Topology for statistical analysis of brain artery images

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joint with

Paul Bendich & Alex Pieloch (Duke Math)
J.S. Marron & Sean Skwerer (Chapel Hill Stat/Oper.Res.)

Young Mathematicians Conference
The Ohio State University

12 August 2017

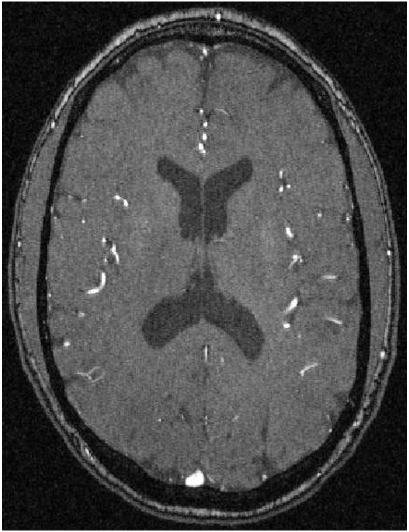
Outline

Brain artery trees

Goal: Statistical analysis taking 3D geometry into account

- predict stroke tendency
- screen for loci of pathology, such as tumors
- explore how age affects vascularization

Magnetic Resonance Angiography (MRA)



from Elizabeth Bullitt, Dept. of Neurosurgery, UNC-CH

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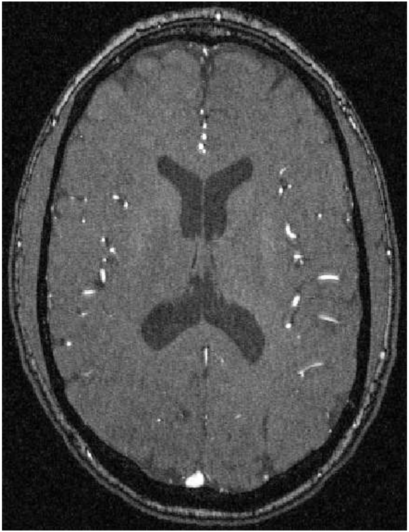

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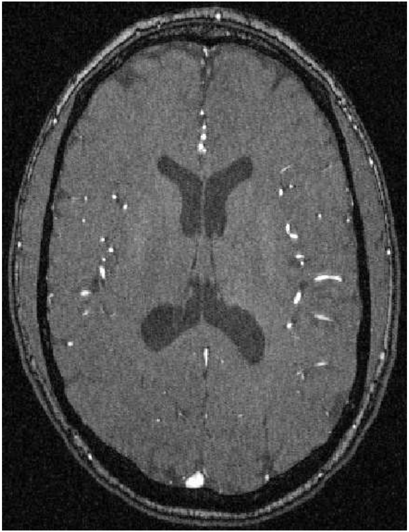

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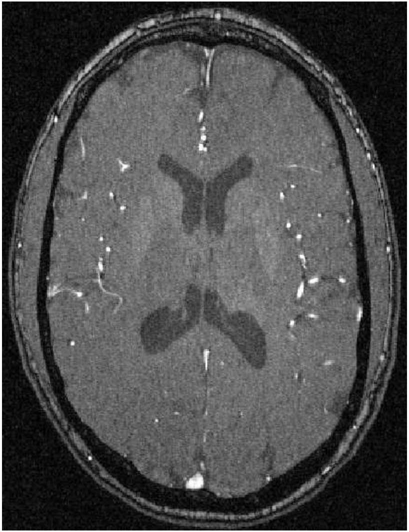

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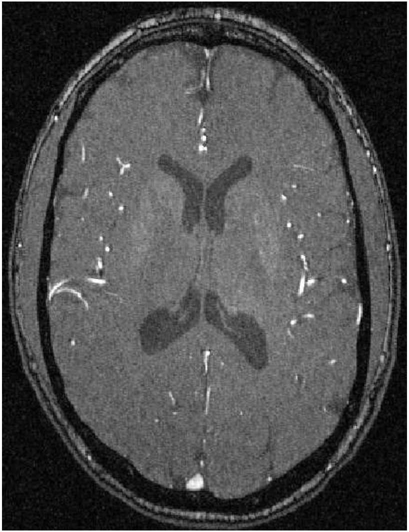

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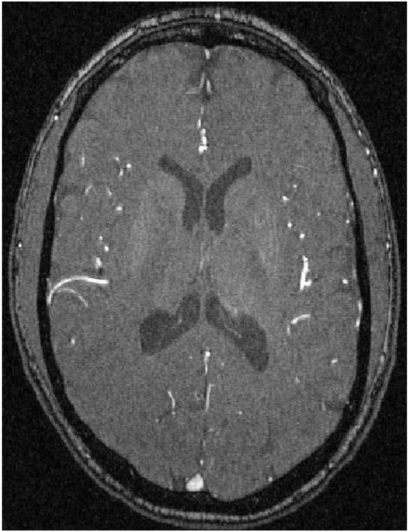

