{b}|-i,\mathbf{c}}(M) = 0$ unless $\mathbf{c} \preceq \mathbf{1}$ by Corollary 3.24. $\square$

Following [BCP99], again, Corollary 4.26 implies that certain Betti numbers of a squarefree module M are equal to those of its Alexander dual $M^{\mathbf{1}}$. To state the result, the following definition is required.

Definition 4.27 *Let $M \in \mathcal{M}_+^{\mathbf{1}}$ be a squarefree module. A Betti number $\beta_{i,\mathbf{b}}(M)$ is called* i-extremal *if $\beta_{i,\mathbf{c}}(M) = 0$ for $\mathbf{c} \succ \mathbf{b}$. Define $\beta_{i,\mathbf{b}}(M)$ to be* extremal *if $\beta_{j,\mathbf{c}}(M) = 0$ whenever $j \geq i$, $\mathbf{c} \succ \mathbf{b}$, and $|\mathbf{c}| - |\mathbf{b}| \geq j - i$, where $|\mathbf{b}| := \sum_{k=1}^{n} b_k$ for $\mathbf{b} \in \mathbb{N}^n$.*

Proposition 4.28 *Let $M \in \mathcal{M}_+^{\mathbf{1}}$ be squarefree. If $\beta_{i,\mathbf{b}}(M^{\mathbf{1}})$ is i-extremal, then $\beta_{i,\mathbf{b}}(M^{\mathbf{1}}) \geq \beta_{|\mathbf{b}|-i,\mathbf{b}}(M)$. If, furthermore, $\beta_{i,\mathbf{b}}(M^{\mathbf{1}})$ is extremal, then $\beta_{i,\mathbf{b}}(M^{\mathbf{1}}) = \beta_{|\mathbf{b}|-i,\mathbf{b}}(M)$.*

Proof: The proof of [BCP99, Theorem 2.8] works here, since they use only the special case [BCP99, Corollary 2.6] of Corollary 4.26. $\square$

This provides another proof of Corollary 4.6. Indeed, the proof of [BCP99, Corollary 2.9] again works here since they use only the special case [BCP99, Theorem 2.8] of Proposition 4.28. Römer [Röm99, Theorem 4.6] also proved the equality of extremal Betti numbers.

4.3 LCM-lattices

Gasharov, Peeva, and Welker [GPW00] introduced the *lcm-lattice* of a monomial ideal $I = \langle m_1, \ldots, m_r \rangle$:

$$L_I := \{\operatorname{lcm}(m_i \in A) \mid A \subseteq \{m_1, \ldots, m_r\}\}.$$

L_I is a join lattice (with atoms $m_1, \ldots, m_r$) inside of $\mathbb{N}^n$, and carries homological information about I in much the same way that the intersection lattice of a subspace arrangement carries homological information about the complement of the arrangement. In particular, one of their main results [GPW00, Theorem 3.3] is that L_I determines the minimal free resolution of S/I, in the following sense.

Theorem 4.29 (Gasharov-Peeva-Welker) *Let I and I' be two monomial ideals in polynomial rings S and S'. All of the nonzero Betti numbers of S/I occur in degrees appearing in L_I. Let $g : L_I \to L_{I'}$ be a map which is a bijection on the atoms and preserves joins. If each label $\mathbf{b}$ in a monomial matrix for the minimal free resolution of S/I is replaced by $g(\mathbf{b})$, then the result is a monomial matrix for a free resolution of S'/I'. If g is an isomorphism, then the relabelled monomial matrix is minimal.*

A natural question to ask is, "what determines the minimal injective resolution?" Duality for resolutions produces the following response. Recall $\mathfrak{m}^{\mathbf{a}+\mathbf{1}} = \langle x_1^{a_1+1}, \ldots, x_n^{a_n+1}\rangle$.

Corollary 4.30 *Let I be a monomial ideal generated in degrees $\preceq \mathbf{a}$. Then $L_{I+\mathfrak{m}^{\mathbf{a}+\mathbf{1}}}$ determines the minimal injective resolution of S/I and the minimal injective resolution of the Alexander dual ideal $I^{[\mathbf{a}]} = (S/I)^{\mathbf{a}}$. In particular, $L_{I+\mathfrak{m}^{\mathbf{a}+\mathbf{1}}}$ determines the minimal free resolutions of both S/I and $S/I^{[\mathbf{a}]}$.*

Proof: The lcm-lattice of $I+\mathfrak{m}^{\mathbf{a}+\mathbf{1}}$ determines the minimal free resolution of $S/(I+\mathfrak{m}^{\mathbf{a}+\mathbf{1}}) = B_{\mathbf{a}}(S/I)$ by Theorem 4.29. The rest follows from Corollary 3.27 and Theorem 3.23. □

The lattices $L_{I+\mathfrak{m}^{\mathbf{a}+\mathbf{1}}}$ and $L_{I^{[\mathbf{a}]}+\mathfrak{m}^{\mathbf{a}+\mathbf{1}}}$ are not very closely related as combinatorial objects. Their relation only emerges when their homology is compared. Therefore, the fact that $L_{I+\mathfrak{m}^{\mathbf{a}+\mathbf{1}}}$ determines the injective resolution of $I^{[\mathbf{a}]}$ is somewhat startling.

Remark 4.31 From the point of view of Proposition 4.23, the injective resolution of $S/I^{[\mathbf{a}]}$ is better than the injective resolution of $I^{[\mathbf{a}]}$. Fortunately, they look pretty much alike. Indeed, the minimal injective resolution of S is the generalized Alexander dual of the Koszul complex (Example 2.20) and has one summand $\underline{E}(S/\mathfrak{m}^F)[F]$ for each $F \subseteq \{1, \ldots, n\}$. Using the exact sequence $0 \to I^{[\mathbf{a}]} \to S \to S/I^{[\mathbf{a}]} \to 0$ and the characterization of Bass numbers in terms of $\underline{\mathrm{Ext}}$ functors (Proposition 4.17), one checks with the long exact sequence for $\underline{\mathrm{Ext}}$ that the only differences between the minimal injective resolutions of $I^{[\mathbf{a}]}$ and $S/I^{[\mathbf{a}]}$, besides a homological shift of 1, come from the indecomposable summands in the injective resolution of S. These get attached either to $S/I^{[\mathbf{a}]}$ or $I^{[\mathbf{a}]}$ according to whether $\mathfrak{m}^F$ does or does not contain $I^{[\mathbf{a}]}$.

Remark 4.32 If $S = S'$, the replacement g in Theorem 4.29 can be viewed as a relabelling function in the sense of Definition 3.12 by letting it be arbitrary on $\mathbb{Z}^n_* \setminus L_I$. But even if $S = S'$, the map $g : L_I \to \mathbb{Z}^n$ is not necessarily induced by a map $\mathbb{Z}^n \to \mathbb{Z}^n$ that is a product of maps $\mathbb{Z} \to \mathbb{Z}$, so it won't have a nice notation as in Definition 3.12.3.

The condition of $I + \mathfrak{m}^{\mathbf{a}+\mathbf{1}}$ and $I' + (\mathfrak{m}')^{\mathbf{a}'+\mathbf{1}'}$ having isomorphic lcm-lattices is much stronger than I and I' having isomorphic lcm-lattices (see Example 4.33). This again comes back to the wealth of information carried by minimal injective resolutions, such as irreducible decompositions and minimal free resolutions over all localizations (Corollary 3.29). A reflection of this is given in Theorem 5.43, where the condition of being a *generic monomial ideal* is shown to be equivalent to having the minimal injective resolution stay constant under deformation of the exponent vectors in the minimal generators of I. In essence, the information missing from the minimal free resolution allows it to stay constant under more lax conditions (Remark 5.45.2), whereas the minimal injective resolution is more rigid.

Example 4.33 Let $n = 3$, with $I = \langle x^2z^2, xyz, y^2z^2\rangle$ and $I' = \langle x^2, xyz, y^2z^2\rangle$. It is easily checked that the lcm-lattices L_I and $L_{I'}$ are isomorphic, although $L_{I+\langle x^3,y^3,z^3\rangle} \not\cong L_{I'+\langle x^3,y^3,z^3\rangle}$. In fact, S/I has 5 irreducible components, while S/I' has only 4. It follows that the injective hulls of S/I and S/I' have different numbers of indecomposable summands; see Proposition 4.23. □

4.4 Equivariant K-theory: the Alexander inversion formula

It is well-known that monomial ideals define the subschemes of projective space which are invariant under the action of the n-dimensional torus scaling the axes. This accounts, for instance, for the geometric interpretation of Gröbner bases via Hilbert schemes [Bay82]. Using this point of view, it is possible to compute or bound various kinds of cohomology of arbitrary subschemes of projective space by deforming the problem into one about monomial ideals. As an example, this has been done (and is classical) for Hilbert functions and degrees of generators [Mac27], and more recently for higher syzygies, as well [Big93, Hul93]. The main point is that the Hilbert function and polynomial remain unchanged by flat deformation. One can also interpret this invariance as the preservation of the $K_\circ$-theory class of the subscheme, and this is where we take our starting point for this section. [The observations here were made in connection with an ongoing joint project with Allen Knutson, and have practical applications to equivariant K-theory of flag manifolds.]

The $K_\circ$-group ("K-homology") of a projective scheme is the group generated by the isomorphism classes of coherent sheaves E, modulo the relations $\{E = E' + E''$ if $0 \to E' \to E \to E'' \to 0$ is an exact sequence$\}$; see [Ful98, Chapter 15] for a formal introduction. A subscheme defines the class of its structure sheaf. In the presence of a torus action on S however, we are interested not in the ordinary $K_\circ$-class of an invariant subscheme of projective space, but in the *equivariant* $K_\circ^T$-class of an invariant subscheme of *affine* space; the definition remains unchanged except that the sheaves must be torus-equivariant.

$K_\circ$ is only one of the possible K-groups. Frequently one encounters the group K° ("K-cohomology"), which has the same definition as $K_\circ$ except that only vector bundles E are allowed. There is a natural map $K^\circ \to K_\circ$ which associates to each vector bundle its sheaf of sections, and in nice cases this map is an isomorphism. The same generalities apply to the equivariant situation, so that torus-equivariant vector bundles generate K_T°. It is a fact that $K_T^\circ \cong K_\circ^T$ for affine space. Moreover, it is not hard to see why: sheaves on affine space are just modules over the polynomial ring, and torus-equivariant means $\mathbb{Z}^n$-graded. The existence of finite $\mathbb{Z}^n$-graded free resolutions guarantees that the $K_\circ^T$-class of every $\mathbb{Z}^n$-graded module is a finite alternating sum of $K_\circ^T$-classes of torus-invariant free modules, and is thus in the image of $K_T^\circ \to K_\circ^T$.

The free modules, in turn, are direct sums of rank-one free modules, and these form a basis for the $\mathbb{Z}$-module K_T°. In particular, the $K_\circ^T$-class of the structure sheaf of every invariant subscheme (i.e. monomial subscheme) of affine space is a $\mathbb{Z}$-linear combination of classes of principal monomial ideals. Which linear combination? Letting the K_T°-class of a principal ideal $\langle \mathbf{x}^{\mathbf{b}}\rangle$ be denoted by $\mathbf{x}^{\mathbf{b}}$, the K_T°-class of S/I for a monomial ideal I is the $\mathbb{N}^n$-graded Euler characteristic of the Taylor complex of I (page 30). More generally:

Proposition 4.34 *The K_T°-class of a $\mathbb{Z}^n$-graded module is the numerator of its $\mathbb{Z}^n$-graded Hilbert series, where the denominator is always assumed to be $\prod_{i=1}^n(1-x_i)$.*

Thus the group of $K_\circ^T$ classes is the group generated by $\mathbb{Z}^n$-graded Hilbert series, under addition. Note that this group only includes very special kinds of Laurent series over $\mathbb{Z}$.

Just as the positive part of K_T° (i.e. the Hilbert series that are actually power series and not just Laurent series) has a basis consisting principal monomial ideals, the positive part of $K_\circ^T$ has a basis consisting of the quotients S/I for irreducible monomial ideals I. So how does one figure out the $K_\circ^T$-class of an $\mathbb{N}^n$-graded module? Alexander duality functors! Given a finitely generated $\mathbb{N}^n$-graded module M, choose any $\mathbf{a}$ for which $M \in \mathcal{M}_+^{\mathbf{a}}$, and calculate a free resolution $\mathbb{F}_{\bullet}$ of the Alexander dual $M^{\mathbf{a}}$. Then the Alexander dual $\mathbb{F}^{\mathbf{a}}$ of this free resolution will be a resolution of M by irreducible quotients S/I, whose alternating sum is the class of M in $K_\circ^T$. This class is always represented in the desired basis, and is therefore independent of the choice of $\mathbf{a}$ (in essence, the $\mathbf{a}$ we used to take $M^{\mathbf{a}}$ cancels with the $\mathbf{a}$ used to take $\mathbb{F}^{\mathbf{a}}$). This procedure can be rephrased as follows.

Theorem 4.35 *the $K_\circ^T$-class of M is obtained from the numerator of the Hilbert series of $M^{\mathbf{a}}$ by replacing each monomial $\mathbf{x}^{\mathbf{b}}$ with the basis element $\langle \mathbf{x}^{\mathbf{b}}\rangle^{\mathbf{a}} = S/\mathfrak{m}^{\mathbf{a}\smallsetminus\mathbf{b}}$ of $K_\circ^T$.*

Said yet another way, for any fixed $\mathbf{a}$ the functor $(-)^{\mathbf{a}}$ can be viewed as a transition matrix taking the basis $\{\langle \mathbf{x}^{\mathbf{b}}\rangle \mid \mathbf{0} \preceq \mathbf{b} \preceq \mathbf{a}\}$ for the positively $\mathbf{a}$-determined part of K_T° to the basis $\{S/\mathfrak{m}^{\mathbf{a}\smallsetminus\mathbf{b}} \mid \mathbf{0} \preceq \mathbf{b} \preceq \mathbf{a}\}$ for the positively $\mathbf{a}$-determined part of $K_\circ^T$. Representing $S/\mathfrak{m}^{\mathbf{a}\smallsetminus\mathbf{b}}$ by the monomial $\mathbf{x}^{\mathbf{a}\smallsetminus\mathbf{b}}$, this change of basis takes {polynomials over $\mathbb{Z}$ whose terms divide $\mathbf{x}^{\mathbf{a}}$} to itself. In the case of $\mathbf{a} = \mathbf{1}$, the change of basis is particularly nice, and has a functional equation given by Theorem 4.36: if $P(\mathbf{x})$ is the K_T°-class of a squarefree (i.e. positively $\mathbf{1}$-determined) module M, then $P(\mathbf{1}-\mathbf{x})$ is the $K_\circ^T$-class of M. The right-hand side in the inversion formula takes care of the denominator polynomials, which are unfounded from the point of view of K-theory. In the theorem, $H(M;\mathbf{x})$ denotes the $\mathbb{N}^n$-graded Hilbert series of M, and $\omega_S = S[-\mathbf{1}]$ is the canonical module of S.

Theorem 4.36 *If $M \in \mathcal{M}_+^{\mathbf{1}}$ is a squarefree module then $H(M^{\mathbf{1}};\mathbf{x}) = H(M;\mathbf{1}-\mathbf{x})H(\omega_S;\mathbf{x})$. This can be restated as the* Alexander inversion formula*:*

$$\frac{H(M^{\mathbf{1}};\mathbf{x})}{H(M;\mathbf{1}-\mathbf{x})} = \prod_{x\in\mathbf{x}} \frac{x}{1-x}\,.$$

Proof: As a function of $M^{\mathbf{1}}$, the series $H(M;\mathbf{1}-\mathbf{x})H(\omega_S;\mathbf{x})$ is additive on short exact sequences $0 \to (M')^{\mathbf{1}} \to M^{\mathbf{1}} \to (M'')^{\mathbf{1}} \to 0$ because Alexander duality is exact. Since M has a finite free resolution, it is enough to check the inversion formula when M is free (and generated in a squarefree degree), and this is easy. □

To get a feeling for the simplification afforded by functoriality and $\mathbb{Z}^n$-grading, compare the statement and proof of [Ter97, Lemma 2.3], which is basically the $\mathbb{Z}$-graded version of the above theorem in the case where M is a radical monomial ideal. It involves careful counting of faces and keeping track of various binomial coefficients. In particular, the Alexander inversion formula pinpoints the relation between the h-vector of a simplicial complex and that of its Alexander dual.

Chapter 5

Cellular resolutions

Labelled cell complexes provide compact vessels for recording the labels and scalar entries in $\mathbb{Z}^n$-graded resolutions of an ideal. Bayer and Sturmfels [BS98] introduce this notion in the context of free resolutions, but only for boundary operators of cell complexes and only for free resolutions. Chapter 2 has already generalized these to include coboundary operators and injective resolutions. What this chapter does is explore the effect of Alexander duality on cellular resolutions, in the spirit of [BPS98], [BS98], and [Stu99].

Theorem 3.23 becomes topological duality on cellular resolutions. After some rumination on the kinds of things possible, the concept of a geometrically defined resolution is broadened to include *relative cocellular free resolutions* (Definition 5.13). In addition, it is shown how irreducible decompositions are specified geometrically by cocellular resolutions (Theorem 5.16). A number of detailed and quite pictorial examples are presented, including permutohedron and tree ideals in Example 5.37.

The culmination of these generalities is a new canonical geometric resolution for monomial ideals called the *cohull resolution* (Definition 5.20), which is defined by applying Alexander duality to the hull resolution of [BS98]. As a special case, the coScarf resolution [Stu99] of a cogeneric monomial ideal is seen to be the cohull resolution (Theorem 5.47), and various numerical consequences result from its relative cocellular nature. Rigidity of injective resolutions is exploited (indirectly) in Section 5.3 to give multiple characterizations of generic ideals. The final Section 5.5 is occupied with the case of $n = 3$ variables, where every ideal has a minimal cellular injective resolution, and another (perhaps different) cocellular one (Theorem 5.60).

5.1 Duality and cellular resolutions

Throughout this chapter, X will denote a cell complex, and all cell complexes are assumed to be finite regular cell complexes (cf. Section 2.3).

Recall from Proposition 2.13 that a cellular monomial matrix supported on X (Definition 2.12) corresponds to a *labelling* of X, each face $G \in X$ having an attached vector $\mathbf{a}_G \in \mathbb{Z}^n_*$. In what follows, a (co)homological complex is called *acyclic* if it is exact everywhere except possibly at its lowest nonzero (co)homological degree.

Definition 5.1 *A* weakly cellular resolution *is an acyclic complex $\mathbb{F}_X$ of flat (resp. $\mathbb{I}^X$ of injective) modules with a given set of generators (resp. cogenerators) such that the monomial matrix induced by Theorem 2.9 is cellular, supported on a labelled cell complex X. The adjective "weakly" may be dropped if the label $\mathbf{a}_G$ on each face $G \in X$ equals the join $\bigvee_{v \in G} \mathbf{a}_v$ of the labels on all vertices $v \in G$.*

Observe that the lowest nonzero (co)homological degree of any complex represented by a cellular monomial matrix is indecomposable, because there is a unique face—the empty set $\varnothing$—of minimal dimension. Weakly cellular resolutions can therefore only resolve a quotient of a localization $S[\mathbf{x}^{-\overline{F}}][\mathbf{b}]$ (if the resolutions is flat) or a submodule of an injective module $\underline{E}(S/\mathfrak{m}^F)[\mathbf{b}]$ (if the resolution is injective). Furthermore, the quotient or submodule is determined by the labels on the vertices. The next lemma makes this precise in the two most important cases.

Lemma 5.2 *Let X be a labelled cell complex.*

1. *The flat complex $\mathbb{F}_X$ is free if and only if $\mathbf{a}_\varnothing \in \mathbb{Z}^n$. If $\mathbf{a}_\varnothing = \mathbf{0}$ and $\mathbb{F}_X$ is a weakly cellular resolution, it resolves $S/\langle \mathbf{x}^{\mathbf{a}_v} \mid v \in X \text{ is a vertex}\rangle$.*
2. *Suppose the injective complex $\mathbb{I}^X$ is a weakly cellular resolution of a nonzero module M. Then M is torsion-free if and only if $\mathbf{a}_\varnothing = *\mathbf{1}$. Given this condition, M is an ideal of S if and only if there is a vector $\mathbf{c} \preceq \mathbf{1}$ such that $\{c_i\mathbf{e}_i + *\overline{\mathbf{e}}_i \mid 1 \leq i \leq n\}$ are among the vertex labels of X.*

Proof: In part 1, $\mathbb{F}_X$ is free if and only if none of the face labels have any $*$ entries, which occurs if and only if $\mathbf{a}_\varnothing$ has no $*$ entries by Proposition 2.13. If $\mathbf{a}_\varnothing = \mathbf{0}$ then the nonzero map in smallest homological degree is $S \leftarrow \bigoplus_{\text{vertices } v \in X} S[-\mathbf{a}_v]$ whose cokernel is as desired.

Every nonzero submodule of every indecomposable injective is torsion, unless the indecomposable injective is $\underline{E}(S) = S[x_1^{-1}, \ldots, x_n^{-1}]$. In any case, the set of faces $X_{(F)} = \{G \in X \mid \mathrm{supp}((\mathbf{a}_G)_{\mathbb{Z}}) \subseteq F\}$ whose integral parts are contained in $F \subseteq \{1, \ldots, n\}$ is a subcomplex of X by Proposition 2.13. Furthermore, $X_{(F)}$ supports an injective resolution of $M_{(F)}$ by Proposition 3.20. Since M is a nonzero ideal if and only if $M_{(F)}$ is a nonzero ideal of $S_{(F)}$ for all $F \subseteq \{1, \ldots, n\}$, it is enough to check the last sentence of the Lemma when $S = k[x]$ has only one variable, and this case is easy. □

Example 5.3 Given that the top labelled cell complex in Figure 2.2 represents a (weakly) cellular injective resolution, the three corners give away the fact that it resolves an ideal. □

As mentioned in Remark 2.16, most (if not all) weakly cellular resolutions that are encountered "in nature" are actually cellular. This is due partially to the next theorem. Nevertheless, there are instances when it is useful not to have to check the join condition on the face labels. This occurs in Theorem 5.30, for instance (but see Conjecture 5.31).

Theorem 5.4 *A minimal weakly cellular free, flat, or injective resolution is cellular.*

Proof: By Matlis duality it is enough to prove this for the cellular flat complex $\mathbb{F}$ supported on a labelled cell complex X. For any $\mathbf{c} \in \mathbb{Z}^n$, let $\mathbf{c} \vee X$ be obtained from X by replacing $\mathbf{a}_G$ with $\mathbf{c} \vee \mathbf{a}_G$ for all faces $G \in X$. Then $\mathbf{c} \vee X$ still determines a cellular monomial matrix.

Furthermore, if $\mathbf{c}_0 \in \mathbb{Z}^n$ is any integer vector $\preceq (\mathbf{a}_G)_\mathbb{Z}$ for all $G \in X$, then one verifies easily that X satisfies the conclusion of the Theorem if and only if $\mathbf{c} \vee X$ does for all integer vectors $\mathbf{c} \preceq \mathbf{c}_0$. Denote by $\mathbf{c} \vee \mathbb{F}$ the complex of flat modules determined by $\mathbf{c} \vee X$.

Fixing $\mathbf{c} \preceq \mathbf{c}_0$, it is enough to show that the conclusion holds for $\mathbf{c} \vee X$. For $\mathbf{b} \in \mathbb{Z}^n$, the complex $\mathbb{F}_\mathbf{b}$ of k-vector spaces of $\mathbb{F}$ in degree $\mathbf{b}$ is the same as the complex $(\mathbf{c} \vee \mathbb{F})_\mathbf{b}$. Indeed, the complexes in question are both given by the *scalar* submatrix matrix whose labels are $\preceq \mathbf{b}$. In particular, $\mathbb{F}$ is acyclic implies that $\mathbf{c} \vee \mathbb{F}$ is acyclic. Shifting everything by $\mathbf{c} \vee \mathbf{a}_\varnothing$, we may assume that $\mathbf{c} \vee \mathbf{a}_\varnothing = \mathbf{0}$, so that $\mathbf{c} \vee \mathbb{F}$ resolves S/I for some ideal I by Lemma 5.2. To finish the Theorem we now prove the following:

> Suppose $(\mathbb{F}_X, \partial)$ represents a cellular free resolution of S/I. If $G \in X$ then $\mathbf{a}_G \succ \bigvee_{v \in G} \mathbf{a}_v$ implies $\mathbb{F}$ is not minimal.

This is vacuous if $\dim G = 0$, so assume G has minimal dimension (necessarily ≥ 1) among all faces satisfying $\mathbf{a}_G \succ \bigvee_{v \in G} \mathbf{a}_v$. Then we can fix an i such that $\mathbf{a}_G - \mathbf{e}_i \succeq \bigvee_{v \in G} \mathbf{a}_v$, whence $\partial(G) = x_i y$ for some $y \in \mathbb{F}_X$ because $\dim G$ is minimal. It follows that $x_i \partial(y) = \partial(x_i y) = 0$, whence $\partial(y) = 0$ because $\mathbb{F}_X$ is torsion-free. Thus $\partial(G) \in x_i \ker(\partial) \subseteq \mathfrak{m} \cdot \ker(\partial)$ does not represent a minimal generator of $\ker(\partial)$ by Nakayama's Lemma for graded modules. □

Remark 5.5 It is possible for a weakly cellular complex that is not cellular to be acyclic, although this doesn't occur in natural situations. Such an example can be artificially constructed starting with a natural but nonminimal cellular resolution. For instance, the complex labelled "hull(I)" in Figure 5.6 on page 72 is cellular but nonminimal (the labels on the faces are the joins of the labels on the vertices, so most of them have been omitted). This cellular complex can be made weakly cellular, without disturbing acyclicity, by changing the displayed facet and edge labels 411 to anything $\succ$ 411, e.g. 511, or 421, or 888.