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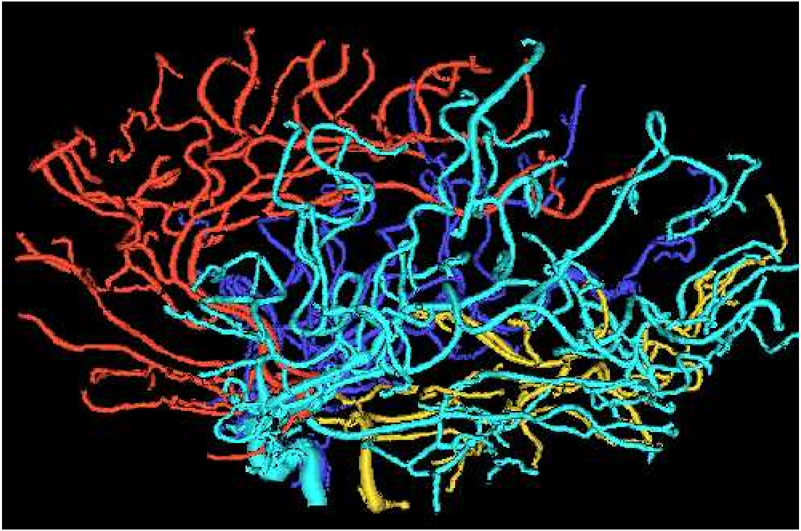
Tube tracking



[Bullitt and Aylward, 2002]

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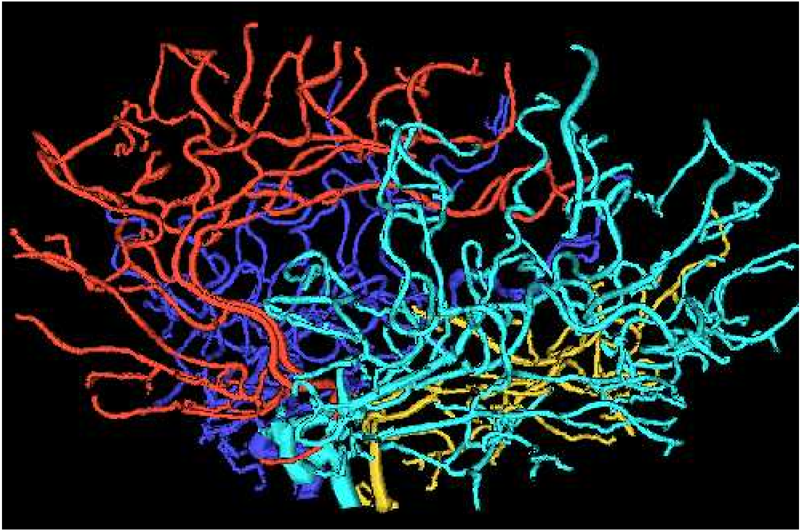

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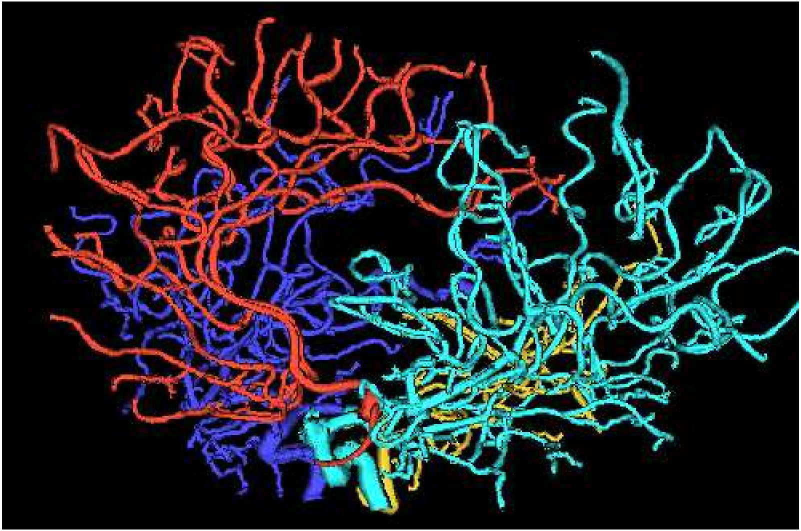