When applied to cellular and cocellular resolutions, Alexander duality becomes topological duality, a transition already summarized in Example 3.28. To get a feel for what the precise statements might look like, the next couple of pages (until Definition 5.13) contain an informal presentation of some relevant examples and definitions; a full cellular restatement of Theorem 3.23 is avoided.

Definition 5.6 *Given a labelled regular cell complex X and a vector $\mathbf{b} \in \mathbb{Z}_*^n$, define the following collections of labelled faces of X:*

1. $X_{\preceq \mathbf{b}} = \{G \in X \mid \mathbf{a}_G \preceq \mathbf{b}\}$, *the* positively bounded *faces with respect to* $\mathbf{b}$
2. $X_{\not\succeq \mathbf{b}} = \{G \in X \mid \mathbf{a}_G \not\succeq \mathbf{b}\}$, *the* negatively unbounded *faces with respect to* $\mathbf{b}$
3. $X_\mathbb{Z} = \{G \in X \mid \mathbf{a}_G \in \mathbb{Z}^n\}$, *the* integer part *of* X

Let $X_U := X_{\not\succeq \mathbf{1}}$ *be simply* the negatively unbounded faces of X.

Example 5.7 Let X cell complex in Figure 2.2, with the labelling as in the top diagram. Then $X_{\preceq 11*}$ is the bottom edge, which supports a monomial matrix representing the injective resolution of the localization $J^{[444]}{}_{(110)}$ of $J^{[444]}$ at $\langle x, y\rangle$ by Theorem 3.30. □

If the labelling on X supports a cellular monomial matrix then both of these collections of faces $X_{\preceq \mathbf{b}}$ and $X_{\not\succeq \mathbf{b}}$ are subcomplexes of X by Proposition 2.13.

Proposition 5.8 *If $\mathbb{F}_X$ is a cellular resolution of a quotient of S, then $\mathbb{F}_{X_{\preceq \mathbf{a}}}$ is a cellular resolution of $S/\langle \mathbf{x}^{\mathbf{a}_G} \mid \mathbf{a}_G \preceq \mathbf{a}\rangle$. In particular, if $\mathbb{F}_X$ resolves $S/(I + \mathfrak{m}^{\mathbf{a}+\mathbf{1}})$ for some $\mathbf{a} \succeq \mathbf{a}_I$ then $\mathbb{F}_{X_{\preceq \mathbf{a}}}$ resolves S/I.*

Proof: $S[-\mathbf{b}]_{\mathbf{a}} \neq 0$ if and only if $\mathbf{b} \preceq \mathbf{a}$, so Lemma 3.3 with $L = \mathbb{F}_X$ implies that $P_{\mathbf{a}}(\mathbb{F}_X)$ is exact and equals $\mathbb{F}_{X_{\preceq \mathbf{a}}}$. That $\mathbb{F}_{X_{\preceq \mathbf{a}}}$ resolves $S/\langle \mathbf{x}^{\mathbf{a}_G} \mid \mathbf{a}_G \preceq \mathbf{a}\rangle$ is Lemma 5.2.1. □

Example 5.9 Let I be as in Example 1.15. The labelled complex X in Figure 5.1 is the *Scarf complex* of $I + \mathfrak{m}^{(5,6,6)}$ (see Example 2.15 or Section 5.3, below), and therefore $\mathbb{F}_X$ supports a cellular minimal free resolution of $S/(I + \mathfrak{m}^{(5,6,6)})$. The label "215" in the diagrams is short for $(2, 1, 5)$. The subcomplex $X_{\preceq 455}$, which is the Scarf complex of the ideal I itself, is also depicted in Figure 5.1; it resolves S/I by Proposition 5.8. □

Example 5.10 Let Y be the labelled complex at the left of Figure 2.1 and X the labelled complex at the bottom of Figure 2.2. The negatively unbounded faces X_U form the boundary of X, which is a subcomplex of X. In contrast, Y_U is the *complement* of a subcomplex of Y. In fact, $|Y \setminus Y_U| = |X_{\preceq 444}|$ is the thickened subcomplex, where $|\cdot|$ means the underlying unlabelled complex. □

Example 5.10 exhibits a couple of general phenomena. First, if X supports a cocellular monomial matrix, then X_U and $X_{\preceq \mathbf{b}}$ are not subcomplexes of X, but their complements are; this follows from Proposition 2.13. Second, if X supports a minimal (cellular *or* cocellular) free resolution of $B_{\mathbf{a}}(M)$, then the faces *not* in X_U are precisely those that contribute to the free resolution of $M^{\mathbf{a}}$; this can be seen by following through the relabelling functions in Theorem 3.23.3′ and Theorem 3.23.4′.

More to the point, when X supports a cocellular injective resolution of S/I (this uses the *chain* complex of $|X|$ for scalar entries), the integer part $|X_{\mathbb{Z}}|$ is a subcomplex of $|X|$. Therefore, applying Theorem 3.23.5′ says that $X_{\mathbb{Z}}$ may be relabelled to give a cellular (not cocellular) free resolution of S/I. What has been observed in Example 5.10 is how one can pick out this same subcomplex of $|X|$ by first relabelling X to be a cellular minimal free resolution of $S/(I + \mathfrak{m}^{\mathbf{a}+\mathbf{1}})$ and then taking $X_{\preceq \mathbf{a}}$ as in Theorem 3.23.1′. In other words, applying part 1, part 3, and then part 4 in Theorem 3.23 can be accomplished instead by simply applying part 5; it's helpful to see this in the diagram above Theorem 3.23.

Example 5.11 Compare integer part of the right labelled cell complex in Figure 2.1 with the bold subcomplex in the bottom of Figure 2.2. □

On the other hand, when X is labelled to support a *cellular* injective resolution of an ideal I (i.e. using the *cochain* complex of $|X|$ for scalar entries), the complement of $X_{\mathbb{Z}}$ is a subcomplex of $|X|$. Theorem 3.23 implies that the cellular pair $(|X|, |X \setminus X_{\mathbb{Z}}|)$ can also be expressed as $(|X'|, |X'_U|)$, where the labelling on X' is gotten from that on X by applying the relabelling function in Theorem 3.23.1′.

Example 5.12 The subcomplex X_U for the labelled cell complex X of Figure 5.1 is depicted in Figure 5.2, along with a representation of the relative cellular complex (X, X_U). However, each label on (X, X_U) has been subtracted from 566. Using this labelling, the

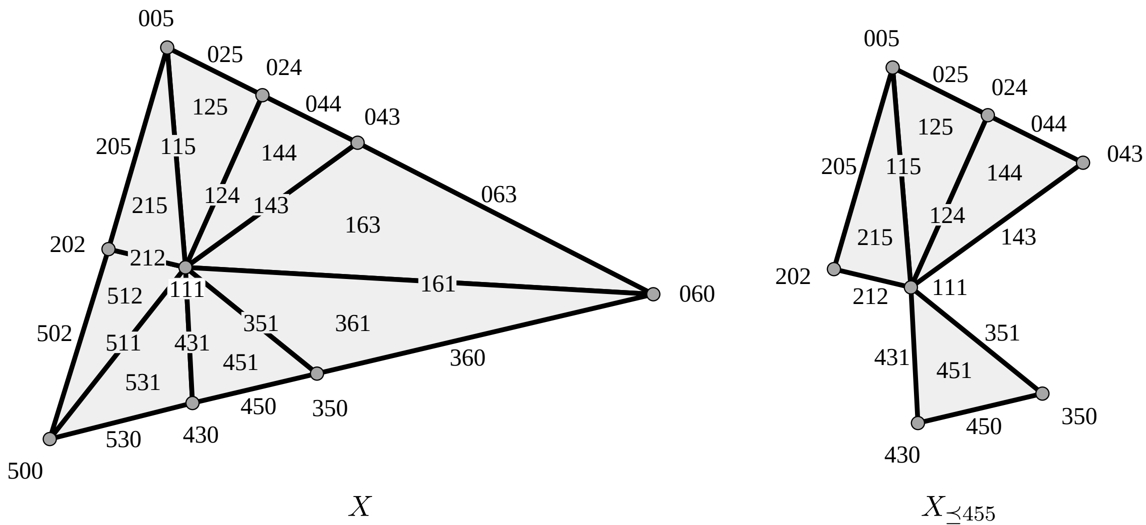
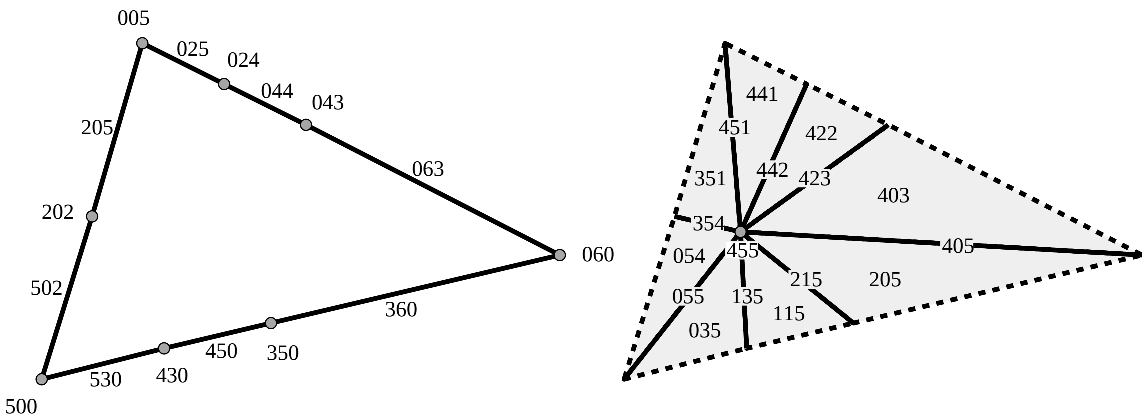


Figure 5.1: The positively bounded subcomplex from Example 5.9

* * * * * * * * *



$X_U = X_{\not\succeq \mathbf{1}}$ $\qquad$ (X, X_U) with labelling subtracted from 566

$$
\begin{array}{c}
\begin{array}{c|cccccccc}
 & 405 & 215 & 135 & 055 & 354 & 451 & 442 & 423 \\
\hline
205 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
115 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
035 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
054 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
351 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
441 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
422 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
403 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
\qquad
\begin{array}{c|c}
 & 455 \\
\hline
405 & 1 \\
215 & 1 \\
135 & 1 \\
055 & 1 \\
354 & 1 \\
451 & 1 \\
442 & 1 \\
423 & 1
\end{array}
\\[1em]
0 \longleftarrow \underset{\text{faces}}{S^8} \longleftarrow \underset{\text{edges}}{S^8} \longleftarrow \underset{\text{vertex}}{S} \longleftarrow 0
\end{array}
$$

Figure 5.2: The negatively unbounded subcomplex from Example 5.12

monomial matrix for the resulting *relative cocellular free resolution* (Definition 5.13) has been written down in full detail; it resolves the ideal $I^\star$ from Example 1.15. To write the matrix, the edges have been oriented towards the center and the faces counterclockwise. The left copy of S^8 represents the 2-cells in clockwise order starting from 361, the right copy of S^8 represents the edges clockwise starting from 161, and the copy of S represents the lone vertex. The other vertices and edges are not considered since they lie in the subcomplex X_U. It is not a coincidence that the negatively unbounded subcomplex of X is the topological boundary of X—see Corollary 5.28. □

Definition 5.13 (Relative Cocellular Complexes) *A monomial matrix Φ whose scalar entries constitute (up to a homological shift) the relative cochain complex $\widetilde{\mathcal{C}}^{\bullet}(X, Y; k)$ of a pair $X \supseteq Y$ of regular cell complexes is called* a relative cocellular *monomial matrix supported on (X, Y). If Φ is free and acyclic, it is denoted $\mathbb{F}^{(X,Y)}$ and called* a relative cocellular free resolution.

To summarize what was done in the examples: A cellular injective resolution supported on X has a differential going the same direction as the coboundary map on X, and the integer part $X_{\mathbb{Z}}$ is really a cellular pair $(X, X \setminus X_{\mathbb{Z}})$. We record for future reference the next result, which is a corollary of Theorem 3.23.5′.

Proposition 5.14 *If X is a labelled cell complex supporting a cellular injective resolution of an ideal I then let $X_{\mathbf{1}-\mathbb{Z}}$ be the cellular pair $X_{\mathbb{Z}}$ with each face label $\mathbf{a}_G$ replaced by $\mathbf{1}-\mathbf{a}_G$. Then $\mathbb{F}^{X_{\mathbf{1}-\mathbb{Z}}}$ is a relative cocellular free resolution of I.*

Now we give a generalization of [BPS98, Theorem 3.7] on reading irreducible decompositions off of geometric resolutions. It will be used in the proof of Theorem 5.43. But first, a lemma.

Lemma 5.15 *If $I \neq S$ is an ideal and $\mathbb{I}_X$ is a minimal cocellular injective resolution of S/I supported on X, then X is pure of dimension $n-1$.*

Proof: If G is a facet then the image of the differential ∂ of $\mathbb{I}_X$ intersects $\underline{E}(S/\mathfrak{m}^F)[\mathbf{a}_G]$ trivially. But ∂ cannot restrict to an injection on $\underline{E}(S/\mathfrak{m}^F)[\mathbf{a}_G]$, else it would split (contradicting minimality) because $\underline{E}(S/\mathfrak{m}^F)[\mathbf{a}_G]$ is injective. Therefore, it must be that G has the highest possible dimension, so that $\underline{E}(S/\mathfrak{m}^F)[\mathbf{a}_G]$ sits in lowest possible cohomological degree: that way, the kernel of ∂ can come from the only cohomology of $\mathbb{I}_X$. □

Theorem 5.16 *If $\mathbb{I}_X$ is a cocellular minimal injective resolution of S/I, then the intersection $\bigcap_G \mathfrak{m}^{\mathbf{1}-\mathbf{a}_G}$ over all facets $G \in X$ is an irredundant irreducible decomposition of I. Here, $\mathfrak{m}^{\mathbf{b}} = \langle x_i^{b_i} \mid b_i \geq 1 \rangle$ is used just as well for vectors $\mathbf{b} \in \mathbb{Z}_*^n$.*

Proof: X is pure of dimension $n-1$ by Lemma 5.15, so the labels $\mathbf{a}_G$ on the facets correspond to zeroth Bass numbers of S/I. The result follows from Proposition 4.23, keeping in mind that $\mathbf{a}_G$ corresponds to a Bass number in degree $-\mathbf{a}_G$ and $1 - * = *$. □

Remark 5.17 Although [BPS98, Theorem 3.7] is stated in terms of free resolutions rather than injective resolutions, Theorem 5.16 really is a generalization, by duality for resolutions.

The fact that the result is about irreducible decompositions means that injective resolutions provide a more natural setting in which to state it: zeroth Bass numbers are to irreducible components as zeroth Betti numbers are to minimal generators. In particular, the arbitrary choices of $\mathbf{a}$ in [BPS98] are no longer needed when injective resolutions are used. See Corollary 5.32 for a generalization of [BPS98, Theorem 3.7] in a different direction.

5.2 The cohull resolution

In the paper that introduced the concept of a cellular free resolution, Bayer and Sturmfels also produced a wealth of examples [BS98]. They show, for instance, that every monomial ideal has a cellular resolution called the *hull resolution*, defined as follows. Let $t > (n+1)!$ and define $t^{\mathbf{b}} := (t^{b_1}, \ldots, t^{b_n})$ for $\mathbf{b} \in \mathbb{Z}^n$. The convex hull of the points $\{t^{\mathbf{b}} \mid \mathbf{x}^{\mathbf{b}} \in I\}$ is a polyhedron P_t whose face poset is independent of t. The hull complex hull(I) is defined to be the bounded faces of P_t, but it may also be described as those faces of P_t admitting a strictly positive inner normal. It is shown in [BS98] that the vertices of P_t are given by those $t^{\mathbf{b}}$ such that $\mathbf{x}^{\mathbf{b}}$ is a minimal generator of I. Each face of the hull complex is then labelled by the join of the labels on its vertices.

Theorem 5.18 (Bayer-Sturmfels) *The free complex* $\mathbb{F}_{\text{hull}(I)}$ *is a cellular resolution of* I*.*

Example 5.19 The Scarf complex of Example 2.15 of a generic ideal I coincides with the hull complex of I by [BS98, Theorem 2.9]. □

The purpose of this section is to apply Alexander duality to the hull resolution. Roughly this means that given an ideal I, the cohull complex of I should be the hull complex of its Alexander dual. Unfortunately, we have to choose an $\mathbf{a} \succeq \mathbf{a}_I$ for this purpose.

Definition 5.20 (The cohull resolution) *Suppose* I *is an ideal and* $\mathbf{a} \succeq \mathbf{a}_I$*. The* cohull complex cohull$^{\mathbf{a}}(I)$ *of* I *with respect to* $\mathbf{a}$ *is* $A^{+,0}_{\mathbf{a}}\text{hull}(I^{[\mathbf{a}]} + \mathfrak{m}^{\mathbf{a}+\mathbf{1}})$*. In other words, the labelled cell complex* cohull$^{\mathbf{a}}(I)$ *is obtained from the labelled cell complex* hull$(I^{[\mathbf{a}]} + \mathfrak{m}^{\mathbf{a}+\mathbf{1}})$ *by applying the relabelling function in Theorem 3.23.2′; it determines a cellular injective resolution of* I*. If* $\mathbf{a}$ *is unspecified then it is assumed to be* $\mathbf{a}_I$*, and the resulting complex* cohull$(I) :=$ cohull$^{\mathbf{a}_I}(I)$ *may be called the* tight *cohull complex of* I*, for emphasis.*

One of the nice properties of the hull resolution is that it involves no choices. Therefore, although it may be nonminimal, it preserves any symmetry inherent in the generators of I. The tight cohull resolution, like the hull resolution, is a possibly nonminimal resolution that preserves some of the symmetry of an ideal. Unfortunately, it may happen that certain cohull complexes of I are unequal (Example 5.23). Even so, cohull$^{\mathbf{a}}(I)$ is only vaguely dependent on $\mathbf{a}$: it is impossible to tell *a priori* which $\mathbf{a}$ was used to define a given cohull complex of I, in general. In fact,

Proposition 5.21 *Every ideal* I *has only finitely many distinct cohull complexes.*

Proof: The labels on the vertices of any cohull complex correspond precisely to the first Bass numbers of I, since the vertices of hull$(I^{[\mathbf{a}]} + \mathfrak{m}^{\mathbf{a}+\mathbf{1}})$ correspond precisely to the minimal generators (a.k.a. *first* Betti numbers) of $S/(I^{[\mathbf{a}]} + \mathfrak{m}^{\mathbf{a}+\mathbf{1}})$. The labels on the rest of the faces of a cohull complex are determined by the labels on the vertices:

Lemma 5.22 *The label on every face of* $\text{cohull}^{\mathbf{a}}(I)$ *is the join of its vertex labels.*

Proof: The relabelling function in Theorem 3.23.2′ preserves joins. (In fact, it is order-preserving and injective as a map $\mathbb{N}^n \to \mathbb{Z}^n_*$.) □

The only thing left to realize is that there are only finitely many regular cell complexes on a finite set of vertices, proving the Proposition. □

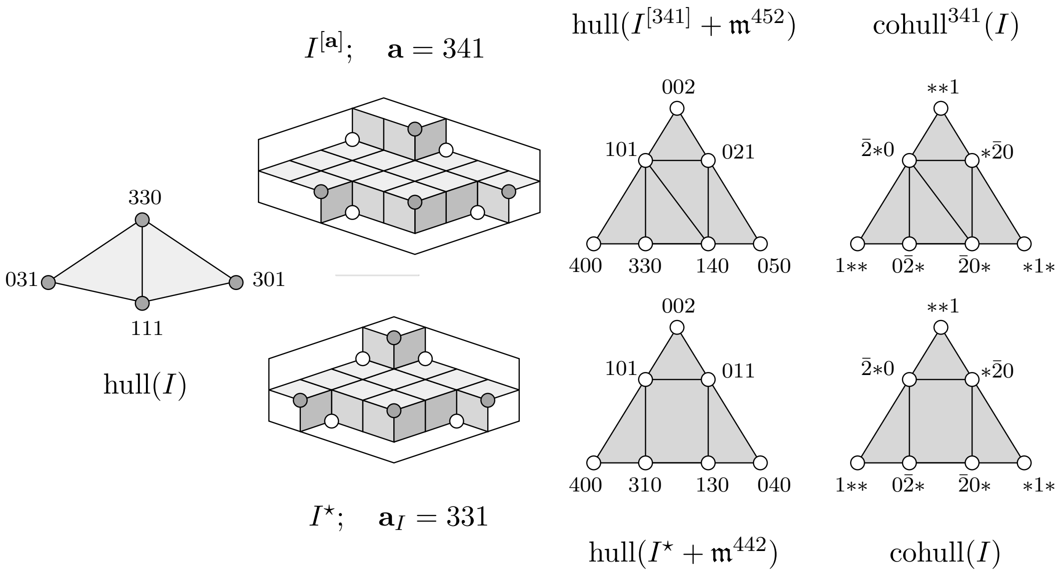


Figure 5.3: The resolutions from Example 5.23

Example 5.23 The tight cohull resolution can differ from other cohull resolutions. Let $I = \langle x^3z, xyz, y^3z, x^3y^3\rangle$, so $\mathbf{a}_I = (3,3,1)$, and let $\mathbf{a} = (3,4,1)$. Then $I^{[\mathbf{a}]} = \langle xz, x^3y^2, xy^4, y^2z\rangle$ and $I^\star = \langle xz, x^3y, xy^3, yz\rangle$. Since hull($I$) is not minimal, we look elsewhere for the minimal resolution of I. But $\text{hull}(I^{[\mathbf{a}]} + \mathfrak{m}^{\mathbf{a}+\mathbf{1}})$ is not minimal either, and the failure of minimality occurs in such a way that $\text{cohull}^{\mathbf{a}}(I)$ is also not minimal. On the other hand, the offending nonminimal edge is not present in $\text{hull}(I^\star + \mathfrak{m}^{\mathbf{a}_I+\mathbf{1}})$, and this resolution is minimal. It follows that cohull(I) is minimal. Note how the passage from $I^{[\mathbf{a}]}$ to $(I^{[\mathbf{a}]})^{\star\star} = I^\star$ "tightens" the hull resolution of $I^{[\mathbf{a}]}$ to make the nonminimal edge disappear in $\text{hull}((I^{[\mathbf{a}]})^{\star\star})$; hence the adjective "tight".

The labelled complexes supporting these resolutions are all depicted in Figure 5.3, where the resolutions with black vertices are drawn "upside down" to make their superimposition on the staircase diagram for I easier to visualize. Observe that a staircase diagram for I can be obtained by turning over the staircase diagram for either $I^{[\mathbf{a}]}$ or $I^\star$, although these result in different "bounding boxes" for I (cf. Remark 1.16).

Remark 5.24 In the cohull complexes of Figure 5.3, as with any cohull complex of any ideal, the irreducible components of I correspond to the vertices of the cohull complex which have no entries equal to 1 in their label. This is independent of the choice of $\mathbf{a}$ by the proof of Proposition 5.21. As in Remark 4.31, vertices with some coordinate equal to 1 come from the long exact sequence of $\underline{\mathrm{Ext}}$ arising from the short exact sequence $0 \to I \to S \to S/I$. Alternatively, use the Alexander dual statement, which is perhaps easier to digest: the generators of $I^{[\mathbf{a}]} + \mathfrak{m}^{\mathbf{a}+\mathbf{1}}$ which have no i^{th} coordinate equal to $a_i + 1$ are the minimal generators of $I^{[\mathbf{a}]}$. The vertices of the cohull complex which have some coordinate 1 correspond to the extra artinian generators of $\mathfrak{m}^{\mathbf{a}+\mathbf{1}}$. □

Question 5.25 Which cell complexes can arise as the cohull complexes for a fixed ideal I? How many can there be? Is there necessarily a "best" one? Is there some way to combine the information from the collection of cohull complexes to come up with a particularly small injective resolution (as close to minimal as possible)?

There are some geometric properties of hull resolutions of artinian ideals that make cohull resolutions a little more tangible. Suppose that J is an artinian monomial ideal, with $x_1^{d_1}, \ldots, x_n^{d_n}$ among its minimal generators. Choose $t > (n+1)!$, and let $v_1, \ldots, v_n$ be the vertices of the polyhedron P_t determined by these minimal generators. The vertices $\{v_i\}$ of P_t span an affine hyperplane which will be denoted by H.

Fix a strictly positive inner normal φ_G for each $G \in \mathrm{hull}(J)$. Recall that P_t is contained in the (closed) polyhedron $\mathbf{1} + \mathbb{R}^n_+$ because monomials in S have no negative exponents. Each face $G \in \mathrm{hull}(J)$ spans an affine space which does not contain the vector $\mathbf{1} \in \mathbb{R}^n$ because the hyperplane containing G and normal to φ_G does not contain $\mathbf{1}$. Therefore the projection π from the point $\mathbf{1}$ to the hyperplane H induces a homeomorphism $\mathrm{hull}(J) \to \pi(\mathrm{hull}(J))$. In fact,

Proposition 5.26 *If J is artinian, $\pi(\mathrm{hull}(J))$ is a regular polyhedral subdivision of the simplex $H \cap P_t$.*

Proof: That $H \cap P_t \subset \mathbf{1} + \mathbb{R}^n_+$ is a simplex follows because it is convex and contains $v_1, \ldots, v_n$. Now π induces a map of the boundary $\partial P_t \to H \cap P_t$ which is obviously surjective. Suppose that $\pi(\mathbf{w})$ is in the interior of $H \cap P_t$ for some $\mathbf{w} \in \partial P_t$. It is enough to show that if a nonzero support functional φ attains its minimum on P_t at $\mathbf{w}$ then φ is strictly positive. All coordinates of φ are ≥ 0 *a priori* because it attains a minimum on P_t; but if the i^{th} coordinate of φ is zero then $\langle \varphi, v_i \rangle < \langle \varphi, \mathbf{w} \rangle$ and φ cannot be minimized at $\mathbf{w}$. □

Remark 5.27 Since the hull complex of a generic monomial ideal is the Scarf complex [BS98, Theorem 2.9], Proposition 5.26 generalizes [BPS98, Corollary 5.5] for generic artinian monomial ideals. Regular subdivisions here are as in [Zie95, Definition 5.3].

We arrive at the following characterization of cohull complexes:

Corollary 5.28 *For any ideal I and $\mathbf{a} \succeq \mathbf{a}_I$, the complex $|\mathrm{cohull}^{\mathbf{a}}(I)|$ is a simplex, and the labelled cellular pair $\mathrm{cohull}^{\mathbf{a}}(I)_{\mathbf{1}-\mathbb{Z}}$ (Proposition 5.14) is the interior.*

Proof: This is equivalent to the Alexander dual statement: if X is the hull complex of an artinian monomial ideal, then $|X|$ is a simplex and the negatively unbounded complex X_U

is the topological boundary of X. By Proposition 5.26, it suffices to show that a face G of the hull complex of any (not necessarily artinian) ideal has a label without full support if and only if it is contained in the topological boundary of the shifted positive orthant $\mathbf{1}+\mathbb{R}^n_+$. But this holds because the i^{th} coordinate of $\mathbf{a}_G$ is zero if and only if every vertex of G (and hence every point in G) has i^{th} coordinate 1. □

Proposition 5.29 *If* $\text{cohull}^{\mathbf{a}}(I)_{(F)}$ *is the restriction of* $\text{cohull}^{\mathbf{a}}(I)$ *to the face of the simplex spanned by* $F \subseteq \{1,\ldots,n\}$*, then* $\text{cohull}^{\mathbf{a}}(I)_{(F)}$ *determines an injective resolution of* $I_{(F)}$.

Proof: It is clear from the definitions that the restriction of $\text{hull}(I^{[\mathbf{a}]} + \mathfrak{m}^{\mathbf{a}+\mathbf{1}})$ to F is $\text{hull}((I^{[\mathbf{a}]} + \mathfrak{m}^{\mathbf{a}+\mathbf{1}})_{[F]})$. The result follows from Theorem 3.30.2. □

Although cohull resolutions are relative cocellular by definition, they can frequently be viewed as cellular resolutions, as well. The next theorem is the main result of the section (aside from Definition 5.20). For notation, recall Proposition 5.14.

Theorem 5.30 *The relative cocellular free resolution* $\mathbb{F}^{\text{cohull}^{\mathbf{a}}(I)_{\mathbf{1}-\mathbb{Z}}}$ *of* I *is weakly cellular for any* $\mathbf{a} \succeq \mathbf{a}_I$*. In particular, if a cohull free resolution is minimal, it is cellular.*

Proof: Let $J = I + \mathfrak{m}^{\mathbf{a}+\mathbf{1}}$ and assume the notation from before Proposition 5.26. Define Q_t to be the intersection of P_t with the closed half-space containing the origin and determined by the hyperplane H. Then Q_t is a polytope which may also be described as the convex hull of (all of) the vertices of P_t. Furthermore, the bounded faces of P_t are simply those faces of Q_t which admit a strictly positive inner normal. Thus $X := \text{hull}(J)$ is a subcomplex of the boundary complex of Q_t, as is the boundary ∂X.

Let $Y \subset \partial Q_t$ be the subcomplex generated by the facets of Q_t whose inner normal is *not* strictly positive. Denote chain and relative cochain complexes over k by $\mathcal{C}_\bullet(-)$ and $\mathcal{C}^\bullet(-,-)$. Then $Y \cap X = \partial X$ and $\mathcal{C}^\bullet(Q_t, Y) = \mathcal{C}^\bullet(X, \partial X)$. For elementary reasons, $\mathcal{C}^\bullet(Q_t, Y) \cong \mathcal{C}_\bullet(X^\star)$ for some subcomplex $X^\star$ of the polar polytope $Q_t^\star$ (use, for instance, the methods of [Zie95, Sections 2.2–2.3]). Note that the isomorphisms will exist regardless of the incidence functions in question, by [BH93, Theorem 6.2.2]. We are done because of the isomorphism $\mathcal{C}_\bullet(X^\star) \cong \mathcal{C}^\bullet(X, \partial X)$, the latter constituting the scalar entries in the cohull free resolution and the former being the boundary complex of a regular cell complex. □

Conjecture 5.31 *All weakly cellular cohull complexes are cellular.*

That is, cellularity of cohull resolutions probably follows without the hypothesis of minimality, because of the convex-geometric nature of its definition. For instance, all cohull resolutions in the examples below are cellular. Cellularity of cohull resolutions has an interesting consequence for hull resolutions—one which will be a key point in the proof of (c) ⇒ (f) in Theorem 5.43.

Corollary 5.32 *Let* J *be a monomial ideal and* $\mathbf{a} \succeq \mathbf{a}_J$*. Using* $\hat{\mathbf{b}} = \sum_{b_i \leq a_i} b_i \mathbf{e}_i$ *for* $\mathbf{b} \in \mathbb{N}^n$*, the intersection* $\bigcap_G \mathfrak{m}^{\hat{\mathbf{a}}_G}$ *over facets* $G \in \text{hull}(J+\mathfrak{m}^{\mathbf{a}+\mathbf{1}})$ *is an irreducible decomposition of* J.

Proof: Let $I = J^{[\mathbf{a}]}$. Applying Lemma 5.2 to the weakly cellular resolution of Theorem 5.30 shows that $I = \langle \mathbf{x}^{\mathbf{a}_H} \mid H \in \text{cohull}^{\mathbf{a}}(I)_{\mathbf{1}-\mathbb{Z}} \text{ is a facet}\rangle$. Although this generating set for

I need not be minimal, it nevertheless follows from Definition 1.2 and Lemma 1.6 that $J = \bigcap\{\mathfrak{m}^{\mathbf{a} \smallsetminus \mathbf{a}_H} \mid H \in \mathrm{cohull}^{\mathbf{a}}(I)_{\mathbf{1}-\mathbb{Z}} \text{ is a facet}\}$. Any labelled facet $H \in \mathrm{cohull}^{\mathbf{a}}(I)_{\mathbf{1}-\mathbb{Z}}$ corresponds to a labelled facet $G \in \mathrm{hull}(J + \mathfrak{m}^{\mathbf{a}+\mathbf{1}})$ which represents the same underlying unlabelled facet of $|\mathrm{hull}(J + \mathfrak{m}^{\mathbf{a}+\mathbf{1}})|$. Since $\mathrm{cohull}^{\mathbf{a}}(I)$ represents an injective resolution of I, the vector $\mathbf{a}_G = \mathbf{a} - \mathbf{a}_H$ is obtained from $\mathbf{1} - \mathbf{a}_H$ by relabelling as in Theorem 3.23.1′ whenever H and G correspond in this way. Thus $\mathbf{a} \smallsetminus \mathbf{a}_H = \hat{\mathbf{a}}_G$ by Definition 1.1. □

Example 5.33 If the ideal J in Corollary 5.32 is taken to be the ideal J from Figure 2.2, then $B_{444}(S/J) = S/(J + \mathfrak{m}^{555})$, and the bottom diagram is $\mathrm{hull}(J + \mathfrak{m}^{\mathbf{a}+\mathbf{1}})$ for $\mathbf{a} = 444$. For an example where Corollary 5.32 applies but Theorem 5.16 does not, take $J = I^{[341]}$ from Figure 5.3 and Example 5.23, with $\mathbf{a} = 452$. □

Remark 5.34 Corollary 5.32 can be a powerful tool for fast calculation of irreducible decompositions of monomial ideals using polyhedral combinatorics and convexity: all one has to do is calculate the facets of $\mathrm{hull}(J + \mathfrak{m}^{\mathbf{a}+\mathbf{1}})$. Just as in Remark 1.8, this observation can be used to calculate polyhedrally the minimal generators of an ideal from its irreducible components.

Corollary 5.35 *If* $\mathrm{hull}(I+\mathfrak{m}^{\mathbf{a}+\mathbf{1}})$ *determines a minimal free resolution of* $S/\mathfrak{m}^{\mathbf{a}+\mathbf{1}}$*, then the label on any interior face is the meet of the labels on the facets that contain it. Equivalently, if* $\mathrm{hull}(I + \mathfrak{m}^{\mathbf{a}+\mathbf{1}})$ *is relabelled to determine a minimal injective resolution of* S/I*, then the label on any interior face is the join of the labels on the facets that contain it.*

Proof: The hull complexes in question are supported on the same purely $(n-1)$-dimensional cell complex by Corollary 5.28. The facets containing a given interior face thus correspond to the vertices contained in a face of the weakly cellular cohull complex of $I^{[\mathbf{a}]}$. But this cohull complex is cellular by Theorem 5.30. The result follows because the relabellings in Theorem 3.23.1′ and Theorem 3.23.5′ are order preserving and reversing, respectively. □

Example 5.36 The free resolutions originating from the cohull complexes in Example 5.23 are both cellular—and not just weakly cellular—as well as relatively cocellular (Figure 5.4). The nonminimal cohull free resolution here actually contains an extra *vertex*, a phenomenon which hasn't occurred yet in any of the examples. □

Example 5.37 *(continuation of Example 1.19)* The minimal resolution of the permutohedron ideal I of Example 1.19 is, by [BS98, Example 1.9], the hull resolution, which is supported on a permutohedron. The minimal resolution of $I + \mathfrak{m}^{(n+1)\mathbf{1}}$ is also the hull resolution, and is supported on the complex X which may described as follows.

There are two kinds of faces of X. The first kind are those that make up the boundary ∂X; these are indexed by the nonzero squarefree vectors $\mathbf{0} \neq F \prec \mathbf{1}$ (i.e. the proper nonempty subsets of $\{1, \ldots, n\}$) and have vertices $t^{(n+1)\mathbf{e}_i} \in P_t$ for $i \in F$. On the other hand, the interior p-faces of X are in bijection with the chains

$$\emptyset \prec F_1 \prec F_2 \prec \cdots \prec F_{n-p} \tag{5.1}$$

where F_{n-p} might (or might not) equal $\mathbf{1}$. Note that the interior faces of X for which $F_{n-p} = \mathbf{1}$ are faces of the permutohedron itself.

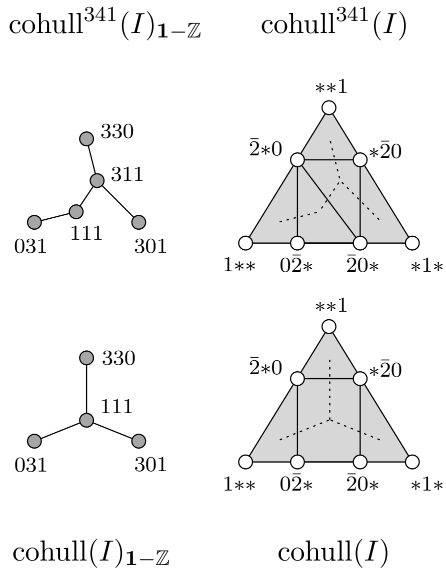


Figure 5.4: Cellular cohull free resolutions from Example 5.36

More generally, an interior p-face G given by (5.1) for which $F_{n-p} \neq \mathbf{1}$ is affinely spanned by the permutahedral $(p-1)$-face $G' : \emptyset \prec F_1 \prec \cdots \prec F_{n-p} \prec \mathbf{1}$ and the "artinian" vertices $\{t^{(n+1)\mathbf{e}_i} \mid i \notin F_{n-p}\}$ of P_t. In fact, a functional which attains its minimum (in P_t) on G may be produced directly. For this purpose, define for any $\mathbf{0} \preceq F \preceq \mathbf{1}$ the functional $F^\dagger$ on $\mathbb{R}^n$ to be the transpose of F; i.e. $\langle F^\dagger, \mathbf{e}_i\rangle = 1$ if $i \in F$ and zero otherwise. Then the functional $\varphi_\epsilon := \mathbf{1}^\dagger + \epsilon \sum_{j=1}^{n-p} F_j^\dagger$ attains its minimum (in P_t) on G' for all $0 < \epsilon \ll 1$. But for $\epsilon \gg 0$ we have $\langle \varphi_\epsilon, t^{(n+1)\mathbf{e}_i}\rangle < \langle \varphi_\epsilon, G'\rangle$ whenever $i \notin F_{n-p}$. Thus we can choose the unique ϵ that makes $\langle \varphi_\epsilon, t^{(n+1)\mathbf{e}_i}\rangle = \langle \varphi_\epsilon, G'\rangle$ for all $i \notin F_{n-p}$, so that φ_ϵ attains its minimum on G.

It is easy to check that the labels on the faces of X are distinct, whence $\mathbb{F}_X$ is a minimal resolution of $I + \mathfrak{m}^{(n+1)\mathbf{1}}$. In particular, the irredundant irreducible components of I are in bijection with the facets of X by Theorem 5.16, and the generators of the tree ideal $I^\star$ are given by $\mathbf{x}^{(n+1)\mathbf{1}-\mathbf{a}_G}$ for facets $G \in X$. This recovers the generators for $I^\star$ in Example 1.19.

Retaining earlier notation, the face G has dimension $1+\dim(G')$. Thus the p-faces of X are in bijection with the collection of p- and $(p-1)$-faces of the permutohedron. In fact, the (unlabelled) pair $(|X|, |\partial X|)$ has the same faces as the pair $\big(\partial(v * Y), v\big)$ consisting of the boundary of the cone over the permutohedron Y rel the apex of the cone. The cellular complex $X^\star$ supporting the cohull resolution of the tree ideal $I^\star$ is therefore easy to describe. Let Y be the permutohedron in $\mathbb{R}^n$ and $Y^\star$ its polar. Then $X^\star$ is the cone over $\partial Y^\star$ from the barycenter of $Y^\star$. The vertices $G^\star$ of $X^\star$, which are labelled by the generators of $I^\star$, almost all correspond to the facets G' of Y (whose labellings are as above). Only the apex of the cone is an exception, corresponding instead to the interior of Y.

The case $n = 3$ is depicted in Figure 5.5. The complex X is the (labelled) regular

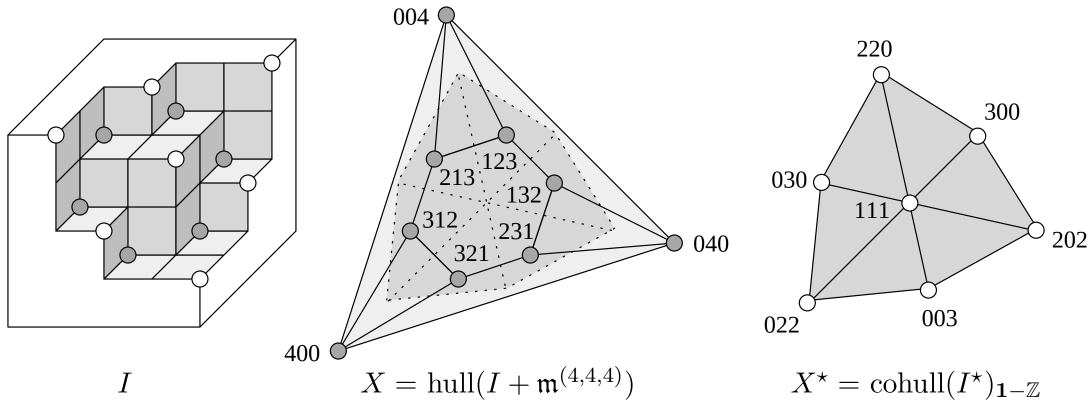


Figure 5.5: I and $I^\star$ are the permutohedron and tree ideals when $n = 3$

polytopal subdivision of the simplex promised by Proposition 5.26. Overlayed on this figure is the dual complex $X^\star$ (without its labelling). At right, $X^\star$ is shown with its labelling, which is $\mathbb{Z}^n$-shifted as per Proposition 5.14. Turn the picture over for the staircase of $I^\star$. It should be noted that the equality $Y = Y^\star$ is only because Y is 2-dimensional, not some more general self-duality. □

Example 5.38 It is possible for the hull and cohull free resolutions to coincide for a given ideal I. For instance, this occurs if $I = \mathfrak{m}$; or if I is simultaneously generic and cogeneric (which turns out to be pretty hard to accomplish!); or if I is a permutohedron ideal. In fact, the hull and cohull resolutions coincide for any cogeneric ideal whose hull resolution is minimal, because its cohull resolution will always be minimal (Section 5.4). Therefore permutohedron ideals of all dimensions have $\mathrm{hull}(I) = \mathrm{cohull}(I)$, since permutohedron ideals are cogeneric. □

Example 5.39 Of course, it is also possible for the hull and cohull free resolutions to be very different. For instance, the cohull free resolution of the ideal $I^\star$ from Examples 1.15, 5.9, and 5.12 is the coScarf resolution (Section 5.4), whose cellular version is supported on an octagon with only one maximal face (dualize the picture in Figure 5.2). On the other hand, $\mathrm{hull}(I^\star)$ is a triangulation of the same octagon. □

Example 5.40 Finally, an example to illustrate that not all cellular resolutions come directly from hull and cohull resolutions. All of the labelled cellular complexes from this example are depicted in Figure 5.6. Let $I = \langle z^2, x^3z, x^4, y^3, y^2z, xyz\rangle$, so that $I^\star = \langle xyz^2, x^2y^3z, x^4y^2z\rangle$. Then $\mathrm{hull}(I)$ and $\mathrm{cohull}(I)_{\mathbf{1}-\mathbb{Z}}$ are not minimal (the offending cells have italic labels); moreover, $\mathrm{cohull}_{\mathbf{a}}(I) = \mathrm{cohull}(I)$ for all $\mathbf{a} \succeq \mathbf{a}_I = (4,3,2)$. Nonetheless, the minimal resolution $\mathbb{F}_X$ of $I^\star + \mathfrak{m}^{(5,4,3)}$ is cellular, so Proposition 5.14 applies, yielding a minimal relative cocellular resolution for I. In fact, this relative cocellular resolution is cellular, supported on the labelled cell complex Y. □

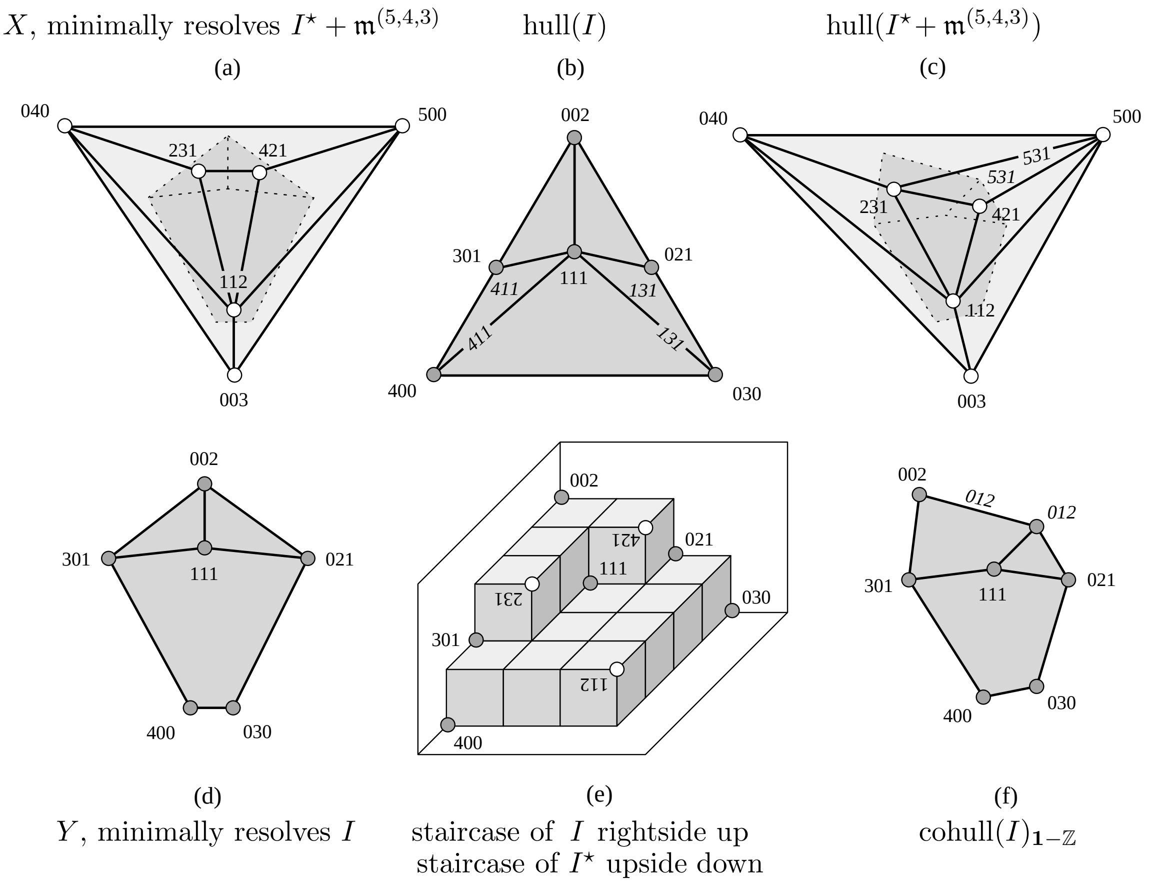


Figure 5.6: The cellular resolutions of Example 5.40

5.3 Genericity

The purpose of this section is use duality for cellular resolutions to give a multifaceted characterization of generic monomial ideals. The theorem in this section was added to [MSY00] after it had been accepted for publication (but before the final proofs were in). A comment from one of the referees and some urging by D. Bayer convinced the author of this dissertation to prove a theorem that gives convincing evidence for the "correctness" of the new definition of genericity, repeated below. Ideals called generic in [BPS98] are called *strongly generic* here. By way of terminology, a monomial m is said to *strictly divide* m' if m divides m' and every variable which divides m' also divides m'/m.

Definition 5.41 *A monomial ideal $I = \langle m_1, \ldots, m_r\rangle$ is called* generic *if the following condition holds: if two distinct minimal generators m_i and m_j have the same positive degree in some variable x_s, there is a third generator m_l which strictly divides $m_{\{i,j\}} = \mathrm{lcm}(m_i, m_j)$.*

Again let $I \subset S$ be the ideal minimally generated by monomials $m_1, \ldots, m_r$. Define $m_\sigma = \mathrm{lcm}(m \in \sigma)$ for any subset $\sigma \subseteq \{m_1, \ldots, m_r\}$, and recall that the *Scarf* *complex* of I,

$$\Delta_I := \{\sigma \subseteq \{m_1, \ldots, m_r\} \mid \text{if } m_\sigma = m_\tau \text{ for some } \tau \subseteq \{m_1, \ldots, m_r\} \text{ then } \tau = \sigma\},$$

consists of the subsets whose least common multiples are uniquely attained. Each face $\sigma \in \Delta_I$ is labelled by the exponent on m_σ. It is known that $\mathbb{F}_{\Delta_I}$ is always a subcomplex of the minimal free resolution of S/I [BPS98, Section 3], although $\mathbb{F}_{\Delta_I}$ is not acyclic in general. However, it will follow from Theorem 5.43 that it is acyclic if I is generic, as was the case for strongly generic ideals.

Since the definition of the Scarf complex depends only on the coordinatewise order of the exponents of the generators, it also makes sense for (formal) monomials with real exponents in $\mathbb{R}^n$. This makes way for the following definition.