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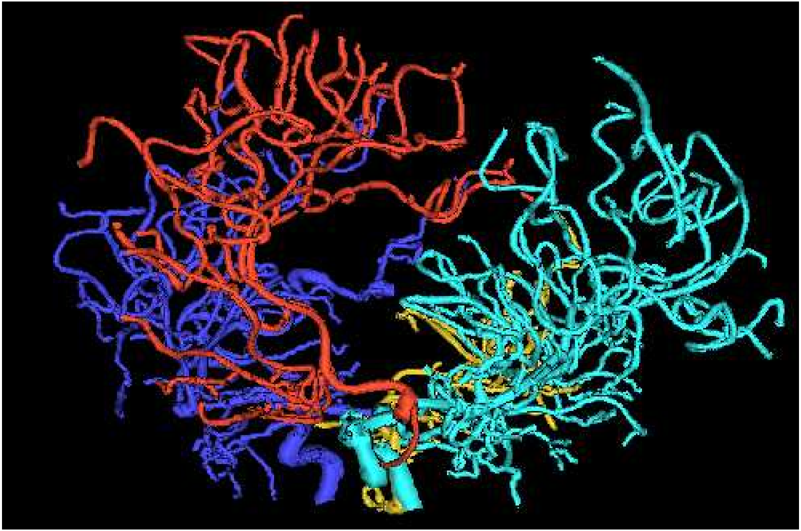

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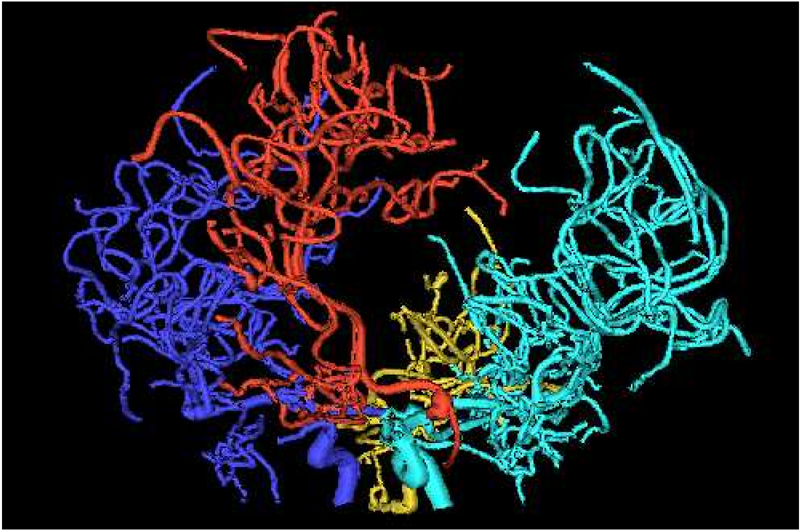

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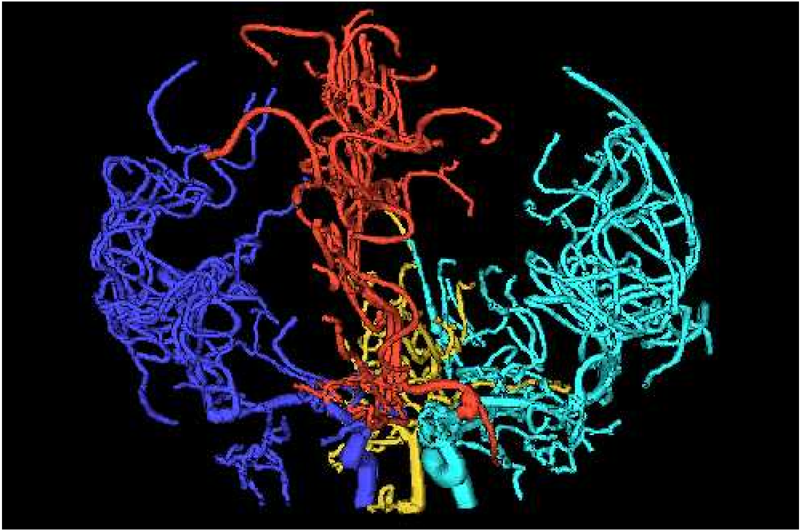

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Brain artery trees

Goal: Statistical analysis taking 3D geometry into account

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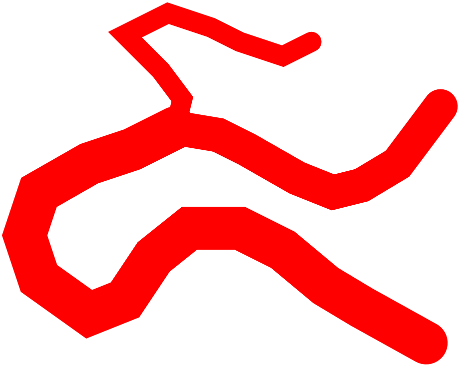