Definition 5.42 *A* deformation ϵ *of a monomial ideal* $I = \langle m_1, \ldots, m_r \rangle \subset S$ *is a choice of vectors* $\epsilon_i = (\epsilon_{i1}, \ldots, \epsilon_{in}) \in \mathbb{R}^n$ *for each* $i \in \{1, \ldots, r\}$ *satisfying*

$$a_{is} < a_{js} \quad \Rightarrow \quad a_{is} + \epsilon_{is} < a_{js} + \epsilon_{js} \qquad \text{and} \qquad a_{is} = 0 \quad \Rightarrow \quad \epsilon_{is} = 0\,,$$

where $\mathbf{a}_i = (a_{i1}, \ldots, a_{in})$ *is the exponent vector of* m_i*. We formally introduce the monomial ideal (in a polynomial ring with real exponents):*

$$I_\epsilon \quad := \quad \langle m_1 \cdot \mathbf{x}^{\epsilon_1}, m_2 \cdot \mathbf{x}^{\epsilon_2}, \ldots, m_r \cdot \mathbf{x}^{\epsilon_r} \rangle = \langle \mathbf{x}^{\mathbf{a}_1+\epsilon_1}, \mathbf{x}^{\mathbf{a}_2+\epsilon_2}, \ldots, \mathbf{x}^{\mathbf{a}_r+\epsilon_r} \rangle.$$

The Scarf complex Δ_{I_ϵ} of the deformation I_ϵ has the same vertex set $\{1, \ldots, r\}$ as Δ_I. For a suitable ϵ, Δ_{I_ϵ} gives a simple (but typically non-minimal) free resolution of I; see [BPS98, Theorem 4.3]. Definition 5.42 is slightly different from the one given by [BPS98, Construction 4.1]. We require that the zeros remain unchanged, but we do not assume that I_ϵ is "(strongly) generic".

It is frequently convenient to add in high powers $x_1^D, \ldots, x_n^D$ of the variables to I, where D is larger than any exponent of any minimal generator of I. One obtains an artinian ideal

$$\widetilde{I} := I + \langle x_1^D, \ldots, x_n^D \rangle\,,$$

the Scarf complex of which is called the *extended Scarf complex* or *Scarf triangulation* $\Delta_{\widetilde{I}}$ of I. Note that if $x_i^d \in I$ for some d then x_i^D is not a minimal generator of $\widetilde{I}$. As a simplicial complex, $\Delta_{\widetilde{I}}$ does not depend on D. The monomial ideal I is generic if and only if $\widetilde{I}$ is generic.

The following theorem provides a justification for our new definition of "generic". It provides appropriate converses to [BPS98, Theorems 3.2 and 3.7] and [BS98, Theorem 2.9]. All statements are independent of the particular choice of D used to define $\widetilde{I}$.

Theorem 5.43 *The following are equivalent for a monomial ideal* I*:*

(a) I *is generic.*

(b) $\mathbb{F}_{\Delta_{\widetilde{I}}}$ *is a minimal free resolution of* $S/\widetilde{I}$.

(c) $\Delta_{\widetilde{I}} = \mathrm{hull}(\widetilde{I})$.

(d) $\widetilde{I} = \bigcap\{\mathfrak{m}^{\mathbf{a}_\sigma} \mid \sigma \in \Delta_{\widetilde{I}},\ \#\sigma = n\}$ *is an irredundant irreducible decomposition.*

(e) For each irreducible component $\mathfrak{m}^{\mathbf{b}}$ *of* $S/\widetilde{I}$*, there is a face* $\sigma \in \Delta_{\widetilde{I}}$ *with* $\mathbf{a}_\sigma = \mathbf{b}$.

(f) $\mathbb{F}_{\Delta_I}$ *is a free resolution of* S/I*, and no variable* x_s *appears with the same non-zero exponent in* m_i *and* m_j *for any edge* $\{i,j\}$ *of the Scarf complex* Δ_I.

(g) If $\sigma \notin \Delta_{\widetilde{I}}$*, then there is some monomial* $m \in I$ *which strictly divides* m_σ.

(h) The Scarf triangulation $\Delta_{\widetilde{I}}$ *does not change under arbitrary deformations of* $\widetilde{I}$.

Proof: The scheme of the proof is

$$(b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (b) \qquad \text{and} \qquad (c) \Rightarrow (f) \Rightarrow (a) \Rightarrow (g) \Rightarrow (h) \Rightarrow (b)\,.$$

(b) $\Rightarrow$ (c): By induction on n; if $n = 2$, this is obvious, so suppose (b) $\Rightarrow$ (c) for $\leq n-1$ variables. The fact that $S/\widetilde{I}$ is artinian implies that $\Delta_{\widetilde{I}}$ is pure of dimension $n-1$ by Lemma 5.15 or [Yan99, Proposition 2.9]. The restriction of $\Delta_{\widetilde{I}}$ to those vertices whose monomial labels are not divisible by x_s is the Scarf complex of the ideal $\widetilde{I}_s = (\widetilde{I} + \langle x_s\rangle)/\langle x_s\rangle$ in $k[x_1,\dots,x_n]/\langle x_s\rangle$. By induction, $\Delta_{\widetilde{I}_s} = \mathrm{hull}(\widetilde{I}_s)$ because $\mathbb{F}_{\Delta_{\widetilde{I}_s}}$ is acyclic by [BPS98, Lemma 2.2]. The topological boundary of $\Delta_{\widetilde{I}}$ is the union of the complexes $\Delta_{\widetilde{I}_s}$, and the topological boundary of $\mathrm{hull}(\widetilde{I})$ is the union of the complexes $\mathrm{hull}(\widetilde{I}_s)$, where s runs over $\{1,\dots,n\}$. On the other hand, by [BS98, Proposition 2.6], we know that the acyclic simplicial complex $\Delta_{\widetilde{I}}$ is a subcomplex of the polyhedral complex $\mathrm{hull}(\widetilde{I})$. The latter being a subdivision of the $(n-1)$-ball, and both complexes containing the boundary of $\mathrm{hull}(\widetilde{I})$, we can conclude that $\Delta_{\widetilde{I}} = \mathrm{hull}(\widetilde{I})$.

(c) $\Rightarrow$ (d): Theorem 5.16 and Corollary 3.27.

(d) $\Rightarrow$ (e): Trivial.

Lemma 5.44 *If* $\mathbf{b} \in \mathbb{Z}^n$ *and* $\beta_{i,\mathbf{b}}(S/\widetilde{I}) \neq 0$ *for some* i*, then there is an irreducible component* $\mathfrak{m}^{\mathbf{a}}$ *of* $S/\widetilde{I}$ *such that* $\mathbf{b} \preceq \mathbf{a}$.

Proof: An ideal $\mathfrak{m}^{\mathbf{a}}$ is an irreducible component of $S/\widetilde{I}$ if and only if $\beta_{n,\mathbf{a}}(S/\widetilde{I}) \neq 0$ by Proposition 4.17 and Corollary 4.18, using the fact that $S/\widetilde{I}$ is artinian which implies that $\mathbf{a}$ has full support. Let $\mathbb{F}_{\bullet}$ be a minimal free resolution of $S/\widetilde{I}$ and $\mathbb{F}^*_{\bullet} := \underline{\mathrm{Hom}}_S(\mathbb{F}_{\bullet}, S)$ its free transpose. Then $\mathbb{F}^*_{\bullet}$ is a minimal free resolution of some $\mathbb{Z}^n$-graded module N with $\beta_{i,\mathbf{b}}(S/\widetilde{I}) = \beta_{n-i,-\mathbf{b}}(N)$ by local duality (Proposition 3.5). It follows that $-\mathbf{b} \succeq -\mathbf{a}$ for some $\mathbf{a}$ with $0 \neq \beta_{0,-\mathbf{a}}(N) = \beta_{n,\mathbf{a}}(S/\widetilde{I})$. □

(e) $\Rightarrow$ (b): Suppose $\beta_{i,\mathbf{a}}(S/\widetilde{I}) \neq 0$. Since the Taylor complex of $\widetilde{I}$ is acyclic, it follows that $\mathbf{a} = \mathbf{a}_\sigma$ for some $\sigma \subset \{1,\dots,r+n\}$. It suffices to prove $\sigma \in \Delta_{\widetilde{I}}$ by [BPS98, Lemma 3.1]. From Lemma 5.44 and (e), there is some $\tau \in \Delta_{\widetilde{I}}$ such that $\mathbf{a}_\sigma (= \mathbf{a}) \preceq \mathbf{a}_\tau$, that is, m_σ divides m_τ. Since $\tau \in \Delta_{\widetilde{I}}$, we have $\sigma \subset \tau$. Thus $\sigma \in \Delta_{\widetilde{I}}$.

(c) $\Rightarrow$ (f): Acyclicity follows from the criterion of [BPS98, Lemma 2.2], because Δ_I is the subcomplex of $\Delta_{\widetilde{I}}$ consisting of the faces whose labels divide $x_1^{D-1}\cdots x_n^{D-1}$. It therefore suffices to show the condition on edges when $I = \widetilde{I}$ (note that Δ_I is a subcomplex of $\Delta_{\widetilde{I}}$).

$$\text{If } \sigma \text{ is any facet of } \Delta_{\widetilde{I}}, \text{ then } \#\sigma = n, \operatorname{supp}(m_\sigma) = \{1, \ldots, n\}, \text{ and}$$
$$\text{each exponent vector } \mathbf{a}_i,\ i \in \sigma, \text{ shares a different coordinate with } \mathbf{a}_\sigma. \qquad (*)$$

Suppose now that $0 \neq \deg_{x_s} m_i = \deg_{x_s} m_j$ and $\{i,j\} \in \Delta_{\widetilde{I}}$ is an edge. Corollary 5.35 says that $m_{\{i,j\}} = \gcd(m_\sigma \mid \sigma \in \Delta_{\widetilde{I}}$ is a facet containing $\{i,j\})$. In particular, there is some facet $\sigma \supseteq \{i,j\}$ with $\deg_{x_s} m_\sigma = \deg_{x_s} m_{\{i,j\}} = \deg_{x_s} m_i = \deg_{x_s} m_j$, contradicting $(*)$.

(f) $\Rightarrow$ (a): For any generator m_i let

$$A_i := \{m_j \mid m_j \neq m_i \text{ and } \deg_{x_s} m_j = \deg_{x_s} m_i > 0 \text{ for some } s\}\,.$$

The set A_i can be partially ordered by letting $m_j \preceq m_{j'}$ if $m_{\{i,j\}}$ divides $m_{\{i,j'\}}$. It is enough to produce a monomial m_l as in Definition 5.41 whenever $m_j \in A_i$ is a minimal element for this partial order. Supposing that m_j is minimal, use acyclicity to write

$$\frac{m_{\{i,j\}}}{m_i} \cdot e_i - \frac{m_{\{i,j\}}}{m_j} \cdot e_j \quad = \sum_{\{u,v\} \in \Delta_I} b_{u,v} \cdot d(e_{\{u,v\}}) \tag{5.2}$$

where we may assume (by picking such an expression with a minimal number of nonzero terms) that the monomials $b_{u,v}$ are 0 unless $m_{\{u,v\}}$ divides $m_{\{i,j\}}$. Here, e_σ is the S-basis vector for the summand of $\mathbb{F}_{\Delta_I}$ corresponding to σ and d is the differential. There is at least one monomial m_l such that $b_{l,j} \neq 0$, and we claim $m_l \notin A_i$. Indeed, m_l divides $m_{\{i,j\}}$ because $m_{\{l,j\}}$ does, so if $\deg_{x_t} m_i < \deg_{x_t} m_j$ (which must occur for some t because m_j does not divide m_i), then $\deg_{x_t} m_l \leq \deg_{x_t} m_j$. Applying the second half of (f) to $m_{\{l,j\}}$ we get $\deg_{x_t} m_l < \deg_{x_t} m_j$, and furthermore $\deg_{x_t} m_{\{i,l\}} < \deg_{x_t} m_{\{i,j\}}$, whence $m_l \notin A_i$ by minimality of m_j. So if $\deg_{x_s} m_{\{i,j\}} > 0$ for some s, then either $\deg_{x_s} m_l < \deg_{x_s} m_j$ by the second half of (f), or $\deg_{x_s} m_l < \deg_{x_s} m_i$ because $m_l \notin A_i$.

(a) $\Rightarrow$ (g): Choose $\sigma \notin \Delta_{\widetilde{I}}$ maximal among subsets with label m_σ. Then $m_\sigma = m_{\sigma \setminus \{i\}}$ for some $i \in \sigma$. If $\operatorname{supp}(m_\sigma/m_i) = \operatorname{supp}(m_\sigma)$, the proof is done. Otherwise, there is some $j \in \sigma \setminus \{i\}$ with $\deg_{x_s} m_i = \deg_{x_s} m_j > 0$ for some x_s. Then neither m_i nor m_j is a power of a variable, so $m_i, m_j \in I$. Since I is generic, there is a monomial $m \in I$ which strictly divides $m_{\{i,j\}}$, which in turn strictly divides m_σ.

(g) $\Rightarrow$ (h): The strict inequalities which define the conditions "m_i does not divide m_σ" and "m_i strictly divides m_σ" persist after deformation. Persistence of the former implies that $\sigma \in \Delta_{\widetilde{I}}$ remains a face in the deformation, while persistence of the latter implies that $\sigma \notin \Delta_{\widetilde{I}}$ remains a non-face.

(h) $\Rightarrow$ (b): By [BPS98, Theorem 4.3], there is a deformation ϵ of $\widetilde{I}$ such that $\Delta_{\widetilde{I}_\epsilon}$ gives a free resolution of $S/\widetilde{I}$. Since $\Delta_{\widetilde{I}} = \Delta_{\widetilde{I}_\epsilon}$, $\mathbb{F}_{\Delta_{\widetilde{I}}}$ is a free resolution. This resolution is automatically minimal because the face labels are distinct. □

Remark 5.45 1. Conditions (b), (d), and (h) in Theorem 5.43 can be more naturally phrased (without referring to algebraic properties of $\widetilde{I}$) in terms of injective resolutions of S/I. For instance, (d) says that the zeroth Bass numbers of S/I are determined by $\Delta_{\widetilde{I}}$, and (b) says that the entire $\mathbb{Z}^n$-graded injective resolution is determined by $\Delta_{\widetilde{I}}$.

2. The equivalence (g) $\Leftrightarrow$ (h) remains true even if every occurrence of $\widetilde{I}$ is replaced by I. However, if $\widetilde{I}$ is replaced by I, then the conditions are not equivalent to genericity. A counterexample is $I = \langle xy, xz, xw \rangle$, whose Scarf complex does not change under deformations, and gives a minimal free resolution of S/I.

5.4 Cogenericity

This section demonstrates applications of some results from Chapter 4, and illustrates how the relative cocellular perspective on the cohull resolution can sometimes give insight that is unavailable from the cellular point of view originating from Theorem 5.30. The example here is the coScarf complex of Sturmfels [Stu99]. His original proof of its acyclicity in the cogeneric case was accomplished by viewing it as the chain complex of bounded faces of a simple polyhedron. On the other hand, the dual interpretation as the *co*chain complex of the interior faces in a (regular) triangulation reveals numerical properties more clearly than the polyhedral description. The results of this section appeared in [MSY00, Section 4], although the presentation is slightly improved here due to the clarification provided by injective resolutions.

First let's review the definition of cogeneric monomial ideal. Recall that cogenericity is equivalent to being Alexander dual to a generic ideal (Proposition 1.20).

Definition 5.46 *A monomial ideal with irreducible decomposition* $I = \bigcap_{i=1}^{r} I_i$ *is called* cogeneric *if the following condition holds: if distinct irreducible components* I_i *and* I_j *have a minimal generator in common, there is an irreducible component* $I_\ell \subset I_i + I_j$ *such that* I_ℓ *and* $I_i + I_j$ *do not have a minimal generator in common.*

Theorem 5.47 *Any cohull injective resolution of a cogeneric monomial ideal is minimal. Moreove,* $\mathrm{cohull}^{\mathbf{a}}(I)$ *is independent of* $\mathbf{a}$*, and the free resolution* $\mathrm{cohull}(I)_{\mathbf{1}-\mathbb{Z}}$ *is cellular.*

Proof: Theorem 5.43, Lemma 5.22, Proposition 1.20. □

Definition 5.48 *The* coScarf triangulation Δ^I *of a cogeneric monomial ideal* I *is the labelled regular triangulation supporting the minimal cellular injective resolution of* I*. The* coScarf complex $\Delta^I_{\mathbf{1}-\mathbb{Z}}$ *is the cellular pair consisting of the interior faces of* $|\Delta^I|$*, labelled to support the minimal relative cocellular free resolution of* I *(Corollary 5.28).*

The origin of coScarf complexes in the context of injective resolutions explains, finally, why the "unbounded faces" had to be left out in [Stu99]: they correspond to injective summands which don't show up in the free resolution.

Remark 5.49 It is possible to define the analogue of the coScarf triangulation for any ideal I as the Scarf complex of $I^{[\mathbf{a}]} + \mathfrak{m}^{\mathbf{a}+\mathbf{1}}$ for any $\mathbf{a} \succeq \mathbf{a}_I$. As a combinatorial object this is independent of $\mathbf{a}$, and the analogue of the coScarf injective labelling on this simplicial complex is unique. It is then straightforward to dualize all of Theorem 5.43, though we do not do this here. In fact, the dual to part (f) of that theorem is applied in the proof of Theorem 5.55.

Example 5.50 The top picture in Figure 2.2 is the coScarf triangulation $\Delta^{J^{[444]}}$ of the cogeneric monomial ideal $J^{[444]}$ there. The coScarf complex $\Delta^{J^{[444]}}_{\mathbf{1}-\mathbb{Z}}$ of $J^{[444]}$ was presented in Example 3.26. □

Remark 5.51 That the coScarf free resolution is cellular as opposed to weakly cellular was assumed in [BS98, Example 1.8] but overlooked in [Stu99].

The first main result of this section, Theorem 5.53, characterizes Cohen-Macaulay cogeneric monomial ideals in terms of the coScarf triangulation. But first, a polyhedral description of depth for cogeneric ideals is necessary.

Lemma 5.52 *Let I be a cogeneric monomial ideal. Then* $\operatorname{depth}(S/I) \leq d$ *if and only if the coScarf complex $\Delta^I_{\mathbf{1}-\mathbb{Z}}$ has a face of dimension d.*

Proof: By Theorem 5.47, the shifted augmentation $\mathbb{F}^{\Delta^I_{\mathbf{1}-\mathbb{Z}}} \to S$ (obtained by including $\operatorname{coker}(\mathbb{F}^{\Delta^I_{\mathbf{1}-\mathbb{Z}}}) = I$ into S and shifting homological degrees up one) is a minimal free resolution of S/I. The coScarf complex $\Delta^I_{\mathbf{1}-\mathbb{Z}}$ has a face of dimension d if and only if this shifted augmented complex is nonzero in homological degree $n-d$. We are done by the Auslander-Buchsbaum formula [Eis95, Theorem 19.9]. □

Recall that a module N satisfies Serre's condition (S_k) if for every prime $\mathfrak{p} \subset S$, $\operatorname{depth}(N_\mathfrak{p}) < k \Rightarrow \operatorname{depth}(N_\mathfrak{p}) = \dim(N_\mathfrak{p})$. Using [BH93, Chapter 2.1] and homogeneous localization, it follows that if S/I satisfies (S_k) then

$$\operatorname{depth}((S/I)_{(F)}) < k \quad \Longrightarrow \quad \dim((S/I)_{(F)}) = \operatorname{depth}((S/I)_{(F)}) \tag{5.3}$$

for $F \subseteq \{1, \ldots, n\}$. Observe that $I_{(F)}$ is cogeneric if I is, since localizing an irreducible ideal either makes it into the unit ideal or leaves its generating set fixed. For condition (c) below, use Remark 4.31 and Proposition 4.23 to identify the irreducible components of I with vertices in the coScarf triangulation. For condition (d) below, define the *excess* $e(\sigma)$ of a face $\sigma \in \Delta^I$ to be $e(\sigma) := \#\operatorname{supp}((\mathbf{a}_\sigma)_\mathbb{Z}) - \#\sigma$. This agrees with Stanley's definition in the more general context of triangulations [Sta92].

Theorem 5.53 *Let $I \subset S$ be a cogeneric monomial ideal of codimension c with the irreducible decomposition $I = \bigcap_{i=1}^r I_i$. Then the following conditions are equivalent.*

(a) S/I is Cohen-Macaulay.

(b) S/I satisfies Serre's condition (S_2).

(c) $\operatorname{codim}(I_i) = c$ *for all i, and* $\operatorname{codim}(I_i + I_j) \leq c + 1$ *for all edges $\{i, j\} \in \Delta^I$.*

(d) Every face of Δ^I has excess $< c$.

(e) $\Delta^I_{\mathbf{1}-\mathbb{Z}}$ has no faces of dimension $< n - c$.

Proof:

(a) $\Rightarrow$ (b) Cohen-Macaulay $\Leftrightarrow$ (S_k) for all k.

(b) $\Rightarrow$ (c) The initial equality follows from [Har62, Remark 2.4.1], so it suffices to prove the inequality. If $c = 1$ this is obvious. We assume $c \geq 2$. Suppose $i \neq j$ with $\{i, j\} \in \Delta^I$. Let F be defined by $\mathfrak{m}^F = \sqrt{I_i + I_j}$. By Proposition 5.29 the coScarf triangulation of $I_{(F)}$ is, as a triangulation of the simplex spanned by F, the restriction $\Delta^I_{(F)}$ of the triangulation Δ^I to F. By our choice of F, the edge $\{i, j\}$ is an interior edge of $\Delta^I_{(F)}$, so Lemma 5.52 implies that $\operatorname{depth}((S/I)_{(F)}) \leq 1$, whence (5.3) implies that $\dim((S/I)_{(F)}) \leq 1$. Equivalently, $\operatorname{codim}(I_i + I_j) \leq c + 1$.

(c) $\Rightarrow$ (d) The purity of the irreducible components means that all vertices have excess $c-1$ or 0, while the condition on the edges implies that the excess of a nonempty face can only decrease or remain the same upon the addition of a vertex.

(d) $\Rightarrow$ (e) In particular, the interior faces have excess less than c.

(e) $\Rightarrow$ (a) Lemma 5.52. □

Remark 5.54 Theorem 5.53 as well as its proof should be compared with Proposition 4.8 and its proof.

Recall that the *type* of a Cohen-Macaulay quotient S/I is the nonzero total Betti number of highest homological degree; if I is cogeneric then this Betti number equals the number of interior faces of minimal dimension in Δ^I by Theorem 5.47.

Theorem 5.55 *Let M be a Cohen-Macaulay cogeneric monomial ideal of codimension ≥ 2. The type of S/I is at least the number of irreducible components of I.*

Along with a demonstration of relative cocellular methods, the proof also uses the results of Section 4.2, in particular Theorem 4.25.

Proof: Let $\mathrm{Irr}\,(S/I)$ denote the set of vectors $\mathbf{b} \in \mathbb{N}^n$ for which $\mathfrak{m}^{\mathbf{b}}$ is an irreducible component of I. For any $\mathbf{c} \in \mathbb{N}^n$, we define

$$\gamma_{\mathbf{c}} \;:=\; \#\{\mathbf{b} \in \mathrm{Irr}\,(S/I) \mid \mathbf{b} = \mathbf{c} \cdot F \text{ for some } F \subseteq \{1, \ldots, n\}\}.$$

Set $d = \mathrm{codim}(I)$. The first aim is to show that

$$\#\mathrm{Irr}\,(S/I) \;\leq\; \sum_{\mathbf{c} \in \mathbb{N}^n} \gamma_{\mathbf{c}} \cdot \beta_{d-1,\mathbf{c}}(I). \tag{5.4}$$

In fact, this inequality holds even if I is not cogeneric, using Alexander duality as follows. If $\mathbf{a} \succeq \mathbf{a}_I$ then the definition of $I^{[\mathbf{a}]}$ implies

$$\#\mathrm{Irr}\,(S/I) \;=\; \sum_{\mathbf{b} \in \mathrm{Irr}\,(S/I)} \beta_{0,\mathbf{a} \smallsetminus \mathbf{b}}(I^{[\mathbf{a}]})$$

Since S/I is Cohen-Macaulay of codimension d, each $\mathbf{b} \in \mathrm{Irr}\,(S/I)$ has precisely d non-zero coordinates, so Theorem 4.25 specializes to

$$\beta_{0,\mathbf{a} \smallsetminus \mathbf{b}}(I^{[\mathbf{a}]}) \;=\; \beta_{1,\mathbf{a} \smallsetminus \mathbf{b}}(S/I^{[\mathbf{a}]}) \;\leq\; \sum_{\mathbf{c} \cdot F = \mathbf{b}} \beta_{d-1,\mathbf{c}}(I)$$

for fixed $\mathbf{b} \in \mathrm{Irr}\,(S/I)$ and $F = \mathrm{supp}(\mathbf{b})$. Summing over all $\mathbf{b} \in \mathrm{Irr}\,(S/I)$ proves (5.4).

The Cohen-Macaulay type of S/I is $\sum_{\mathbf{c} \in \mathbb{N}^n} \beta_{d-1,\mathbf{c}}(I)$, so it suffices to prove that if $\beta_{d-1,\mathbf{c}}(I) \neq 0$ then $\gamma_{\mathbf{c}} \leq 1$. Suppose the opposite, that is, $\gamma_{\mathbf{c}} \geq 2$ and $\beta_{d-1,\mathbf{c}}(I) \neq 0$. Then there are sets $F, F' \subseteq \{1, \ldots, n\}$ such that $\mathbf{c} \cdot F, \mathbf{c} \cdot F' \in \mathrm{Irr}\,(S/I)$ are distinct. They correspond to labels $\mathbf{a}_v = \mathbf{1} - (\mathbf{c} \cdot F) + *\overline{F}$ and $\mathbf{a}_{v'} = \mathbf{1} - (\mathbf{c} \cdot F') + *\overline{F'}$ for the vertices $v, v' \in \Delta^I$. Since the coScarf complex $\Delta^I_{\mathbf{1}-\mathbb{Z}}$ determines a minimal free resolution of I and $\beta_{d-1,\mathbf{c}}(I) \neq 0$, there is a face $\sigma \in \Delta^I_{\mathbf{1}-\mathbb{Z}}$ with label $\mathbf{c}$. Hence the corresponding face $\sigma \in \Delta^I$

has label $\mathbf{a}_\sigma = \mathbf{1} - \mathbf{c}$. Since $\mathbf{a}_v$ and $\mathbf{a}_{v'}$ are both $\preceq \mathbf{a}_\sigma$ by construction, we find that σ contains both v and v'. In particular, $\{v, v'\}$ is an edge of Δ^I. Now S/I is Cohen-Macaulay of codimension ≥ 2, so $\mathbf{c} \cdot F$ and $\mathbf{c} \cdot F'$ share a coordinate by Theorem 5.53 (c). This contradicts the following Alexander dual to Theorem 5.43(f): if I is a cogeneric monomial ideal then no irreducible components which share a generator can form an edge in Δ^I. □

Recall that S/I is *Gorenstein* if its Cohen-Macaulay type equals 1. Theorem 5.55 implies:

Corollary 5.56 *Let I be a cogeneric monomial ideal. Then S/I is Gorenstein if and only if I is either a principal ideal or an irreducible ideal.*

Remark 5.57 In the generic monomial ideal case, we have the opposite inequality to the one in Theorem 5.55. More precisely, if I is Cohen-Macaulay and generic then

$$\begin{aligned}\text{Cohen-Macaulay type of } S/I &= \#\{\text{facets of the Scarf complex } \Delta_I\}\\ \leq \quad \#\{\text{facets of } \Delta_{\tilde{I}}\} &= \#\{\text{irreducible components of } I\}\,,\end{aligned}$$

because the map $\Delta_{\tilde{I}} \to \Delta_I$ taking $\sigma \mapsto \{\text{vertices } v \in \sigma \mid \mathbf{a}_v \preceq (D, \ldots, D)\}$ is surjective on facets. Also here, S/I is Gorenstein if and only if it is complete intersection [Yan99, Corollary 2.11].

Remark 5.58 There is another proof of Theorem 5.55 by M. Bayer [Bay99] using Stanley's theory of *local h-vectors* [Sta92]. Bayer casts the inequality as a statement purely about triangulations of a simplex and proves the inequality for quite general types of triangulations.

5.5 Planar graphs

Every monomial ideal (in any number of variables) has a cellular free resolution—the hull resolution. On the other hand, it is not true that every monomial ideal has a minimal cellular resolution. In the case of 3 variables however, minimality can be attained in a cellular way. This section gives a complete proof of this result, announced in [MS99, Theorem 3]. The idea is to choose a generic deformation I_ϵ of the ideal $I \subset k[x, y, z]$, so that specializing $I_\epsilon \rightsquigarrow I$ step by step makes the nonminimal edges in the Scarf triangulation disappear one at a time. The verb *specialize* is used here to indicate that a deformation (generization) is being reversed; thus I is a *specialization* of I_ϵ if the latter is a deformation of the former.

Any minimal cellular free resolution of a monomial ideal in 3 variables is supported on a cell complex of dimension ≤ 2, by Theorem 3.23 (or the Hilbert Syzygy Theorem). For arbitrary I it turns out to be a little cleaner to look for a minimal cellular injective resolution instead of a cellular free resolution. This affords more control over the kinds of cell complexes in question, since minimal cellular injective resolutions are pure of dimension 2 by Lemma 5.15 and Corollary 3.27. However, we can restrict our attention to free resolutions of artinian quotients of $k[x, y, z]$ by Alexander duality.

As nice a condition as "pure of dimension 2" is, we shouldn't be satisfied with it. After all, the hull resolution of an artinian ideal is even better: it is *planar* by Proposition 5.26. Likewise, it would be better to have a minimal cellular injective resolution that is planar. What is a purely 2-dimensional planar regular cell complex? The following definition and resulting notions, taken nearly verbatim from [Tro92, Section 6.3], describe all of the planar cell complexes which come up in this section.

Definition 5.59 *A* planar map *is a finite connected planar graph G together with a plane drawing of G, i.e. a representation of G by points and arcs in the plane $\mathbb{R}^2$ in which there are no edge crossings.*

A vertex (resp. edge) of G is not distinguished from the corresponding point (resp. arc) in the plane. Deleting the vertices and edges of G from the plane leaves several connected components whose closures are the *regions* of the planar map. The unique unbounded face is called the *exterior* region. The only graphs that come up here are simple, i.e. without loops or multiple edges. The distinction between the graph G and the embedded planar map will be blurred, to allow phrases such as "a region of the graph G".

Given a planar graph G, the above discussion tells us how to label G (without its exterior region) so that it supports a cellular monomial matrix, by Proposition 2.13. Thus it makes sense to say that G supports a cellular injective resolution of I. The graphs that appear as minimal injective resolutions satisfy a condition slightly weaker than being 3-connected; this condition is related to that found in [Tro92, Chapter 6] involving "normal families of paths" (and the relation is certainly no coincidence). Basically, adding a single vertex and some edges to G makes it 3-connected. More precisely, given a planar map G and three vertices α, β, γ in the exterior region of G, let $G_\infty(\alpha, \beta, \gamma)$ be the planar map obtained from G by connecting each of α, β, γ to a point ∞ in the exterior region. Then, for suitable α, β, γ, the graph $G_\infty(\alpha, \beta, \gamma)$ is 3-connected.

Theorem 5.60 *Let $I \subset k[x, y, z]$ be a monomial ideal.*

1. *I has a minimal cellular injective resolution supported on a planar graph G. In any such graph there are unique vertices $\widetilde{x}$, $\widetilde{y}$, and $\widetilde{z}$ whose labels are of the form $(d_x, *, *)$, $(*, d_y, *)$, and $(*, *, d_z)$, where $\mathbf{1} \succeq \mathbf{d} = (d_x, d_y, d_z) \in \mathbb{Z}^3$.*
2. *$k[x, y, z]/I$ has a minimal cocellular injective resolution supported on a planar graph G. In any such graph there are unique vertices $\widetilde{x}$, $\widetilde{y}$, and $\widetilde{z}$ whose labels are of the form $(d_x, 1, 1)$, $(1, d_y, 1)$, and $(1, 1, d_z)$, where $\mathbf{0} \succeq \mathbf{d} = (d_x, d_y, d_z) \in \mathbb{Z}^3_*$.*
3. *If $k[x, y, z]/I$ is artinian then it has a minimal cellular free resolution supported on a planar graph G. In any such graph there are unique vertices $\widetilde{x}$, $\widetilde{y}$, and $\widetilde{z}$ whose labels are of the form $(d_x, 0, 0)$, $(0, d_y, 0)$, and $(0, 0, d_z)$, where $\mathbf{d} = (d_x, d_y, d_z) \in \mathbb{N}^3$.*

Any of the graphs $G_\infty(\widetilde{x}, \widetilde{y}, \widetilde{z})$ above are 3-connected.

Example 5.61 A large proportion of the examples in this dissertation fall under the umbrella of this theorem, especially those examples that come with pictures (see the List of Figures, after the Table of Contents). A particularly good example is Figure 5.6 on page 72, since there is otherwise no general reason why the minimal resolutions of the ideals there should be cellular. □

Example 5.62 The graphs G in Theorem 5.60 may fail to be 3-connected. This occurs, for instance, when $I = \langle x, y, z \rangle^2$. If one connects the midpoints of the edges of a triangle, the minimal cellular free resolution of I is gotten by removing any interior edge. □

The three parts of Theorem 5.60 are equivalent by Theorem 3.23 and Corollary 3.27, so the proof below only treats part 3 and the 3-connectedness. But first we present an algorithm for finding G, postponing its justification until later. The algorithm constructs a particularly nice strongly generic ideal I_ϵ which specializes to I. Then it "undeforms" I_ϵ back to I by specializing each coordinate of each exponent vector in an orderly fashion, noting the effect on the minimal graph resolution. The graph at each stage minimally resolves the partially specialized version of I_ϵ. Each time a strict inequality $a_{iu} + \epsilon_{iu} < a_{ju} + \epsilon_{ju}$ is made into an equality $a_{iu} = a_{ju}$, we will find that either the minimal graph resolution remains minimal, or it becomes minimal after precisely one edge is removed. For notation, $\mathbf{a}_i^\epsilon = \mathbf{a}_i + \epsilon_i$, and vectors $\mathbf{b} \in \mathbb{R}^3$ are written $\mathbf{b} = (b_x, b_y, b_z)$.

Algorithm 5.63

INPUT An artinian ideal $I \subset k[x,y,z]$ with minimal generating exponents $\mathbf{a}_1, \ldots, \mathbf{a}_r$

OUTPUT A planar graph G supporting a minimal cellular free resolution of $k[x,y,z]/I$

ϵ := a strong generization $\mathbf{a}_i \mapsto \mathbf{a}_i^\epsilon$ of I satisfying the following condition:

> If $0 < a \in \mathbb{N}$ and $\{i_1, \ldots, i_s\} = \{i \mid a_{ix} = a\}$ are such that $a_{i_1 y} > \cdots > a_{i_s y}$, then $a = a^\epsilon_{i_1 x} < \cdots < a^\epsilon_{i_s x}$; and the same holds when the ordered pair (x, y) is replaced by the pairs (y, z) and (z, x) induced by cyclic permutation of $\{x, y, z\}$.

G := Scarf graph of I_ϵ with vertices labelled by the $\mathbf{a}_i^\epsilon$

WHILE $I_\epsilon \neq I$ DO

Choose $i \in \{1, \ldots, r\}$ and $u \in \{x, y, z\}$ such that $a^\epsilon_{iu} \neq a_{iu}$ and a^ϵ_{iu} is minimal. For ease of notation, assume $u = x$, with the understanding that if $u \neq x$, a *cyclic* permutation of $\{x, y, z\}$ should be applied, translating x to u.

c_x := a_{ix}

c_y := $\min_j \{a_{jy} \mid a_{jx} \leq a_{ix}$ and $a_{jz} \leq a_{iz}$ and $j \neq i\}$

c_z := $\min_j \{a_{jz} \mid a_{jx} \leq a_{ix}$ and $a_{jy} < c_y$ and $j \neq i\}$

ℓ := unique index in $\{1, \ldots, r\}$ with $a_{\ell y} = c_y$ and $a_{\ell z} < c_z$ and $a_{\ell z}$ maximal

REDEFINE G by replacing the x-coordinate a^ϵ_{ix} of the label on the i^{th} vertex of G by a_{ix}

REDEFINE ϵ by replacing the x-coordinate ϵ_{ix} of the i^{th} deformation vector by 0

IF $a_{\ell x} = a_{ix}$

THEN remove an edge whose label is $\mathbf{c} = (c_x, c_y, c_z)$

ELSE leave G unchanged

END IF-THEN-ELSE

END WHILE-DO

OUTPUT G

Remark 5.64 Before delving into the proof of correctness, here are some elementary observations which should help in parsing the algorithm.

1. The prescribed generization specifies a total order on all of the x, y, and z coordinates of the deformed exponent vectors $\mathbf{a}_i^\epsilon$. Therefore, I_ϵ is essentially unique—that is, the condition determines a unique deformation equivalence class.
2. Although ϵ_i may of course be chosen to be in $\mathbb{Q}^3$ for all i, it may be impossible to choose $\epsilon_i \in \mathbb{N}^3$ for all i, because of the condition $a = a^\epsilon_{i_1 u}$ for $u \in \{x, y, z\}$. However, nothing essential changes if the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_r$ and $\mathbf{a}_1^\epsilon, \ldots, \mathbf{a}_r^\epsilon$ are scaled simultaneously. Therefore, it is possible to deal only with integer exponent vectors.

In fact, the algorithm makes perfect sense without choosing any actual vectors ϵ_i at all; only the total ordering as in item 1 above is essential. This abstraction makes the algorithm hairy to state, but perhaps easier to implement.

3. The set used to define c_y is nonempty because there is always an artinian generator of I in the set.
4. $c_y > a_{iy}$ else the generator m_j whose exponent has y-coordinate c_y divides m_i, contradicting minimality of the generating exponents.
5. The set used to define c_z is nonempty because $c_y > 0$ by the previous item, so an artinian generator of I is always in the set.
6. $c_z > a_{iz}$ by minimality of c_y.
7. The set $\mathcal{S} = \{j \mid a_{jy} = c_y \text{ and } a_{jz} < c_z\}$ used to define ℓ is nonempty since the index of the generating exponent whose y-coordinate is c_y is in $\mathcal{S}$.
8. The index $\ell \in \mathcal{S}$ with $a_{\ell z}$ maximal is unique by minimality of the generating exponents.
9. Relabelling G yields a cellular free resolution of the resulting ideal with the new ϵ by Theorem 4.29, though it need not be minimal.

Example 5.65 If $I = \langle x^2, xy, xz, y^2, yz, z^2\rangle$ as in Example 5.62, then the strongly generic deformation $I_\epsilon = \langle x^2, xy^{3/2}, x^{3/2}z, y^2, yz^{3/2}, x^2\rangle$ satisfies the condition of Algorithm 5.63. Furthermore, the Scarf graph of I_ϵ is the triangle with its edge-midpoints connected, as in Example 5.62. If Algorithm 5.63 is run on this I, then one of the three nonminimal edges is removed on the first iteration of the WHILE loop. Precisely which of the nonminimal edges is removed depends on which $u \in \{x, y, z\}$ is chosen first; any u will work, not just $u = x$. In the remaining two iterations of the WHILE loop, no further edges are removed. It is instructive to work out this example by hand; see Figure 5.7 for help with pictures. □

The proof of Theorem 5.60 rests on a tool which allows the Betti numbers in degree $\mathbf{b} \in \mathbb{Z}^n$ of a monomial ideal in n variables to be read off of local properties of the ideal near the vector $\mathbf{b}$. (See [BCP99] for an exposition and proofs.)

Definition 5.66 *The* Koszul simplicial complex *of $I \subseteq k[x_1, \dots, x_n]$ at $\mathbf{b} \in \mathbb{Z}^n$ is*

$$K_{\mathbf{b}}(I) = \{F \subseteq \{1, \dots, n\} \mid \mathbf{x}^{\mathbf{b}-F} \in I\},$$

where it is to be interpreted that $\mathbf{x}^{\mathbf{b}-F} \notin I$ whenever $\mathbf{b} - F \notin \mathbb{N}^n$.

Proposition 5.67 ([Hoc77, Roz70]) $\beta_{i,\mathbf{b}}(I) = \dim_k \widetilde{H}_{i-1}(K_{\mathbf{b}}(I); k)$ *is the dimension of the $(i-1)^{\text{st}}$ reduced simplicial homology of $K_{\mathbf{b}}(I)$ with coefficients in k.*

In particular, there is only one way to have a nonzero $(n-1)^{\text{st}}$ Betti number.

Lemma 5.68 $\beta_{n-1,\mathbf{b}}(I) \neq 0$ *if and only if $K_{\mathbf{b}}(I)$ is the boundary of the entire simplex.*

Proof: Any $(n-2)$-cycle in a simplicial complex Δ with n vertices is a linear combination of facets of the full simplex, and this linear combination has no boundary if and only if every facet is involved the same number of times (mod the characteristic of k). In order for $\widetilde{H}_{n-2}(\Delta; k) \neq 0$, it must therefore be that Δ contains the boundary of the simplex. Clearly Δ cannot contain the interior of the simplex, because all of its homology would vanish. □

Lemma 5.69 *If $I \subset k[x,y,z]$ is an ideal then there is at most one nonzero Betti number of I in degree $\mathbf{b} \in \mathbb{Z}^3$. Any nonzero Betti number equals 1 unless $K_{\mathbf{b}}(I)$ consists precisely of 3 vertices and $\varnothing$, in which case the Betti number is 2.*

Proof: List all of the simplicial complexes on 3 vertices. □

Most of the gruntwork in proving that the algorithm accomplishes its goal is contained in the following two technical lemmas, whose hypotheses are made to be satisfied by the deformation taking place in one pass of the WHILE loop (after a cyclic permutation of (x,y,z) and an application of Remarks 5.64.1 and 5.64.2, perhaps).

Lemma 5.70 *Suppose $I = \langle \mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_r} \rangle \subset k[x,y,z]$ is an artinian monomial ideal and i is an index with $a_{ix} \neq 0$ such that $a_{jx} = a_{ix} \Rightarrow a_{jy} \geq a_{iy}$. Suppose further that $\epsilon = \{\epsilon_j\}_{j=1}^r$ is a deformation of I with $\epsilon_j = \mathbf{0}$ for $j \neq i$ and $\epsilon_i = (1,0,0) =: \mathbf{e}_x$. Let*

$$c_y = \min_{j \neq i} \{a_{jy} \mid a_{jx} \leq a_{ix} \text{ and } a_{jz} \leq a_{iz}\}.$$

Then $K_{\mathbf{b}}(I_\epsilon) = K_{\mathbf{b}}(I)$ unless

$$a_{iz} \leq b_z \quad \text{and} \quad a_{iy} \leq b_y \leq c_y \quad \text{and} \quad b_x \in \{a_{ix}, 1 + a_{ix}\}. \qquad (*)$$

If $b_y \neq c_y$ and $\mathbf{b}$ satisfies $(*)$ with $b_x = a_{ix}$ then $K_{\mathbf{b}+\mathbf{e}_x}(I_\epsilon) = K_{\mathbf{b}}(I)$ while both $K_{\mathbf{b}}(I_\epsilon)$ and $K_{\mathbf{b}+\mathbf{e}_x}(I)$ have no reduced homology. (The case where $\mathbf{b}$ satisfies $(*)$ and $b_y = c_y$ will be covered in Lemma 5.71.)

The idea comes from a relatively simple picture. In Figure 5.7, the grey dots represent minimal generators of I, the white dots represent second syzygies (irreducible components) of I, and the black dots represent first syzygies of I. Looking from far down the x-axis, the generator $\mathbf{x}^{\mathbf{a}_i+\mathbf{e}_x}$ of I_ϵ has a vertical wall (the big medium-grey wall, parallel to the yz-plane) behind it. Pushing this generator back to $\mathbf{x}^{\mathbf{a}_i}$ moves that vertical wall back a single unit. The only places where the topology of $K_{\mathbf{b}}(I)$ can possibly change are at lattice points $\mathbf{b}$ which sit either on the original wall in I_ϵ or its pushed-back image in I; these are the vectors $\mathbf{b}$ described in $(*)$. For vectors $\mathbf{b} + \mathbf{e}_x$ that sit on the original wall but off its right-hand edge (i.e. those with $b_y \neq c_y$), the Koszul simplicial complex $K_{\mathbf{b}+\mathbf{e}_x}(I_\epsilon)$ gets carried along for the ride to $K_{\mathbf{b}}(I)$ as the wall gets pushed back; the empty circles denote where the filled (black and white) dots get moved to. On the other hand, if $\mathbf{b}$ sits on the pushed-back image of the wall in I (as the empty circles strictly to the left of c_y do), then the topology of I_ϵ is translation-invariant in the x-direction near $\mathbf{b}$, making $K_{\mathbf{b}}(I_\epsilon)$ a cone; the same goes for $K_{\mathbf{b}+\mathbf{e}_x}(I)$. And cones have no reduced homology. Having this geometry in mind, here's the official proof of the lemma.

Proof: First we treat the cases when $\mathbf{b}$ fails to satisfy $(*)$. To start off with, $\mathbf{a}_j \preceq \mathbf{b} - F$ if and only if $\mathbf{a}_j^\epsilon \preceq \mathbf{b} - F$ for all $j \neq i$, so the only possible differences in the simplicial complexes $K_{\mathbf{b}}(I)$ and $K_{\mathbf{b}}(I_\epsilon)$ come from the placement of $\mathbf{a}_i$ and $\mathbf{a}_i^\epsilon$ relative to the vectors $\mathbf{b} - F$. If $b_x \notin \{a_{ix}, 1 + a_{ix}\}$ then clearly $\mathbf{a}_i \preceq \mathbf{b} - F$ if and only if $\mathbf{a}_i + \mathbf{e}_x \preceq \mathbf{b} - F$. And if $b_y < a_{iy}$ or $b_z < a_{iz}$ then neither $\mathbf{a}_i$ nor $\mathbf{a}_i + \mathbf{e}_x$ is $\preceq \mathbf{b} - F$ for any F. The only remaining case of $\mathbf{b}$ not satisfying $(*)$ has $b_x \geq a_{ix}$ and $b_y > c_y$ and $b_z \geq a_{iz}$. Suppose that $\mathbf{a}_\ell$ is the

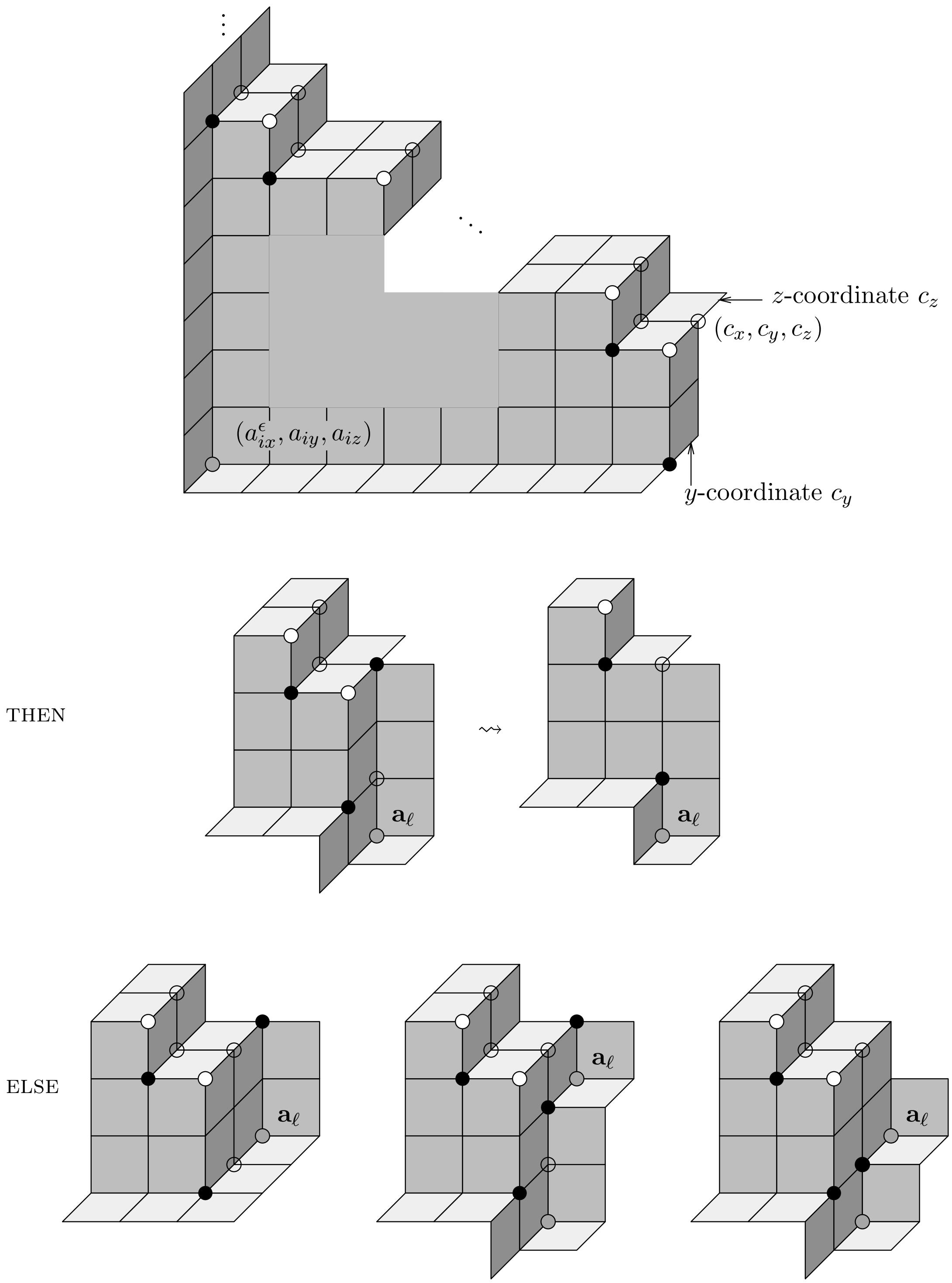


Figure 5.7: The geometry of Algorithm 5.63

exponent vector whose y-coordinate attained the minimum defining c_y. Then $K_{\mathbf{b}}(I)$ and $K_{\mathbf{b}}(I_\epsilon)$ are both contained in $K_{\mathbf{b}}(\langle \mathbf{x}^{\mathbf{a}_\ell}\rangle)$, and are thus independent of $\mathbf{a}_i$ and $\mathbf{a}_i + \mathbf{e}_x$.