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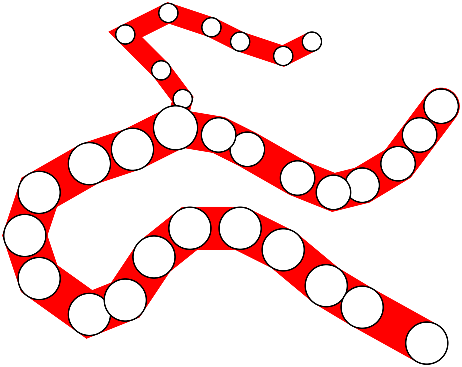

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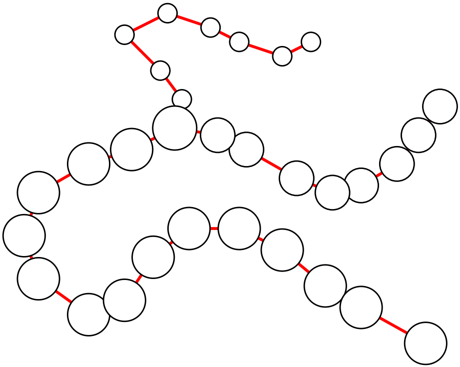

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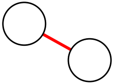

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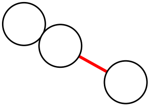

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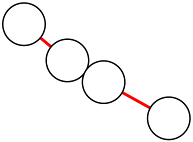

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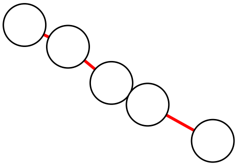

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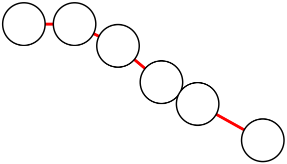

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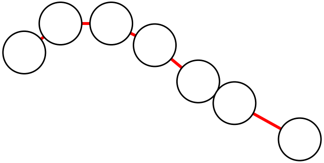

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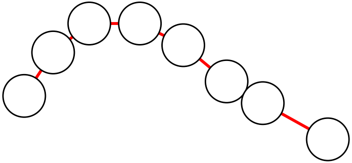

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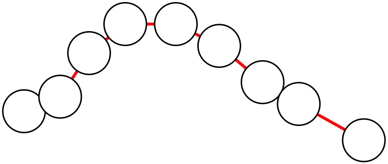

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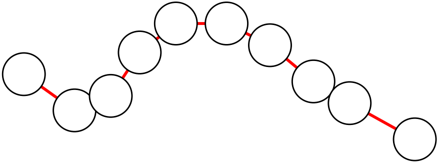

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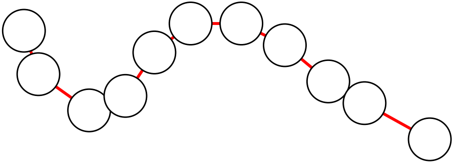

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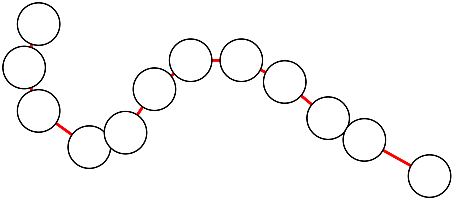

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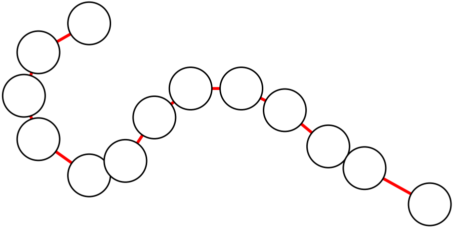

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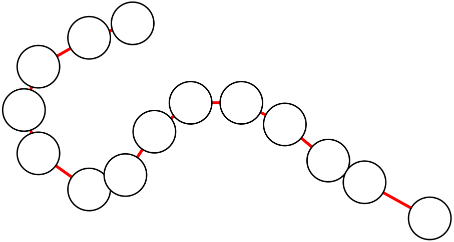

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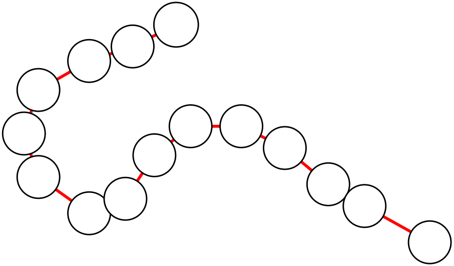

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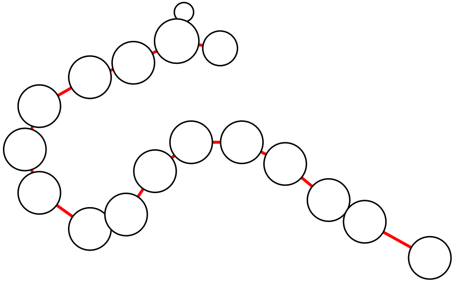