Now assume $\mathbf{b}$ satisfies $(*)$ and $b_y < c_y$. Then $\mathbf{a}_j \preceq \mathbf{b} - F$ if and only if $\mathbf{a}_j \preceq \mathbf{b} - (F \cup \mathbf{e}_x)$ whenever $j \neq i$ by the assumption $a_{jx} = a_{ix} \Rightarrow a_{jy} \geq a_{iy}$. If $b_x = a_{ix}$ then $\mathbf{a}_i^\epsilon \not\preceq \mathbf{b}$ and thus $K_{\mathbf{b}}(I_\epsilon)$ is a cone with vertex $\mathbf{e}_x$. If $b_x = 1 + a_{ix}$ then $\mathbf{a}_i \preceq \mathbf{b} - F$ if and only if $\mathbf{a}_i \preceq \mathbf{b} - (F \cup \mathbf{e}_x)$, so $K_{\mathbf{b}}(I)$ is another cone with vertex $\mathbf{e}_x$. Finally, suppose that $b_x = a_{ix}$ in addition to $(*)$ and $b_y < c_y$. Then $\mathbf{a}_j \preceq \mathbf{b} + \mathbf{e}_x - F$ if and only if $\mathbf{a}_j \preceq \mathbf{b} - F$ for $j \neq i$ by the assumption $a_{jx} = a_{ix} \Rightarrow a_{jy} \geq a_{iy}$. And clearly $\mathbf{a}_i^\epsilon \preceq \mathbf{b} + \mathbf{e}_x - F$ if and only if $\mathbf{a}_i \preceq \mathbf{b} - F$, since $\mathbf{a}_i^\epsilon = \mathbf{a}_i + \mathbf{e}_x$. Therefore $K_{\mathbf{b}+\mathbf{e}_x}(I_\epsilon) = K_{\mathbf{b}}(I)$ in this case. □

What does Lemma 5.70 mean? If $\mathbb{F}^\epsilon_{\bullet}$ is a minimal free resolution of I_ϵ and $\mathbb{F}_{\bullet}$ is its specialization to a resolution of I as in Theorem 4.29, then the only possible nonminimal summands of $\mathbb{F}_{\bullet}$ occur in degrees (a_{ix}, c_y, b_z) or $(1 + a_{ix}, c_y, b_z)$ for some value of $b_z \geq a_{iz}$, because the other Betti numbers of I and I_ϵ are in bijection by Lemma 5.70. Furthermore, nonminimal summands cannot come from zeroth syzygies of I_ϵ, since these are in bijection with those of I (no generators of I_ϵ disappear when the ϵ is removed). Therefore, nonminimal syzygies in $\mathbb{F}_{\bullet}$ can only be first or second syzygies.

Such nonminimal summands come in pairs (s_1, s_2) consisting of a first and second syzygy, arising from minimal first and second syzygies $(s_1^\epsilon, s_2^\epsilon)$ of I_ϵ. Since $\deg(s_1) = \deg(s_2)$ but $\deg(s_1^\epsilon) \neq \deg(s_2^\epsilon)$ and the only change is occurring in the x-direction, it must be that $\deg(s_2^\epsilon) = \mathbf{e}_x + \deg(s_1^\epsilon)$. Lemma 5.70 therefore implies that a minimal syzygy s_2^ϵ becoming nonminimal in $\mathbb{F}_{\bullet}$ must have $\deg(s_2^\epsilon)$ along the vertical ray $(1 + a_{ix}, c_y, b_z)$ for varying $b_z \geq a_{iz}$. Furthermore, there can be at most one value c_z for b_z, since there can be only one second syzygy along any line parallel to an axis. Indeed, second syzygies of artinian ideals I_ϵ correspond to irreducible components of I_ϵ by Corollary 4.18 and Proposition 4.23, and the irreducible components are incomparable. This proves all but the last sentence of the next lemma.

Lemma 5.71 *Assume the hypotheses and notation from Lemma 5.70, let $c_x = a_{ix}$, and let*

$$c_z = \min_{j \neq i}\{a_{jz} \mid a_{jx} \leq a_{ix} \text{ and } a_{jy} < c_y\},$$

which exists because I is artinian. Denote by $\mathbb{F}^\epsilon_{\bullet}$ a minimal free resolution of I_ϵ and by $\mathbb{F}_{\bullet}$ the specialized free resolution of I via Theorem 4.29. Then at most two syzygies of $\mathbb{F}^\epsilon_{\bullet}$ become nonminimal in $\mathbb{F}_{\bullet}$: a second syzygy s_2^ϵ in degree $\mathbf{c} + \mathbf{e}_x = (1 + c_x, c_y, c_z)$ and a first syzygy s_1^ϵ in degree $\mathbf{c}$. Choose the unique index ℓ with $a_{\ell y} = c_y$ such that $a_{\ell z} < c_z$ is maximal. Then the specializations (s_1, s_2) of $(s_1^\epsilon, s_2^\epsilon)$ are nonminimal if and only if $a_{\ell x} = c_x$.

Proof: The two specialized syzygies are nonminimal if and only if either one of them is. The specialization of s_2^ϵ is a second syzygy in degree $\mathbf{c}$ that is minimal if and only if $K_{\mathbf{c}}(I)$ is the boundary of a triangle by Lemma 5.68. In any case, $\mathbf{x}^{\mathbf{c}-\mathbf{1}} \notin I$ by minimality of c_z. Suppose $a_{\ell x} \neq c_x$. Then $(0,1,1) \in K_{\mathbf{c}}(I)$ because $c_z > a_{iz}$ and $c_y > a_{iy}$. Any minimal generator $\mathbf{a}_j$ whose z-coordinate was used to define c_z has $a_{jx} < a_{ix}$ by the assumption "$a_{jx} = a_{ix} \Rightarrow a_{jy} \geq a_{iy}$"; thus $(1,1,0) \in K_{\mathbf{c}}(I)$. And $(1,0,1) \in K_{\mathbf{c}}(I)$ because of $\mathbf{a}_\ell$. Therefore, $K_{\mathbf{c}}(I)$ is the boundary of the triangle when $a_{\ell x} \neq c_x$ (in this case, there is no first syzygy of I_ϵ in degree $\mathbf{c}$ waiting to cancel s_2^ϵ as it specializes to s_2). Finally, if $a_{\ell x} = c_x$, then $(1,0,1) \notin K_{\mathbf{c}}(I)$, whence $K_{\mathbf{c}}(I)$ cannot be the boundary of a triangle. □

Example 5.72 Some possible combinatorial types for $K_{\mathbf{b}}(I)$, where $\mathbf{b} = \deg(s_1)$ is the degree of the specialized first syzygy of Lemma 5.71, are depicted in Figure 5.7. The headings "ELSE" and "THEN" correspond to the cases in Algorithm 5.63. Observe that in the single THEN case, the white dot s_2^ϵ at $(1 + c_x, c_y, c_z)$ gets smushed into the vertical plane during specialization and cancels the black dot s_1^ϵ at (c_x, c_y, c_z). On the other hand, the topology remains constant in the first two ELSE cases. In the final ELSE case, two of the black dots merge to become a "double" black dot, since the resulting Koszul simplicial complex (3 disjoint vertices), has 2-dimensional $\widetilde{H}_0$ after the wall is pushed back. □

Proposition 5.73 *At every iteration of the line* END WHILE-DO*, the labelled graph G provides a* minimal *cellular free resolution of I_ϵ.*

Proof: This has two parts, of course: THEN and ELSE. Both follow from Lemma 5.71, given Remark 5.64.9. Indeed, removing the unique nonminimal edge automatically destroys the unique nonminimal region by merging it with an adjacent region. It is here that the precise condition on the deformation ϵ in Algorithm 5.63 is used in an essential way. □

Proof of Theorem 5.60: By the previous proposition, Algorithm 5.63 produces a free resolution as in part 3, so the only thing left to check is that $G_\infty(\widetilde{x}, \widetilde{y}, \widetilde{z})$ is 3-connected. This is really due to a local phenomenon. Given a minimal generating exponent $\mathbf{a}_i$ of I let

$$c_y = \min_{j \neq i}\{a_{jy} \mid a_{jx} \leq a_{ix} \text{ and } a_{jz} \leq a_{iz}\}$$

and $\mathbf{b} = (a_{ix}, c_y, a_{iz})$ (this is the same c_y as in Lemma 5.70, but the only hypothesis there that is also assumed here is that I is artinian). Again, the minimum exists because $y^{d_y} \in I$ for some $d_y \in \mathbb{N}$. Then $\beta_{1,\mathbf{b}}(I) \neq 0$ since $K_{\mathbf{b}}(I)$ is disconnected: $(0, 1, 0) \in K_{\mathbf{b}}(I)$ because of $\mathbf{a}_i$, but neither $(1, 1, 0)$ nor $(0, 1, 1)$ is in $K_{\mathbf{b}}(I)$ by minimality of c_y; one of $(1, 0, 0)$ and $(0, 0, 1)$ is in $K_{\mathbf{b}}(I)$ because of the generating exponent whose y-coordinate is c_y. Choose an edge whose label is $\mathbf{b}$ (there may be two of them by Lemma 5.69). The other vertex label $\mathbf{a}_j$ of this edge has $a_{jx} \leq a_{ix}$ and $a_{jz} \leq a_{iz}$, but $c_y = a_{jy} > a_{iy}$. Therefore, choosing such an edge at each generator makes a path in G from $\mathbf{a}_i$ that eventually reaches the vertex $\widetilde{y} \in G$. There are similar paths from each vertex of G to $\widetilde{x}$ and $\widetilde{z}$. Moreover, the 3 paths from any fixed vertex of G to $\widetilde{x}$, $\widetilde{y}$, and $\widetilde{z}$ do not intersect except at their source vertex, since the labels on the vertices in the 3 paths are strictly increasing in x, y, and z, respectively, but weakly decreasing in all other directions. We conclude that every vertex in $G_\infty(\widetilde{x}, \widetilde{y}, \widetilde{z})$ has three independent paths to ∞, whence $G_\infty(\widetilde{x}, \widetilde{y}, \widetilde{z})$ is 3-connected. □

Theorem 5.60 has the following corollary (which is really what [MS99, Theorem 3] was after). To say that a cell complex X with monomial labels provides an *irredundant* formula for the numerator of the Hilbert series means that the numerator of the Hilbert series with denominator $(1 - x)(1 - y)(1 - z)$ is the alternating sum of the labels on X, and no cancellation occurs.

Corollary 5.74 *Every ideal $I \subset k[x, y, z]$ has a cellular minimal free resolution. It gives an irredundant formula for the numerator of the Hilbert series.*

Proof: The existence of a cellular minimal free resolution follows from Proposition 5.8 and Theorem 5.60.3. The non-cancellation follows from Lemma 5.69. □

Chapter 6

Local cohomology

Increasing attention has been paid recently to properties of local cohomology modules $H_I^\bullet(M)$ of modules M over the polynomial ring S with support on a monomial ideal I [Ter99, Mus00a, Yan00b]. In part, this interest is due to the fact that graded pieces of certain such modules can be interpreted as cohomology groups of sheaves on a toric variety [EMS00, Mus00b]. But another motivation comes from the local cohomology community itself, which generates questions (such as those in [Hun92]) that beg for concrete examples.

When $M = S$, the local cohomology can be described quite concretely. The theory in large part owes its existence to the fundamental theorem of Hochster (Theorem 6.18), which in contrast considers $H_{\mathfrak{m}}^\bullet(S/I_\Delta)$, where $\mathfrak{m} = \langle x_1, \ldots, x_n\rangle$ is the maximal ideal and I_Δ is the Stanley-Reisner ideal of some simplicial complex Δ. Using the Čech complex on $x_1, \ldots, x_n$, Hochster gives an explicit formula for the Hilbert series in terms of the simplicial cohomology of Δ. Subsequent calculations by the authors above concerning $H_{I_\Delta}^\bullet(S)$ have all been accomplished using either Hochster's numerical results, or his Čech complex methods, or both. These calculations for $H_{I_\Delta}^\bullet(S)$ appear to be somewhat more complicated than Hochster's, mostly because the Čech complex on the generators of I_Δ has too many indecomposable summands, so that the resulting simplicial homologies come from simplicial complexes with more vertices than Δ.

The initial aim of this chapter is to demonstrate that there are better (e.g. smaller) complexes of localizations with which to compute local cohomology. The recent structure theorems mentioned above are then presented, at which point injective resolutions are shown to simplify and clarify significantly the calculations. But once injective resolutions enter the picture, a wide variety of new results emerge. It turns out, for example, that the modules $H_{\mathfrak{m}}^\bullet(S/I_\Delta)$ and $H_{I_\Delta}^{n-\bullet}(S)$ are (up to a shift by $\mathbf{1}$) generalized Alexander duals (see Section 3.1), whence all of their module structure and numerical information are actually *equivalent* (Section 6.2). Furthermore, this generalized Alexander duality can be put into a much more general setting, as a $\mathbb{Z}^n$-graded analog of *Greenlees-May duality*, which is an adjointness in the derived category markedly generalizing the usual Grothendieck-Serre local duality (Section 6.3).

6.1 The canonical Čech complex

Throughout this chapter, $I \subset S$ will denote a fixed radical monomial ideal of height d.

Local cohomology $H_I^{\bullet}(M)$ with support on I is usually defined either via an injective resolution of the module M or the Čech complex on a set of generators for I. Here we show that there are lots of complexes which can replace the Čech complex for this purpose. For notation in what follows, recall the *Čech hull* functor $\check{C}$ from Definition 1.30.3, and let ω_S denote the canonical module, which is the $\mathbb{Z}^n$-graded shift $S[-\mathbf{1}] \cong x_1 \cdots x_n S$.

Definition 6.1 *If $\mathbb{F}_{\bullet}$ is a free resolution of S/I and $\mathbb{F}^{\bullet} = \underline{\mathrm{Hom}}(\mathbb{F}_{\bullet}, \omega_S)$, define*

$$\mathbb{C}_{\mathbb{F}}^{\bullet} = (\check{C}\mathbb{F}^{\bullet})[\mathbf{1}]$$

to be the generalized Čech complex *determined by $\mathbb{F}_{\bullet}$. Give names to two special cases:*

1. *When $\mathbb{F}_{\bullet}$ is minimal, call $\mathbb{C}_{\mathbb{F}}^{\bullet}$ the* canonical Čech complex of I, *and denote it by $\mathbb{C}_I^{\bullet}$.*
2. *When $\mathbb{F}_{\bullet} = \mathbb{T}_{\bullet}(I)$ is the Taylor resolution of S/I (see Example 6.3, below), call $\mathbb{C}_{\mathbb{T}(I)}^{\bullet}$ simply* the Čech complex of I *(Proposition 6.4 justifies this name).*

Example 6.2 The canonical Čech complex of $\mathfrak{m}$ is the usual Čech complex; that is, $\mathbb{C}_{\mathfrak{m}}^{\bullet} = \mathbb{C}^{\bullet}(x_1, \ldots, x_n)$ is the complex

$$\mathbb{C}_{\mathfrak{m}}^{\bullet}:\ 0 \to S \to \bigoplus_{i=1}^{n} S[x_i^{-1}] \to \cdots \to \bigoplus_{|F|=\ell} S[\mathbf{x}^{-F}] \to \cdots \to S[x_1^{-1}, \ldots, x_n^{-1}] \to 0\,,$$

where $\mathbf{x}^F = \prod_{i \in F} x_i$ for $F \subseteq \{1, \ldots, n\}$. Since the Koszul complex is self-dual, the Cech hull of the Koszul complex is the shift by $[-\mathbf{1}]$ of the Čech complex (see Example 2.5).

This example is the reason for the term "Čech hull". The Čech hull appeared in [Mil98] before Theorem 6.7 was known, although the use of the term there was to indicate that it transformed merely the homology of the Koszul complex into the homology of the Čech complex, as opposed to transforming the complexes themselves. □

Example 6.3 The previous example generalizes as follows. Recall from Section 2.3 that the *Taylor resolution* $\mathbb{T}_{\bullet}(I)$ for the ideal $I = \langle \mathbf{x}^{F_1}, \ldots, \mathbf{x}^{F_r} \rangle$ is the cellular free resolution of S/I supported on the simplex X with vertices $\{1, \ldots, r\}$ labelled by $F_1, \ldots, F_r$. (The label on each face $G \in X$ is defined to be the join of the vertex labels in G.) The terminology in Definition 6.1 agrees with common practice:

Proposition 6.4 *$\mathbb{C}_{\mathbb{T}(I)}^{\bullet}$ is the usual Čech complex on the generators of I. That is, $\mathbb{C}_{\mathbb{T}(I)}^{\bullet}$ is the cocellular complex of flat modules*

$$\mathbb{C}_{\mathbb{T}(I)}^{\bullet}: 0 \to S \to \bigoplus_{i=1}^{n} S[\mathbf{x}^{-F_i}] \to \cdots \to \bigoplus_{\substack{G \in X \\ |G|=\ell}} S[\mathbf{x}^{-F_j} \mid j \in G] \to \cdots \to S[\mathbf{x}^{-F_1}, \ldots, \mathbf{x}^{-F_r}] \to 0$$

supported on the same simplex as $\mathbb{T}_{\bullet}(I)$ except that each label F_i has been replaced by $*F_i$.

Proof: Each summand $S[-F]$ in $\mathbb{T}_\bullet(I)$ becomes a summand $S[-\overline{F}] = \underline{\mathrm{Hom}}_S(S[-F], \omega_S)$ in $\underline{\mathrm{Hom}}_S(\mathbb{T}_\bullet(I), \omega_S)$, where $\overline{F} = \{1, \ldots, n\} \setminus F$. By Example 1.31.1, $\check{C}(S[-\overline{F}]) \cong S[\mathbf{x}^{-F}][-\overline{F}]$, and this is isomorphic to $S[\mathbf{x}^{-F}][-\mathbf{1}]$ because x_i acts as a unit whenever $i \in F$. Shifting by $[\mathbf{1}]$ yields the result. One can also check everything directly using operations on monomial matrices (Section 3.3). □

Example 6.5 The proof of Proposition 6.4 says how to describe any generalized Čech complex $\mathbb{C}^\bullet_{\mathbb{F}}$ in more familiar terms. After choosing bases in the complex $\mathbb{F}^\bullet = \underline{\mathrm{Hom}}_S(\mathbb{F}_\bullet, \omega_S)$, simply replace each summand $S[-\overline{F}]$ by the localization $S[\mathbf{x}^{-F}]$. This description is easiest to effect on monomial matrices, particularly cellular monomial matrices (Chapter 2).

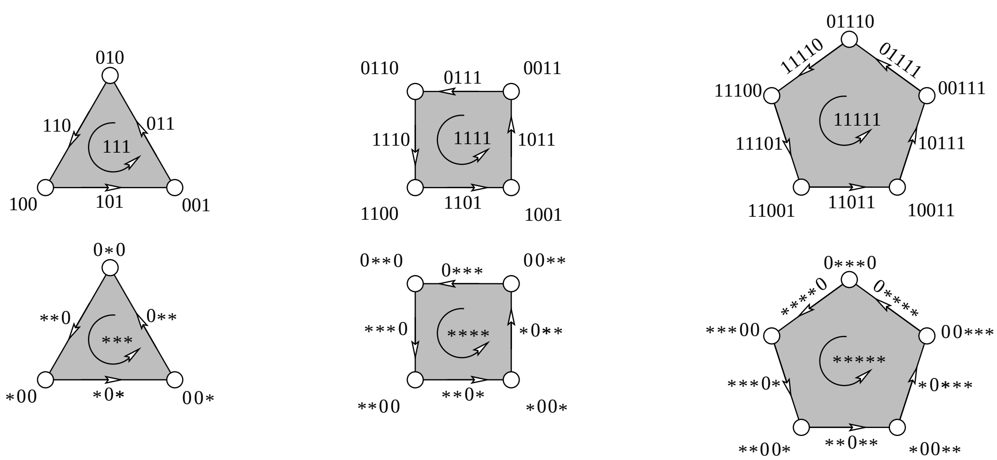


Figure 6.1: Some cellular resolutions and cocellular canonical Čech complexes

Take, for instance, the triangle, square, and pentagon in Figure 6.1, which can be regarded as polyhedral cell complexes with orientations as shown. When labelled as in the top diagrams, these respectively determine cellular minimal free resolutions of

$$S/\langle a, b, c\rangle, \quad S/\langle ab, bc, cd, ad\rangle, \quad \text{and} \quad S/\langle abc, bcd, cde, ade, abe\rangle$$

for appropriate S. The associated canonical Čech complexes are depicted in the bottom diagrams, which determine *cocellular* monomial matrices (Proposition 2.13). The empty set is labelled by $0 \cdots 0$ in all six pictures. Therefore, if $I = \langle m_1, \ldots, m_r\rangle$, the first map in the canonical Čech complex is $S \to \bigoplus_{i=1}^r S[m_i^{-1}]$, just like the usual Čech complex. Observe that the bottom triangle actually is the usual Čech complex, whereas the top triangle is the Koszul complex (see Examples 2.5 and 6.2).

This example works more generally for the *irrelevant ideal I* of the *Cox homogeneous coordinate ring* of a smooth projective toric variety V (see [Ful93] and [Cox95] for the relevant background). One is particularly interested in computing local cohomology with support on I in this case, because it is important for sheaf cohomology on V [EMS00]. The minimal free resolution of I [MP00] is a linear cellular resolution supported on the moment polytope P of V, which is simple. By [EMS00], these examples provide yet another polyhedral way to think of cohomology of line bundles on smooth projective toric varieties.

Interestingly, these cellular "moment resolutions" are also cocellular, supported on the simplicial polytope Q polar to the moment polytope P. Therefore, the canonical Čech complex can be regarded either as a cellular or a cocellular flat complex. This is particularly evident for the cases above, where the polar polytope can be inscribed. $\square$

Remark 6.6 The canonical Čech complex of I depends (up to isomorphism of $\mathbb{Z}^n$-graded complexes) only on I, not on any system of generators of I. The term "canonical Čech complex" refers both to this freedom from choices and the fact that it is, up to $\mathbb{Z}^n$-graded and homological shift, the Čech hull of a free resolution of the canonical module $\omega_{S/I}$ when S/I is Cohen-Macaulay (see Theorem 6.7). When talking about canonical Čech complexes, a little caution is warranted because $\mathbb{C}_I^{\bullet}$, like the minimal free resolution of S/I, does *not* necessarily depend only on the Stanley-Reisner complex of I. For instance, it can depend on the characteristic of S.

If one is convinced that the minimal free resolution of S/I is in any sense "better" than the Taylor resolution, then one should be equally convinced that the canonical Čech complex $\mathbb{C}_I^{\bullet}$ is similarly "better" than the usual Čech complex $\mathbb{C}^{\bullet}_{\mathbb{T}(I)}$. Indeed, the next result shows how the canonical Čech complex upstages $\mathbb{C}^{\bullet}_{\mathbb{T}(I)}$ even in the role for which the latter was created. The adjective "positively $\mathbf{1}$-determined" is presented formally in Definition 1.25, but in the theorem here it just means that all of the summands in $\mathbb{F}_{\bullet}$ are generated in $\mathbb{Z}^n$-graded degrees contained in $\{0,1\}^n$.

Theorem 6.7 *If $\mathbb{F}_{\bullet}$ is a positively $\mathbf{1}$-determined free resolution of S/I and M is an object in the category $\mathcal{M}$ of $\mathbb{Z}^n$-graded modules, then*

$$H_I^i(M) \cong H^i(M \otimes_S \mathbb{C}_{\mathbb{F}}^{\bullet}).$$

In particular, if S/I is Cohen-Macaulay then

$$H_I^i(M) \cong \underline{\mathrm{Tor}}^S_{d-i}(M, \check{C}(\omega_{S/I}))[\mathbf{1}]\,.$$

Proof: Let Λ be a matrix for $\mathbb{F}_{\bullet}$ with row labels $\{\mathbf{b}_{p\cdot}\}$ and column labels $\{\mathbf{b}_{\cdot q}\}$. Lemmas 3.15 and 3.16 say that the induced matrix for $\mathbb{F}^{\bullet} := \underline{\mathrm{Hom}}(\mathbb{F}_{\bullet}, \omega_S)$ has row labels $\{\overline{\mathbf{b}}_{\cdot q}\}$ and column labels $\{\overline{\mathbf{b}}_{p\cdot}\}$. The complex $\check{C}\mathbb{F}^{\bullet}$ is the direct limit (union) over $t \in \mathbb{N}$ of the free complexes $\mathbb{L}_t := (\check{C}\mathbb{F}^{\bullet})_{\succeq -t\cdot\mathbf{1}}$. Now $\check{C}(S[-\overline{F}]) \cong S[\mathbf{x}^{-F}][-\overline{F} + *F]$ whenever $F \subseteq \{1,\ldots,n\}$ by Example 1.31.1, and there is an easy isomorphism $S[\mathbf{x}^{-F}][-\overline{F}+*F]_{\succeq -t\cdot\mathbf{1}} \cong S[-\overline{F}+t\cdot F]$. Thus the induced matrix for $\mathbb{L}_t$ has row labels $\{\overline{\mathbf{b}}_{\cdot q} - t\,\mathbf{b}_{\cdot q}\}$ and column labels $\{\overline{\mathbf{b}}_{p\cdot} - t\,\mathbf{b}_{p\cdot}\}$.

If I is has generators $m_1,\ldots,m_r$, then the t^{th} *Frobenius power* of I for $t \in \mathbb{N}$ is the ideal $I^{[t]} = \langle m_1^t,\ldots,m_r^t\rangle$. The resolution $\mathbb{F}_{\bullet}$ of S/I induces a resolution $\mathbb{F}_{\bullet}^{[t]}$ of $S/I^{[t]}$ which is obtained from $\mathbb{F}_{\bullet}$ by substituting x_s^t for every occurrence of the variable x_s. The effect on the matrix Λ for $\mathbb{F}_{\bullet}$ is to multiply all of the labels of Λ by t. This, in turn, induces a matrix on $\mathbb{F}^{\bullet}_{[t]}[-\mathbf{1}] := \underline{\mathrm{Hom}}(\mathbb{F}_{\bullet}^{[t]}, S[-\mathbf{1}])$ whose row labels are $\{\mathbf{1} - t\,\mathbf{b}_{\cdot q}\}$ and whose column labels are $\{\mathbf{1} - t\,\mathbf{b}_{p\cdot}\}$ by Lemmas 3.15 and 3.16. Comparing this with the matrix for $\mathbb{L}_t$ above, we conclude that $\mathbb{F}^{\bullet}_{[t]}[-\mathbf{1}] \cong \mathbb{L}_{t-1}$.

The free complex $\mathbb{F}^{\bullet}_{[t]}[-\mathbf{1}]$ has homology $\underline{\mathrm{Ext}}^{\bullet}_S(S/I^{[t]}, S[-\mathbf{1}])$, so we can calculate

$$\begin{aligned} H^i_I(M) &\cong \varinjlim_t \underline{\mathrm{Ext}}^i_S(S/I^{[t]}, M) \\ &\cong \varinjlim_t H^i(\mathbb{F}^{\bullet}_{[t]}[-\mathbf{1}] \otimes_S M)[\mathbf{1}] \\ &\cong \varinjlim_t H^i(\mathbb{L}_{t-1} \otimes_S M)[\mathbf{1}] \\ &\cong H^i \varinjlim_t (\mathbb{L}_{t-1} \otimes_S M)[\mathbf{1}] \\ &\cong H^i((\varinjlim_t \mathbb{L}_{t-1}) \otimes_S M)[\mathbf{1}] \\ &\cong H^i(\check{C}\mathbb{F}^{\bullet} \otimes_S M)[\mathbf{1}] \\ &\cong H^i(\mathbb{C}^{\bullet}_{\mathbb{F}} \otimes_S M)\,, \end{aligned}$$

proving the first equation. The remaining statement holds because $\mathbb{F}^{\bullet}$ is a free resolution of $\omega_{S/I}$ when S/I is Cohen-Macaulay, and because the Čech hull, which takes free modules to flat modules by Lemma 3.3, is exact. In particular, $\check{C}\mathbb{F}^{\bullet}$ is a flat resolution of $\check{C}(\omega_{S/I})$, which tensored with M calculates the $\underline{\mathrm{Tor}}$ module in question. □

Example 6.8 The proof above shows how the transformation Taylor complex ⇝ Čech complex is the result of applying direct limits to [Mus00a, Theorem 1.1]. There, limits were applied to the homology of $\underline{\mathrm{Hom}}(\mathbb{F}, S)$, but not to the complex itself. □

Remark 6.9 The canonical Čech complex calculates local cohomology in any grading (including the non-grading) for which the variables are homogeneous. This is because no properties of M were used in the proof of Theorem 6.7 and the limit works just as well in any grading. For instance, gradings by quotients of $\mathbb{Z}^n$ appear in calculations of sheaf cohomology on toric varieties [EMS00]. See Example 6.5 for the canonical Čech complex in the case of a smooth projective toric variety.

It is worth noting that Theorem 6.7 is still true without assuming $\mathbb{F}_{\bullet}$ is positively $\mathbf{1}$-determined. Unfortunately, the notation in the proof becomes worse if Λ has labels that are not in $\{0, 1\}^n$. But there is another proof of Theorem 6.7 that has no such limitations, and is perhaps even more revealing because it pinpoints the relation between the canonical Čech complex and the usual Čech complex as being analogous to the relation between the minimal free resolution of S/I and the Taylor resolution. In what follows, a *quism* (also known as a quasi-isomorphism) of complexes is a homomorphism inducing an isomorphism on homology. The second proof of Theorem 6.7 is mainly the following quite simple observation.

Proposition 6.10 *There is a quism $\mathbb{C}^{\bullet}_{\mathbb{F}} \to \mathbb{C}^{\bullet}_{\mathbb{T}(I)}$ for any finite free resolution $\mathbb{F}_{\bullet}$ of S/I.*

Proof: This is induced by applying $(\check{C}\underline{\mathrm{Hom}}(-, \omega_S))[\mathbf{1}]$ to any quism $\mathbb{T}_{\bullet}(I) \to \mathbb{F}_{\bullet}$. These exist because there are quisms $\mathbb{F}_{\bullet} \to \mathbb{F}'_{\bullet}$ and $\mathbb{F}'_{\bullet} \to \mathbb{T}_{\bullet}$ where $\mathbb{F}'_{\bullet}$ is the minimal free resolution of S/I. Indeed, $\mathbb{F}'_{\bullet}$ is a split subcomplex of every free resolution of S/I. □

Second proof of Theorem 6.7: The two complexes in Proposition 6.10 are bounded, so the following lemma (whose proof we provide, even though it is standard) applies. □

Lemma 6.11 *If $\mathbb{L}_\bullet \to \mathbb{L}'_\bullet$ is a quism of cohomologically bounded above complexes of flat modules and M is any module, then $M \otimes \mathbb{L}_\bullet \to M \otimes \mathbb{L}'_\bullet$ is also a quism.*

Proof: Let $\mathbb{F}_\bullet$ be a projective resolution of M. The spectral sequence obtained from $\mathbb{F}_\bullet \otimes \mathbb{L}_\bullet$ by first taking homology in the $\mathbb{L}$ direction has E^1_{pq} term $F_q \otimes H_p\mathbb{L}_\bullet \cong F_q \otimes H_p\mathbb{L}'_\bullet$ because $\mathbb{F}_\bullet$ is flat and $\mathbb{L} \to \mathbb{L}'_\bullet$ is a quism. Therefore, the natural morphism of these spectral sequences $\{E^r\} \to \{(E')^r\}$ induced by $\mathbb{L}_\bullet \to \mathbb{L}'_\bullet$ is an isomorphism for $r \geq 1$, whence $\mathrm{Tot}(\mathbb{F}_\bullet \otimes \mathbb{L}_\bullet) \to \mathrm{Tot}(\mathbb{F}_\bullet \otimes \mathbb{L}'_\bullet)$ is a quism [Mac95, Theorem XI.3.4]. On the other hand, the augmentations $\mathrm{Tot}(\mathbb{F}_\bullet \otimes \mathbb{L}_\bullet) \to M \otimes \mathbb{L}_\bullet$ and $\mathrm{Tot}(\mathbb{F}_\bullet \otimes \mathbb{L}'_\bullet) \to M \otimes \mathbb{L}'_\bullet$ are quisms because of the spectral sequences obtained by first taking homology in the $\mathbb{F}$ direction, whose E^1 terms are $M \otimes \mathbb{L}_\bullet$ and $M \otimes \mathbb{L}'_\bullet$ by flatness of $\mathbb{L}_\bullet$ and $\mathbb{L}'_\bullet$. The lemma follows since the homomorphism between the augmentations is induced by the homomorphism of total complexes. □

Remark 6.12 What this second proof of Theorem 6.7 really says is that $\mathbb{C}^\bullet_{\mathbb{F}}$ and $\mathbb{C}^\bullet_{\mathbb{T}(I)}$ are isomorphic in the derived category $D_b(\mathcal{M})$ of bounded complexes, so that the right derived functor $\mathbb{R}\Gamma_I(-)$ equals $-\otimes \mathbb{C}^\bullet_{\mathbb{T}(I)}$ and $-\otimes \mathbb{C}^\bullet_{\mathbb{F}}$ as functors on $D_b(\mathcal{M})$. This point of view will be revisited in Section 6.3, and is the content of the next proposition, which is standard.

Proposition 6.13 *If $\mathbb{L}_\bullet \to \mathbb{L}'_\bullet$ is a quism of bounded-below complexes of flat modules and $\mathbb{B}$ is any bounded complex of modules, then* $\mathrm{Tot}(\mathbb{B} \otimes \mathbb{L}) \to \mathrm{Tot}(\mathbb{B} \otimes \mathbb{L}')$ *is also a quism.*

Proof: The natural morphism of spectral sequences obtained by first taking homology in the $\mathbb{L}$ and $\mathbb{L}'$ directions is an isomorphism $\mathbb{B} \otimes H\mathbb{L} \xrightarrow{\sim} \mathbb{B} \otimes H\mathbb{L}'$ on the E^1 terms by Lemma 6.11. Now use [Mac95, Theorem XI.3.4]. □

Theorem 6.7 and the Čech hull can be used to answer, in the special case of $\mathbb{Z}^n$-graded local cohomology with monomial support, four questions of Huneke [Hun92]:

1. When are $H^r_I(M) = 0$?
2. When are $H^r_I(M)$ finitely generated?
3. When are $H^r_I(M)$ artinian?
4. If M is finitely generated, is the number of associated primes of $H^r_I(M)$ always finite?

Observe that the restriction to $\mathbb{Z}^n$-graded modules is not so bad, since the polynomial ring S is $\mathbb{Z}^n$-graded, and a fair amount of the behavior of the functor H^r_I is governed by its values on S. Terai [Ter99, Section 3] used his Hilbert series calculation (Theorem 6.20, below) to answer the first three of these questions for $M = S$. The crux of his reasoning is to deduce the answer to each of the questions from a correpsonding property of $H^{n-r}_{\mathfrak{m}}(S/I)$ or $\underline{\mathrm{Ext}}^r_S(S/I, \omega_S)$.

Since Terai does such a complete job of answering Questions (1)–(3) for $M = S$ (equivalently, $M = \omega_S$), we give just a survey here, focusing more on what happens for general $\mathbb{Z}^n$-graded modules M. Question (4) is answered more precisely.

1. The canonical Čech complex $\mathbb{C}^\bullet_I$ of I has length $\mathrm{proj.dim}(S/I)$, so $H^r_I(M)$ automatically vanishes for $r \geq \mathrm{proj.dim}(S/I)$. This result was originally proved by Lyubeznik [Lyu84]. Of course, the result follows from the case $M = S$ by the usual generalities, and is true for any module M (graded or not; finitely generated or not), by Remark 6.9.

2. Very rarely are these local cohomology modules finitely generated. This is because, up to shift, the local cohomology modules can be expressed as Čech hulls:

   Lemma 6.14 *If M is finitely generated then some $\mathbb{Z}^n$-graded shift of $H^r_I(M)$ is* **a***-determined for some* $\mathbf{a} \in \mathbb{N}^n$ *(Definition 1.25).*

   Proof: By $\mathbb{Z}^n$-graded shifting, assume M is $\mathbb{N}^n$-graded and **a**-determined (it will be necessary to shift M to the *interior* of $\mathbb{N}^n$ for this to occur). Then every localization of M is also **a**-determined. Calculating $H^r_I(M)$ by the usual Čech complex or the canonical Čech complex yields the desired result. □

   So the problem comes down to seeing what kinds of **a**-determined modules can be finitely generated, and which of those can occur as local cohomology modules. The first half of this is an easy lemma, whose proof follows from the definition of **a**-determined.

   Lemma 6.15 *An* **a***-determined module N is finitely generated if and only if it is contained in the interior of* $\mathbb{N}^n$.

   As for which modules occur as local cohomology, note that any nonzero localization of any finitely generated module M fails to be contained in the interior of $\mathbb{N}^n$. This usually causes $H^r_I(M)$ to be infinitely generated. For instance, when $M = \omega_S$, the local cohomology is never finitely generated if it is nonzero [Ter99, Corollary 3.1].

3. This is the hardest of the four, in general. Let us be content with Terai's result [Ter99, Corollary 3.7]: $H^r_I(S)$ is artinian if and only if $H^{n-r}_{\mathfrak{m}}(S/I)$ is a finite-dimensional k-vector space. It follows easily from Theorem 6.22, below.

4. Yes: the set of associated primes is finite whenever M is finitely generated; in fact, the Bass numbers are finite, too. All of this follows from Lemma 6.14, since an **a**-determined module is the Čech hull of its $\mathbb{N}^n$-graded part (which is finitely generated) and the Čech hull is an essential extension. The statement about Bass numbers fails for semigroup-graded ideals and modules over nonsimplicial affine semigroup rings (see Remark 6.17, and [HM00] for details).

Remark 6.16 The possible uses of the material in this section (and Section 6.3, below) toward effective computation of local cohomology with monomial support have not been seriously explored. Such effective computation is important for calculations of sheaf cohomology on toric varieties [EMS00]. The canonical Čech complex should lend itself better to computation than the usual Čech complex because the former is often much smaller; it is certainly shorter. One possibility uses the that the canonical Čech complex is a complex of holonomic modules over the Weyl algebra when char$(k) = 0$. This follows from [Yan00b, Remark 2.13], because the canonical Čech complex $\mathbb{C}^{\bullet}_I$ is, like the usual Čech complex $\mathbb{C}^{\bullet}_{\mathbb{T}(I)}$, a direct sum of localizations of squarefree shifts of S. Thus, Walther's D-module algorithms for computation of $H^{\bullet}_J(M)$ when $J \subset S$ is any (not necessarily monomial) ideal [Wal99] might be useful (even for I in the nongraded case, cf. Remark 6.9).

Remark 6.17 The methods in this section relate local cohomology modules to finitely generated Ext modules via the Čech hull. This sort of process generalizes to affine semigroup algebras (rings whose spectra are affine toric varieties), except that the Čech hull is no longer exact. Using its derived functors however, one obtains a spectral sequence involving certain kinds of iterated Ext modules which converge to the local cohomology. Such calculations give insight into the ill-understood relationship between local cohomology modules and the Ext modules of which they are limits [EMS00], and puts the example of Hartshorne of an infinite-dimensional socle [Har70] into a general framework. This point of view also bridges the gap between local cohomology with monomial support in polynomial rings and Stanley's calculations of local cohomology with maximal support in affine semigroup rings [Sta82], while generalizing techniques developed by Yanagawa [Yan00c]. This project is currently being developed jointly with David Helm [HM00].

6.2 Hilbert series and module structures

Hilbert series for $\mathbb{Z}$-graded rings are classical and well-studied objects in commutative algebra and algebraic geometry. Beginning with the work of Stanley and Hochster [Sta96, Sta82, Hoc77], writing down formulas for $\mathbb{Z}^n$-graded Hilbert series of various modules has become increasingly popular. The starting point for this section is the following theorem of Hochster, which he never published, though it finally appeared in [Sta96, Theorem II.4.1]. For notation, the $\mathbb{Z}^n$-graded Hilbert series of a module M is $H(M;\mathbf{x})$; the *link* of of face G in a simplicial complex Δ is

$$\text{link}_\sigma\Delta := \{\tau \in \Delta \mid \tau \cap \sigma = \varnothing \text{ and } \tau \cup \sigma \in \Delta\};$$

and $\widetilde{H}^\bullet$ is reduced cohomology.

Theorem 6.18 (Hochster) *Let $I = I_\Delta$ be the Stanley-Reisner ideal of a simplicial complex Δ on $\{1,\ldots,n\}$. Then*

$$H(H^r_{\mathfrak{m}}(S/I_\Delta);\mathbf{x}) = \sum_{\sigma\in\Delta} \dim_k \widetilde{H}^{r-|\sigma|-1}(\text{link}_\sigma\Delta;k) \prod_{j\in\sigma} \frac{x_j^{-1}}{1-x_j^{-1}}\,.$$

Hochster's formula gives only the dimensions of the $\mathbb{Z}^n$-graded degrees of $H^r_{\mathfrak{m}}(S/I_\Delta)$. But a paper of Gräbe (which seems to have been overlooked by many recent studies) gives the module structure of $H^r_{\mathfrak{m}}(S/I_\Delta)$ in terms of simplicial homology [Grä84].

Theorem 6.19 (Gräbe) *There are isomorphisms $H^r_{\mathfrak{m}}(S/I_\Delta)_{\mathbf{b}} \cong \widetilde{H}^{r-|\sigma|-1}(\text{link}_\sigma\Delta;k)$ for each graded degree $\mathbf{b} \preceq \mathbf{0}$ and $\sigma = \text{supp}(\mathbf{b})$ such that given $\sigma\in\Delta$ and $i \notin \sigma$, the maps*

$$\widetilde{H}^{r-|\sigma\cup i|-1}(\text{link}_{\sigma\cup i}\Delta;k) \xrightarrow{\cdot x_i} \widetilde{H}^{r-|\sigma|-1}(\text{link}_\sigma\Delta;k)$$

for $b_i = -1$ are induced by the differential of the reduced cochain complex $\widetilde{\mathcal{C}}^\bullet(X;k)$ via the isomorphisms $\widetilde{H}^{r-1-|\sigma|}(\text{link}_\sigma\Delta) \cong H^{r-1}(\Delta,\text{cost}_\sigma\Delta;k)$ (see Equation (6.1) on page 96).

Recently, Terai wrote down the Hilbert series of the modules $H^i_I(S)$, inspired by Hochster's formula [Ter99]. Terai used his formula to answer, in the special case of squarefree monomial ideals, some questions of Huneke [Hun92] concerning when local cohomology with support on a (not necessarily monomial) ideal is zero, finitely generated, or artinian. His methods were similar to those used by Hochster, exploiting simplicial complexes appearing in various $\mathbb{Z}^n$-graded degrees of the usual Čech complex.

Theorem 6.20 (Terai) *The $\mathbb{Z}^n$-graded Hilbert series of $H^r_{I_\Delta}(S)$ is*

$$H(H^r_{I_\Delta}(S);\mathbf{x}) = \sum_{\sigma\in\Delta} \dim_k \widetilde{H}_{n-r-|\sigma|-1}(\mathrm{link}_\sigma\Delta;k) \prod_{i\in\overline{\sigma}} \frac{x_i^{-1}}{1-x_i^{-1}} \prod_{j\in\sigma} \frac{1}{1-x_j}.$$

While Terai made his calculation, M. Mustaţă simultaneously and independently gave the module structure in terms of simplicial cohomology [Mus00a]. His proof relied on calculating the local cohomology as a limit of $\underline{\mathrm{Ext}}$ modules, and then applying Hochster's formula on the smallest necessary finite pieces. We present here a new and direct proof of Mustaţă's result from scratch, using a cellular injective resolution. For notation, $\Delta|_\sigma = \{\tau\in\Delta \mid \tau\subseteq\sigma\}$ is the *restriction of* Δ *to* σ; the complex $\Delta^\star$ is the *Alexander dual* of Δ (page 9); the complement of σ in $\{1,\ldots,n\}$ is $\overline{\sigma}$; and $\mathbf{b}_-$ is the meet $\mathbf{b}\wedge\mathbf{0}$ for $\mathbf{b}\in\mathbb{Z}^n$.

Theorem 6.21 (Mustaţă) *Each graded piece $H^r_{I_\Delta}(S)_\mathbf{b}$ is isomorphic to $\widetilde{H}^{r-2}(\Delta^\star|_{\overline{\sigma}};k)$, where $\mathbf{b}\in\mathbb{Z}^n$ and $\mathrm{supp}(\mathbf{b}_-)=\overline{\sigma}$. For $i\in\overline{\sigma}$, the maps on simplicial cohomology*

$$\widetilde{H}^{r-2}(\Delta^\star|_{\overline{\sigma}};k) \longrightarrow \widetilde{H}^{r-2}(\Delta^\star|_{\overline{\sigma}\setminus i};k)$$

induced by the inclusions $\Delta^\star|_{\overline{\sigma}\setminus i}\to\Delta^\star|_{\overline{\sigma}}$ agree with multiplication by x_i on the graded piece $H^r_\mathfrak{m}(S/I_\Delta)_\mathbf{b}$ whenever $b_i=-1$. If $b_i\neq -1$, then multiplication by x_i is an isomorphism.

Proof: It is more natural and notationally easier to calculate $H^\bullet_{I_\Delta}(\omega_S) = H^\bullet_{I_\Delta}(S)[-\mathbf{1}]$, observing that $\mathrm{supp}(\mathbf{b}_-)=\overline{\sigma}$ if and only if $\mathrm{supp}((\mathbf{b}+\mathbf{1})_+)=\sigma$, where $\mathbf{b}_+=\mathbf{b}\vee\mathbf{0}$.

The minimal injective resolution $\mathbb{J}^\bullet$ of ω_S is the Matlis dual of $\mathbb{C}^\bullet_\mathfrak{m}$ (cohomologically shifted so the nonzero cohomology is in degree 0). $\mathbb{J}^\bullet$ is cocellular, supported on an $(n-1)$-simplex X with each face $\sigma\in X$ labelled by $*\sigma$. An injective summand $\underline{E}(S/\mathfrak{m}^{\overline{\tau}}) = \underline{E}(S/\mathfrak{m}^{\overline{\tau}})[*\tau]$ contributes a nonzero k-vector space in degree $\mathbf{b}$ if and only if $\tau\supseteq\mathrm{supp}(\mathbf{b}_+)$. Therefore, in any degree $\mathbf{b}$ with $\mathrm{supp}(\mathbf{b}_+)=\sigma$, the complex $\mathbb{J}^\bullet_\mathbf{b}$ is the *local chain complex*

$$\mathcal{L}^\sigma_\bullet X(n-1) := \widetilde{\mathcal{C}}_\bullet(X;k)/\widetilde{\mathcal{C}}_\bullet(\mathrm{cost}_\sigma X;k)(n-1) = \mathcal{C}_\bullet(X,\mathrm{cost}_\sigma X;k)(n-1)$$

of σ in the simplex, spanned as a k-vector space by basis vectors e_σ for all faces $\supseteq\sigma$. The homological shift $(n-1)$ means that $(\mathbb{J}^\bullet)_\mathbf{b} = \mathcal{L}^\sigma_{n-1-\bullet}X$; in other words, $(\mathbb{J}^r)_\mathbf{b} = (\mathbb{J}_{-r})_\mathbf{b} = \mathcal{L}^\sigma_\bullet(n-1)_{-r} = \mathcal{L}^\sigma_{n-1-r}X$.

Applying Γ_{I_Δ} takes the subcomplex of $\mathbb{J}^\bullet$ whose labels $*\sigma$ have $\sigma\in\Delta$. The complex $(\Gamma_{I_\Delta}\mathbb{J}^\bullet)_\mathbf{b}$ is therefore $\mathcal{L}^\sigma_{n-1-\bullet}\Delta$, where $\sigma=\mathrm{supp}(\mathbf{b}_+)$. The natural map $\cdot x_i : (\Gamma_{I_\Delta}\mathbb{J}^\bullet)_\mathbf{b} \to (\Gamma_{I_\Delta}\mathbb{J}^\bullet)_{\mathbf{b}+\mathbf{e}_i}$ is either the identity map on $\mathcal{L}^\sigma_{n-1-\bullet}\Delta$ (if $b_i\neq 0$), or else it is the natural restriction map $\mathcal{L}^\sigma_{n-1-\bullet}\Delta \to \mathcal{L}^{\sigma\cup i}_{n-1-\bullet}\Delta$ (if $b_i=0$) which takes the quotient modulo the subcomplex spanned as a k-vector space by $\{e_\tau \mid \tau\supseteq\sigma \text{ but } \tau\not\supseteq\sigma\cup i\}$.

We could finish the proof here, having succeeded in writing the module structure in simplicial terms, but it seems we have a formula more closely related to Gräbe's theorem than Mustaţă's. To recover Mustaţă's formulation, we use the commutative diagram

$$\begin{array}{ccccccc} H_{n-1-r}(\mathcal{L}^{\sigma}_{\bullet}\Delta) & = & \widetilde{H}_{n-1-r-|\sigma|}(\mathrm{link}_{\sigma}\Delta) & = & \widetilde{H}^{r-2}((\mathrm{link}_{\sigma}\Delta)^{\star}) & = & \widetilde{H}^{r-2}(\Delta^{\star}|_{\overline{\sigma}}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{n-1-r}(\mathcal{L}^{\sigma\cup i}_{\bullet}\Delta) & = & \widetilde{H}_{n-1-r-|\sigma\cup i|}(\mathrm{link}_{\sigma\cup i}\Delta) & = & \widetilde{H}^{r-2}((\mathrm{link}_{\sigma\cup i}\Delta)^{\star}) & = & \widetilde{H}^{r-2}(\Delta^{\star}|_{\overline{\sigma}\setminus i}) \end{array} \tag{6.1}$$

of natural isomorphisms (written as equalities) and maps. The right equalities are because the simplicial complexes inside the parentheses are equal; the middle equalities are by simplicial Alexander duality, Equation (1.1), inside (e.g. in the top row) the support of the simplex $\overline{\sigma}$; and the left equalities are gotten at the level of chains by sending (e.g. in the top row) $e_{\tau} \mapsto \mathrm{sign}(\sigma, \tau \smallsetminus \sigma)\, e_{\tau\smallsetminus\sigma}$, where $\mathrm{sign}(\sigma, \tau \smallsetminus \sigma)$ is the sign of the permutation putting the list $(\sigma, \tau \smallsetminus \sigma)$ into increasing order. The natural vertical map second from the left (between the homologies of links) sends $e_{\tau\smallsetminus\sigma} \mapsto (-1)^{|\sigma|}\mathrm{sign}(i, \tau)\, e_{\tau\smallsetminus(\sigma\cup i)}$. □

We now proceed to give complete proofs of all 4 of the theorems before this point in the section. For instance, it follows from Equation (6.1) that Mustaţă's theorem implies Terai's formula. Our aim, then, is to show as a consequence of Theorem 6.7 that the formulas of Terai and Hochster are equivalent, as are the module structures of Mustaţă and Gräbe. The moral of the story is that no matter how the calculation is accomplished, there really is only one set of simplicial (co)homology floating around—that of Δ—and all of the objects whose structure can be expressed in terms of it are equivalent in one way or another via a generalized Alexander duality functor.