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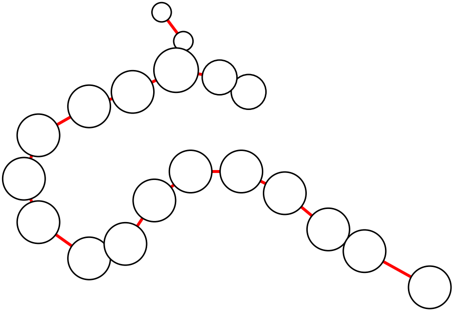

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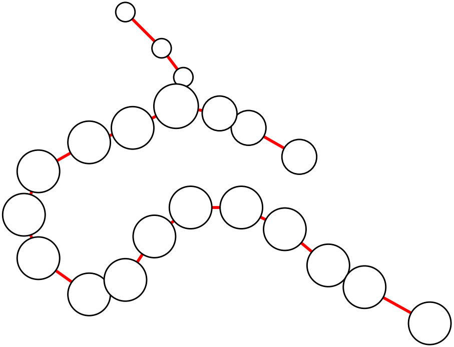

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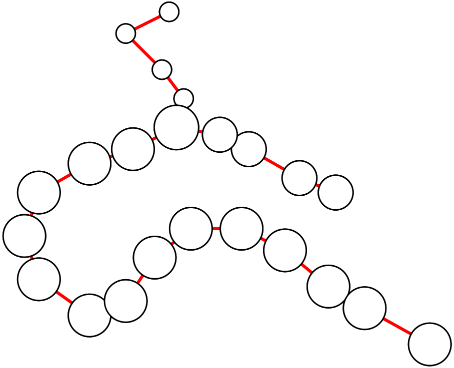

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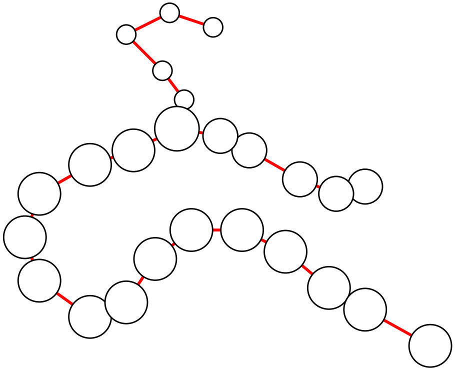

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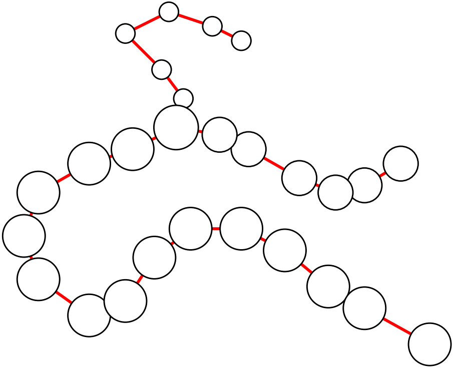

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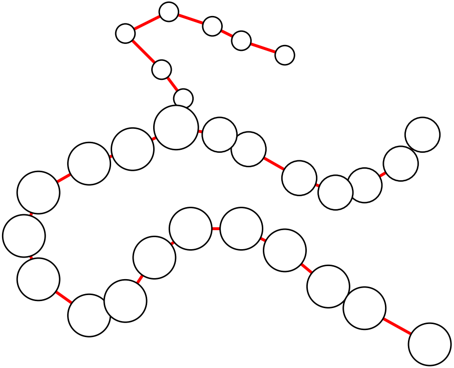

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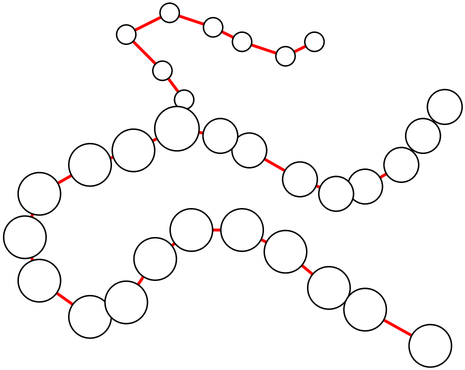

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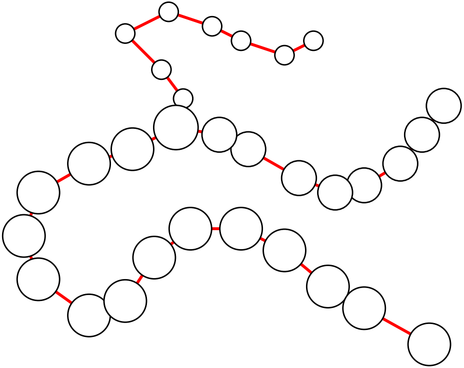