Theorem 6.22 *The module $H^{n-r}_{\mathfrak{m}}(S/I_{\Delta})[-\mathbf{1}]$ is negatively $\mathbf{1}$-determined, and its Alexander dual in $\mathcal{M}^{\mathbf{1}}$ is $H^r_{I_{\Delta}}(S)[-\mathbf{1}]$. Equivalently,*

$$H^r_{I_{\Delta}}(S) \;\cong\; \check{C}(H^{n-r}_{\mathfrak{m}}(S/I_{\Delta})^{\vee})[\mathbf{1}] \;\cong\; (\check{C}\underline{\mathrm{Ext}}^r_S(S/I_{\Delta}, \omega_S))[\mathbf{1}]\,.$$

Proof: If $\mathbb{F}_{\bullet}$ is a free resolution of S/I_{Δ} then the cohomology of the complex $\underline{\mathrm{Hom}}_S(\mathbb{F}_{\bullet}, \omega_S)$ is $\underline{\mathrm{Ext}}^r_S(S/I_{\Delta}, \omega_S)$. Exactness of $\check{C}(-)[\mathbf{1}]$ implies that it commutes with taking cohomology, so setting $M = S$ in Theorem 6.7 yields $H^r_{I_{\Delta}}(S) \cong \check{C}(\underline{\mathrm{Ext}}^r_S(S/I_{\Delta}, \omega_S))[\mathbf{1}]$. The remaining isomorphism in the first equation will follow from local duality, whose easy proof we postpone until Corollary 6.25. The equivalence of the first sentence with the displayed equation is by the definition $M \mapsto \check{C}(M[\mathbf{1}]^{\vee})$ of the appropriate generalized Alexander duality.

Corollary 6.23 *The theorem of Terai (6.20) is equivalent to that of Hochster (6.18).*

Proof: $\underline{\mathrm{Ext}}^r_S(S/I_{\Delta}, \omega_S) \cong H^{n-r}_{\mathfrak{m}}(S/I_{\Delta})^{\vee}$ by Corollary 6.25. Applying this to Theorem 6.18 yields

$$H(\underline{\mathrm{Ext}}^r_S(S/I_{\Delta}, \omega_S); \mathbf{x}) = \sum_{\sigma\in\Delta} \dim_k \widetilde{H}_{n-r-|\sigma|-1}(\mathrm{link}_{\sigma}\Delta; k) \prod_{j\in\sigma} \frac{x_j}{1 - x_j}$$

because Matlis duality reverses the grading. Note that we replace the superscripts on $\widetilde{H}$ by subscripts when we take Matlis duals; we will see why in the proof of the next corollary. As a $\mathbb{Z}^n$-graded k-vector space, the Čech hull of an $\mathbb{N}^n$-graded module is obtained by tensoring the graded piece in degree $\mathbf{b}$ with the inverse polynomial ring $S[x_i^{-1} \mid i \notin \text{supp}(\mathbf{b})]$. Thus

$$H(\check{C}\underline{\text{Ext}}^r_S(S/I_\Delta, \omega_S); \mathbf{x}) = \sum_{\sigma \in \Delta} \dim_k \widetilde{H}_{n-r-|\sigma|-1}(\text{link}_\sigma \Delta; k) \prod_{j \in \sigma} \frac{x_j}{1-x_j} \prod_{i \in \overline{\sigma}} \frac{1}{1-x_i^{-1}}\,.$$

Shifting the input module by $[\mathbf{1}]$ multiplies this whole expression by $x_1^{-1} \cdots x_n^{-1} = \mathbf{x}^{-\sigma}\mathbf{x}^{-\overline{\sigma}}$, and gives the Hilbert series of $H^r_{I_\Delta}(S)$ by Theorem 6.22. All of these steps are reversible, so Hochster's formula can similarly be derived from Terai's. □

Corollary 6.24 *The theorem of Mustaţă (6.21) is equivalent to that of Gräbe (6.19).*

Proof: The proof is the same as the previous corollary, except that we need to keep track of the multiplication maps between the $\mathbb{Z}^n$-graded degrees of the modules in question. The crux is that the homomorphisms

$$\widetilde{H}^{n-r-|\sigma \cup i|-1}(\text{link}_{\sigma \cup i} \Delta; k) \longrightarrow \widetilde{H}^{n-r-|\sigma|-1}(\text{link}_\sigma \Delta; k)$$

coming from Gräbe's theorem become the transpose homomorphisms

$$\widetilde{H}_{n-r-|\sigma|-1}(\text{link}_\sigma \Delta; k) \longrightarrow \widetilde{H}_{n-r-|\sigma \cup i|-1}(\text{link}_{\sigma \cup i} \Delta; k)$$

in the Matlis dual, by definition. Then, by Alexander duality in the form of Equation (6.1), these agree with the maps

$$\widetilde{H}^{r-2}(\Delta^\star|_{\overline{\sigma}}; k) \longrightarrow \widetilde{H}^{r-2}(\Delta^\star|_{\overline{\sigma} \setminus i}; k)$$

in Mustaţă's theorem. □

6.3 Greenlees-May duality

Alexander duality contains and is much stronger than local duality for objects M in the category $\mathcal{M}$ of $\mathbb{Z}^n$-graded modules. This is because local duality works on the level of cohomology (or at best, in the derived category), whereas the Alexander duality functors work on the level of resolutions. This strength is demonstrated by the ease with which local duality can be extracted directly from Theorem 3.23:

Corollary 6.25 (Local duality at the maximal ideal) *If $M \in \mathcal{M}$ is finitely generated, then $H^i_\mathfrak{m}(M)^\vee \cong \underline{\text{Ext}}^{n-i}_S(M, \omega_S)$, where $\omega_S = S[-\mathbf{1}]$ is the canonical module.*

Proof: Shifting $\mathbb{Z}^n$-degrees and choosing $\mathbf{a}$ large enough, we may assume that $M \in \mathcal{M}^\mathbf{a}_+$. Using the notation of Theorem 3.23, the matrix $\Lambda_\mathbb{Z}$ in Theorem 3.23.5′ represents simultaneously the cohomological complexes $H^0_\mathfrak{m}(\mathbb{I})$ and $\underline{\text{Hom}}_S(\mathbb{F}', S[-\mathbf{1}])(n)$, where $\mathbb{F}' = \underline{\text{Hom}}_S(\mathbb{I}^\vee, S)[\mathbf{1}]$ is a minimal free resolution of M. Corollary 2.10 says that the complexes $\underline{\text{Hom}}_S(\mathbb{F}', S[-\mathbf{1}])(n)$ and $H^0_\mathfrak{m}(\mathbb{I})$ are therefore Matlis dual, so their homology is, as well. □

Local duality is just the beginning. Using Theorem 6.22, local duality can be generalized to the case of H^i_I for any radical monomial ideal I, while removing the restriction of $M \in \mathcal{M}$ being finitely generated.

Theorem 6.26 (Local duality with monomial support) *If $\mathbb{F}.$ is a minimal free resolution of S/I and $\mathbb{C}^{\bullet}_{\mathbb{F}}$ is the generalized Čech complex determined by $\mathbb{F}.$, then for arbitrary $M \in \mathcal{M}$,*

$$H^i_I(M)^\vee \cong H^{-i}\underline{\mathrm{Hom}}_S(M, (\mathbb{C}^{\bullet}_{\mathbb{F}})^\vee)\,.$$

If S/I is Cohen-Macaulay, then (with $\omega^{\mathbf{1}}_{S/I}$ being the Alexander dual *of $\omega_{S/I}$ as in Definition 1.36),*

$$H^i_I(M)^\vee \cong \underline{\mathrm{Ext}}^{d-i}_S(M, \check{C}(\omega^{\mathbf{1}}_{S/I})).$$

When $\mathbb{F}.$ is minimal, $(\mathbb{C}^{\bullet}_{\mathbb{F}})^\vee = (\mathbb{C}^{\bullet}_I)^\vee$ is called the Greenlees-May complex *of S/I.*

Proof: This is the Matlis dual of Theorem 6.7. Precisely,

$$\begin{aligned} H^i_I(M)^\vee &\cong H^i(\mathbb{C}^{\bullet}_{\mathbb{F}} \otimes_S M)^\vee \\ &\cong H^{-i}((M \otimes_S \mathbb{C}^{\bullet}_{\mathbb{F}})^\vee) \\ &\cong H^{-i}\underline{\mathrm{Hom}}_k(M \otimes_S \mathbb{C}^{\bullet}_{\mathbb{F}}, k) \\ &\cong H^{-i}\underline{\mathrm{Hom}}_S(M, (\mathbb{C}^{\bullet}_{\mathbb{F}})^\vee) \end{aligned}$$

proving the first statement. In any case, $(\mathbb{C}^{\bullet}_{\mathbb{F}})^\vee$ is a complex of injectives by Lemma 1.21. If $\mathbb{F}.$ is a minimal free resolution of a Cohen-Macaulay codimension d quotient S/I, then $\underline{\mathrm{Hom}}_S(\mathbb{F}, \omega_S)$ is a minimal free resolution of $\omega_{S/I}$ (with $\omega_{S/I}$ in homological degree $-d$). Hence $(\mathbb{C}^{\bullet}_I)^\vee$ is an injective resolution of $(\check{C}\omega_{S/I})[\mathbf{1}]^\vee = \check{C}(\omega^{\mathbf{1}}_{S/I})$, with $\check{C}(\omega^{\mathbf{1}}_{S/I})$ in cohomological degree $-d$ (the equality follows from the categorical equivalences in Theorem 1.34 and Table 1.1; see also Section 1.6). The result follows because $\underline{\mathrm{Ext}}^{\bullet}_S(-, N)$ can be calculated from an injective resolution of N, and $-i - (-d) = d - i$. □

Example 6.27 Of course, Corollary 6.25 is a special case of Theorem 6.26, because their left sides are equal when $I = \mathfrak{m}$ and M is finitely generated. To see the connection directly, observe that $\omega_{S/\mathfrak{m}}$ is just $S/\mathfrak{m}$ and $\mathrm{codim}(\mathfrak{m}) = n$. Now use $\check{C}(\omega^{\mathbf{1}}_{S/\mathfrak{m}}) \cong \check{C}((S/\mathfrak{m})^{\mathbf{1}}) \cong \check{C}(S[-\mathbf{1}]) \cong S[-\mathbf{1}] \cong \omega_S$. □

Remark 6.28 It is also possible to define $(\mathbb{C}^{\bullet}_{\mathbb{F}})^\vee$ in terms of the generalized Alexander duality functor $A^{+,0}_{\mathbf{1}} : \mathcal{M}^{\mathbf{1}}_+ \to \mathcal{M}^{\mathbf{1}}$ from Definition 3.1.2, which is defined as $(\check{C}\,-)[\mathbf{1}]^\vee : \mathcal{M}^{\mathbf{a}}_+ \to \mathcal{M}^{\mathbf{a}}$. With this terminology, $(\mathbb{C}^{\bullet}_{\mathbb{F}})^\vee = A^{+,0}_{\mathbf{1}}\mathbb{F}^{\bullet}$, where $\mathbb{F}^{\bullet} = \underline{\mathrm{Hom}}_S(\mathbb{F}, \omega_S)$, as above.

The first equation in Theorem 6.26 says that the complex $(\mathbb{C}^{\bullet}_{\mathbb{F}})^\vee$ of $\mathbb{Z}^n$-graded injective modules plays for $H^{\bullet}_I$ a role similar to that of a dualizing complex for $H^{\bullet}_{\mathfrak{m}}$. Indeed, setting $M = S$ yields $H^i_I(S)^\vee \cong H^{-i}(\mathbb{C}^{\bullet}_{\mathbb{F}})^\vee$, just as the dualizing complex is a complex of injectives whose cohomology is Matlis dual to local cohomology of S with support on $\mathfrak{m}$. This comparison with dualizing complexes suggests that Theorem 6.26 should be put into the language of derived categories. It turns out that doing so makes a connection to some of the recent literature on generalizations of local duality.

The starting point is a result of Greenlees and May [GM92, Proposition 3.8], which contains a spectral sequence $E^{p,q}_2 = \mathrm{Ext}^p(N, DH^{n-q}_I(R)) \Rightarrow DH^{\bullet}_I(N)$. Here R is any commutative ring, I is any finitely generated ideal, N is any R-module, and $D(-) =$

Hom$(-, Q)$ where Q is an injective R-module. This specializes to the more common local duality spectral sequence when R is a complete noetherian local ring, I is maximal, N is finitely generated, and $Q = E(R/I)$ is the injective hull of the residue field. Of course, this result has a more natural formulation in terms of the derived category, replacing the spectral sequence by an isomorphism of functors. Numerous such reformulations were published in [AJL97].

To describe an appropriate derived categorical statement, some notations are required. To begin with, the local cohomology functor $H_I^\bullet$ is replaced by the right derived functor $\mathbb{R}\Gamma_I$. Recall what this entails: application of $\mathbb{R}\Gamma_I$ to a complex $\mathcal{F}$ means that $\Gamma_I = H_I^0$ is applied to an injective resolution of $\mathcal{F}$. This injective resolution is, by definition, a complex of injectives $\mathcal{I}$ along with map $\mathcal{F} \to \mathcal{I}$ that is a quism. Thus $\mathbb{R}\Gamma_I$ outputs another complex $\Gamma_I(\mathcal{I})$, called $\mathbb{R}\Gamma_I(\mathcal{F})$, which is well-defined up to isomorphism in the derived category. In the case where $\mathcal{F} = M$ is a module, the cohomology modules of the complex $\mathbb{R}\Gamma_I(M)$ are the local cohomology modules $H_I^\bullet(M)$.

The same considerations apply to the functor Ext, which is superseded by $\mathbb{R}$Hom. The right derived functor $\mathbb{R}\mathrm{Hom}(\mathcal{F}, -)$ is calculated on a complex $\mathcal{E}$ by taking an injective resolution $\mathcal{E} \to \mathcal{I}$ and applying $\mathrm{Hom}(\mathcal{F}, -)$ to it. This yields a double complex whose total complex $\mathrm{Tot}(\mathrm{Hom}(\mathcal{F}, \mathcal{I}))$ is $\mathbb{R}\mathrm{Hom}(\mathcal{F}, \mathcal{E})$. As is the case with Ext, $\mathbb{R}\mathrm{Hom}(\mathcal{F}, \mathcal{E})$ may also be defined as the total complex of $\mathrm{Hom}(\mathcal{P}, \mathcal{E})$ for a projective resolution of $\mathcal{F}$—i.e. a quism $\mathcal{P} \to \mathcal{F}$ of a complex of projectives to $\mathcal{F}$. Note that if $\mathcal{E}$ is already injective (in particular, if it is an injective module), then the $\mathbb{R}$ may be removed from $\mathbb{R}$Hom; and similarly if $\mathcal{F}$ is already projective.

The final player in the Greenlees-May theorem is the one to which the theorem owes its existence in the first place (Greenlees and May needed it for equivariant K-theory). Just as Γ_I takes the (co)limit of submodules annihilated by powers of I, there is another familiar functor Λ_I, otherwise known as *I-adic completion*, which takes the limit of quotients annihilated by I. Precisely, $\Lambda_I(M)$ is the limit of the inverse system $M/I^{t+1}M \to M/I^tM$. This functor is neither left-exact nor right exact in general. For instance, completion at an ideal in a noetherian ring R is exact on *finitely generated* modules since such completion is obtained by tensoring with the flat module $\hat{R}_I$; and of course, tensoring with $\hat{R}_I$ is still exact on all modules. However, it doesn't necessarily produce I-adic completion on infinitely generated modules. Nonetheless, one can define the left derived functor $\mathbb{L}\Lambda_I$ on a complex $\mathcal{E}$ by taking a projective resolution of $\mathcal{E}$ and applying Λ_I to it. The catch is that if $\mathcal{E} = M$ is a module, the zeroth homology of $\mathbb{L}\Lambda_I(M)$ need not be isomorphic to $\Lambda_I(M)$. In other words, the natural map $\hat{M}_I \to L_0\Lambda_I(M)$ need not be an isomorphism (as it would be if Λ_I were right exact).

A heuristic way to describe Greenlees-May duality is as an adjointness in the derived category between taking support on I and completion along its zero set. It all really comes down to the adjointness of Hom and $\otimes$. Indeed, one would like to have something as close to $\mathrm{Hom}(\mathrm{Hom}(S/I, M), N) \cong \mathrm{Hom}(M, S/I \otimes N)$ as possible, but already we need to cope with the nonexactness at this stage. Mix this up with a few limits (direct on the Hom side and inverse on the $\otimes$ side), and you get $\Gamma_I M$ on the left and $\hat{N}_I$ on the right. Passing to the derived category allows one to work with complexes whose pieces are modules for which Hom and $\otimes$ are exact, so that there is some hope of being able to mimic duality for vector spaces.

Theorem 6.29 (Greenlees-May duality) *Let $\mathbb{R}\Gamma_I$ be the right derived functor of taking elements with support on I and $\mathbb{L}\Lambda_I$ the left derived functor of completion with respect to the I-adic filtration. Then*

$$\mathbb{R}\mathrm{Hom}(\mathbb{R}\Gamma_I(\mathcal{F}), \mathcal{E}) \cong \mathbb{R}\mathrm{Hom}(\mathcal{F}, \mathbb{L}\Lambda_I(\mathcal{E}))$$

for complexes $\mathcal{F}$ and $\mathcal{E}$.

Keep in mind the near lack of hypotheses for this theorem mentioned above (actually, more conditions than those above are needed—see [AJL97] for precision; in any case it is enough to assume the ring in question is noetherian). This remarkable theorem greatly generalizes Grothendieck-Serre local duality as is (see above), and unifies a number of other dualities after it has been appropriately sheafified; the paper [AJL97] contains an explanation of these assertions, along with a host of other reformulations.

In order to establish the connection between Theorem 6.29 and Theorem 6.26, the latter result needs to be translated into derived categories. First, observe that there is no reason, in view of the proof of Theorem 6.26, why $\underline{E}(k)$ should be used in preference to any other injective module Q. Indeed, Theorem 6.26 could just as well have been written

$$DH^i_I(\mathcal{F}) \cong H^{-i}\underline{\mathrm{Hom}}_S(M, D(\mathbb{C}^\bullet_{\mathbb{F}}))$$

for $D = \underline{\mathrm{Hom}}(-, Q)$. The point is that the complex $D(\mathbb{C}^\bullet_{\mathbb{F}})$ is a complex of injectives (proof: $\underline{\mathrm{Hom}}(-, \underline{\mathrm{Hom}}(\text{flat}, \text{injective})) = \underline{\mathrm{Hom}}(- \otimes \text{flat}, \text{injective})$ is an exact functor of "$-$"). Therefore the $\underline{\mathrm{Hom}}$ on the right hand side, without the $\mathbb{R}$ in front, already computes $\mathbb{R}\underline{\mathrm{Hom}}$. At this point, the functors $H^\bullet_I$ and $H^{-i}\underline{\mathrm{Hom}}$ can both be replaced with their derived categorical versions, producing

$$D(\mathbb{R}\Gamma(\mathcal{F})) \cong \mathbb{R}\underline{\mathrm{Hom}}(\mathcal{F}, D(\mathbb{C}^\bullet_{\mathbb{F}})), \tag{6.2}$$

where Remark 6.12 has been used to justify replacing M by a complex $\mathcal{F}$.

Taking $\mathcal{F} = S$, we find that $D(\mathbb{R}\Gamma_I S) \cong D(\mathbb{C}^\bullet_{\mathbb{F}})$. (When $Q = \underline{E}(k)$, a more functorial way of stating this isomorphism is via Remark 6.28, which implies that $\mathbb{R}\Gamma_I(S)^\vee \cong A^{+,0}_{\mathbf{1}}\mathbb{R}\underline{\mathrm{Hom}}(S/I, \omega_S)$, since $\underline{\mathrm{Hom}}_S(\mathbb{F}, \omega_S) \cong \mathbb{R}\underline{\mathrm{Hom}}(S/I, \omega_S)$.) Making this replacement, Equation (6.2) becomes

$$\underline{\mathrm{Hom}}(\mathbb{R}\Gamma_I(\mathcal{F}), Q) \cong \mathbb{R}\underline{\mathrm{Hom}}(\mathcal{F}, D(\mathbb{R}\Gamma_I S)), \tag{6.3}$$

In this guise, Theorem 6.26 is a $\mathbb{Z}^n$-graded analogue of Greenlees-May duality, quite close to the way Greenlees and May stated it.

To see why the ungraded analogue of Equation (6.3) can be derived from Theorem 6.29, note first that by taking $\mathcal{F} = S$ and $\mathcal{E} = Q$, Theorem 6.29 says that

$$D(\mathbb{R}\Gamma_I S) \cong \mathbb{L}\Lambda_I(Q).$$

Now substituting $\mathcal{E} = Q$ and $D(\mathbb{R}\Gamma_I S)$ for $\mathbb{L}\Lambda_I(Q)$ in Theorem 6.29 with arbitrary $\mathcal{F}$ yields back the ungraded Equation (6.3), since the left $\mathbb{R}\mathrm{Hom}$ can again be made into Hom by injectivity of Q.

In this general context, the novelty of Theorem 6.26 (which was discovered independently from Theorem 6.29) is, aside from its $\mathbb{Z}^n$-grading, the identification of the specific representative $A_{\mathbf{1}}^{+,0}\mathbb{R}\underline{\mathrm{Hom}}(S/I, \omega_S)$ for the object $\mathbb{R}\Gamma_I(S)^\vee$ in the derived category of $\mathcal{M}$. That this object (which is isomorphic to the even more mysterious $\mathbb{L}\Lambda_I\,\underline{E}(k)$) is the generalized Alexander dual of a *finitely generated* complex is quite surprising.

In view of the importance of dualizing complexes and the contribution of Greenlees and May, it seems appropriate to make the following definition.

Definition 6.30 *The object $\mathbb{R}\Gamma_I(S)^\vee$ in the derived category is a* Greenlees-May *complex.*

So a Greenlees-May complex with I being a maximal ideal $\mathfrak{m}$ is a dualizing complex. And $A_{\mathbf{1}}^{+,0}\mathbb{R}\underline{\mathrm{Hom}}(S/I, \omega_S) \cong (\mathbb{C}_I^{\boldsymbol{\cdot}})^\vee$ is a Greenlees-May complex for the $\mathbb{Z}^n$-graded category. In the case where $I = \mathfrak{m}$, [Har66, Proposition 3.4] says that $\mathbb{J}^{\boldsymbol{\cdot}}$ is a dualizing complex for a noetherian local ring with residue field k if and only if there is an integer d such that $\mathrm{Ext}^i(k, \mathbb{J}^{\boldsymbol{\cdot}}) = 0$ for $i \neq d$ and k for $i = d$.

Question 6.31 Is there a characterization of Greenlees-May complexes analogous to that of [Har66, Proposition 3.4]?

Another contribution of Theorem 6.26 is the explicit description of the *Greenlees-May module* $\check{C}(\omega_{S/I}^{\mathbf{1}})$ when S/I is Cohen-Macaulay. In general there won't be a Greenlees-May module even when there is a Greenlees-May complex, since such a complex will not in general be a resolution of anything—there may be more than one nonvanishing cohomology, or the cohomology may fail to sit at the end.

Remark 6.32 It would be interesting to know if Theorem 6.26 has an impact on explicit computation of local cohomology with monomial support, especially for the purpose of sheaf cohomology on toric varieties [EMS00]. In particular, the interaction with the finiteness conditions there should be investigated: what happens to local duality with monomial support when the grading is coarser than the $\mathbb{Z}^n$-grading? There is always the general (ungraded) form of Greenlees-May duality, but it's unclear how to make that finite enough for computation.

Judging from the relation between the usual local duality theorem and Serre duality for projective schemes, it seems natural, in view of the connections made in [EMS00, Mus00b] between local cohomology with monomial support in polynomial rings and sheaf cohomology on toric varieties, to ask:

Question 6.33 What is the precise relation between local duality with monomial support and Serre duality on toric varieties?

We have one final comment on the interpretation of Theorem 6.22 (which is a special case of local duality with monomial support) in this context. It can be alternatively read as

$$H_I^r(S)^\vee[\mathbf{1}] \quad \cong \quad \check{C}(H_{\mathfrak{m}}^{n-r}(S/I)^\vee)^\vee \quad \cong \quad P_{\mathbf{1}}(H_{\mathfrak{m}}^{n-r}(S/I)[-\mathbf{1}]).$$

In the case of a local ring $(R, \mathfrak{p})$, the Matlis dual of $H_I^r(R)$ agrees with the local cohomology at $\mathfrak{p}$ of the formal scheme obtained by completion of $\mathrm{Spec}(R)$ along $\mathrm{Spec}(R/I)$ [Ogu73, Proposition 2.2]. Thus the positive extension functor $P_{\mathbf{1}}$ mimics, in the $\mathbb{Z}^n$-graded setting, the transition from local cohomology of $\mathrm{Spec}(R/I)$ to local cohomology of the formal completion, and the Čech hull is Matlis dual to this operation. It would be interesting to know the precise connection, which is very likely related to the globalization of Greenlees-May duality via formal schemes [AJL00].

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