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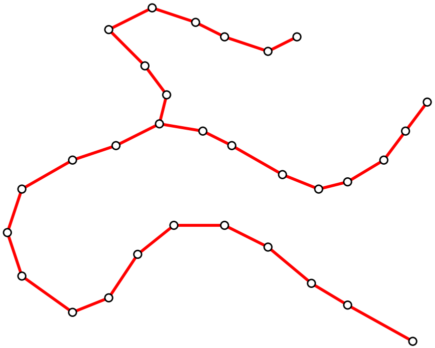

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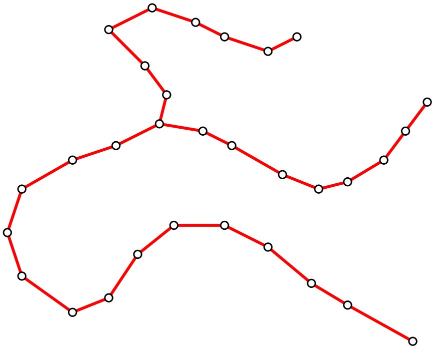

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Topological space $X \rightsquigarrow$ homology $H_i X$ for each dimension i.

- vector space that measures "i-dimensional holes" in X

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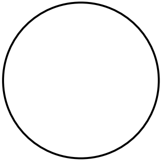

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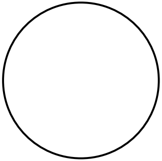

$\dim(H_1) = 1$

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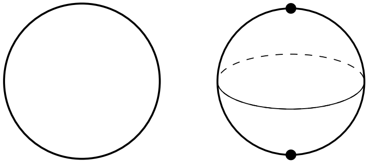

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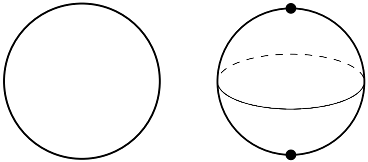

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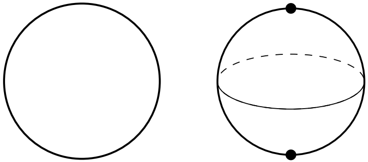

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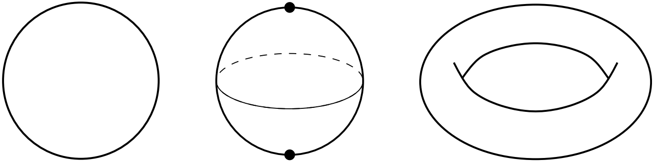

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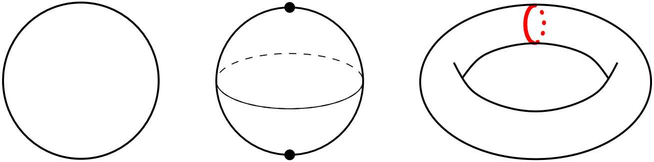

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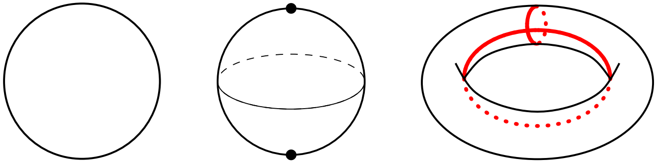

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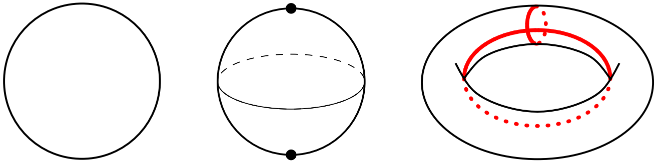

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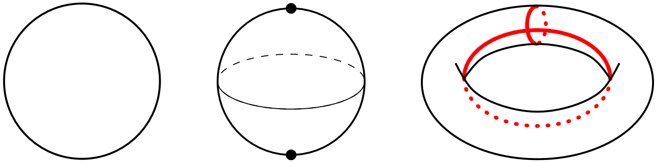

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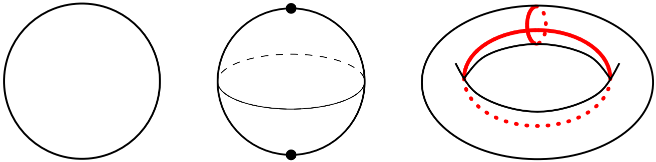

$\dim(H_1) = 1$ $\dim(H_1) = 0$ $\dim(H_1) = 2$

$\dim(H_2) = 1$ $\dim(H_2) = 1$

- $i = 0$ case: basis for $H_i \leftrightarrow$ connected components of X

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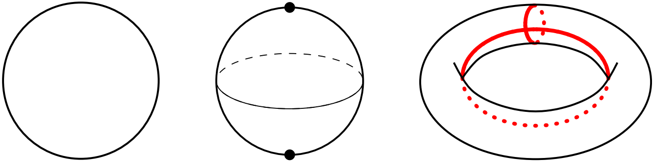

$\dim(H_0) = 1$	$\dim(H_0) = 1$	$\dim(H_0) = 1$
$\dim(H_1) = 1$	$\dim(H_1) = 0$	$\dim(H_1) = 2$
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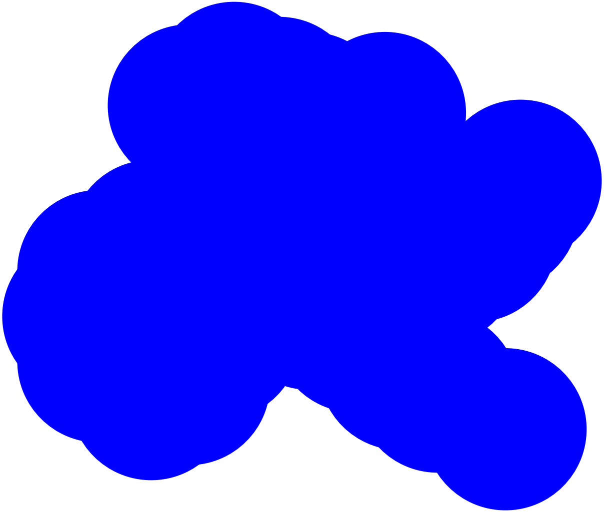
Example: expanding balls



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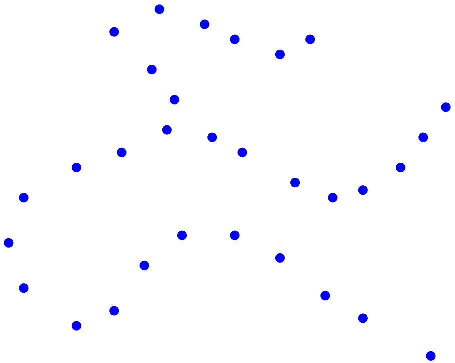

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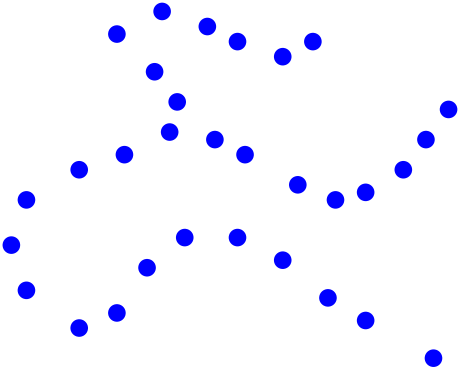

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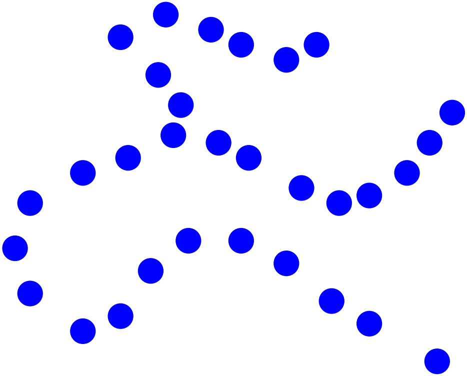

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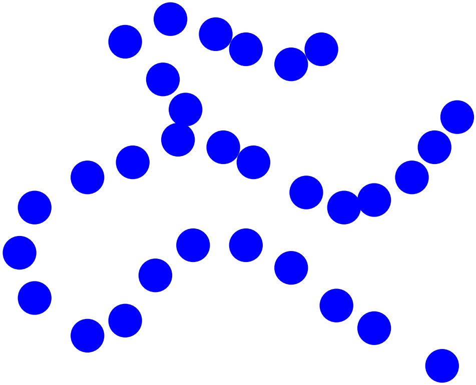

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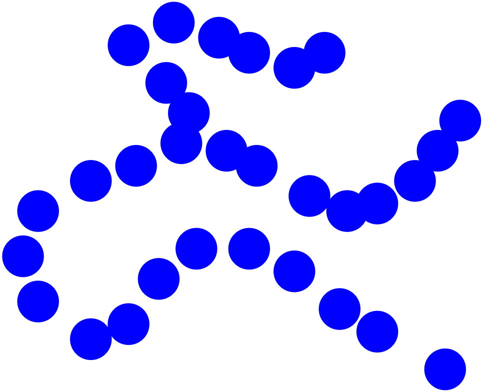

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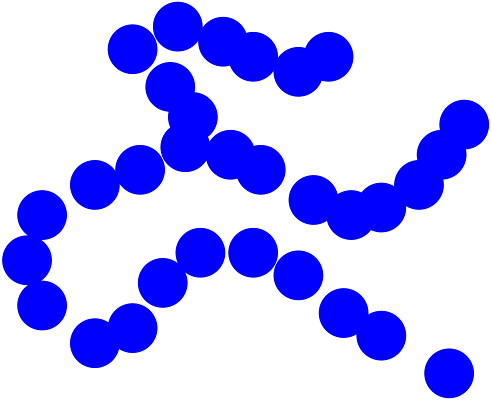

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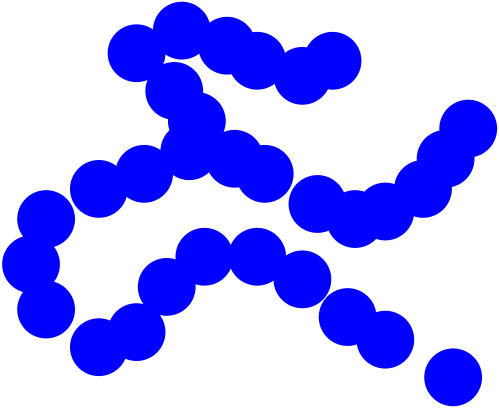

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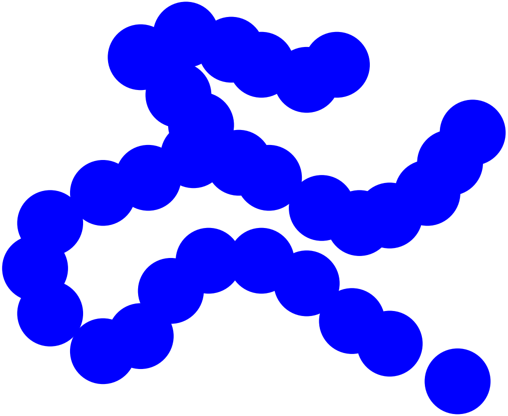

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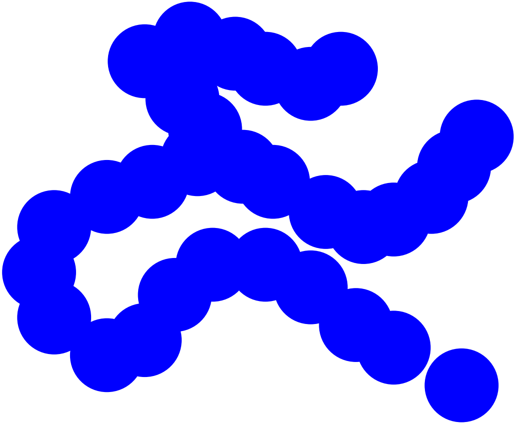

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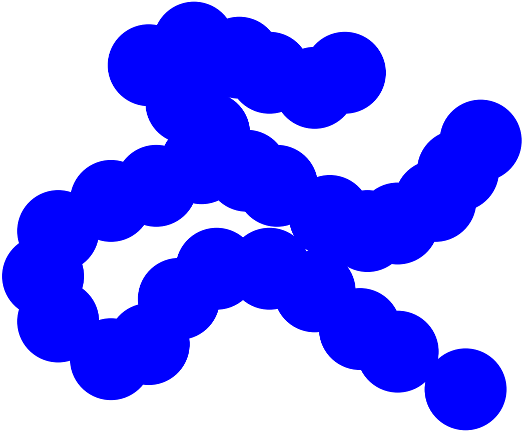

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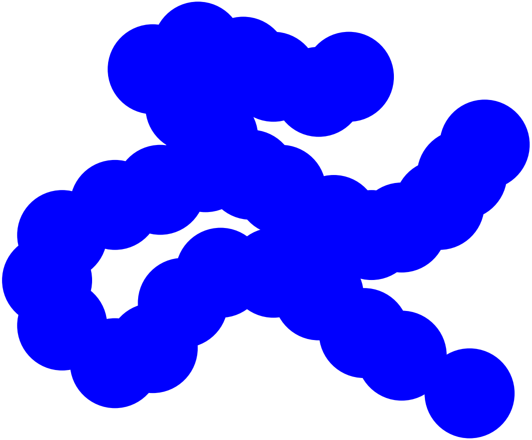

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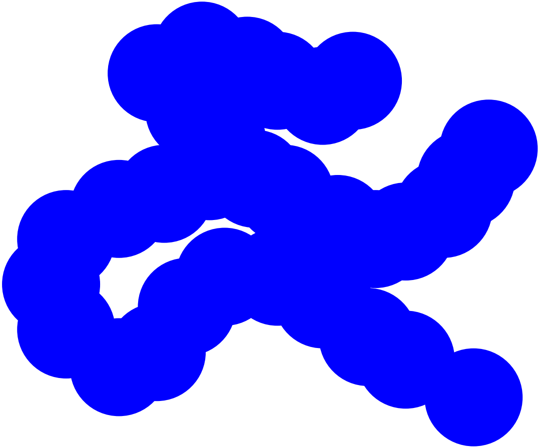

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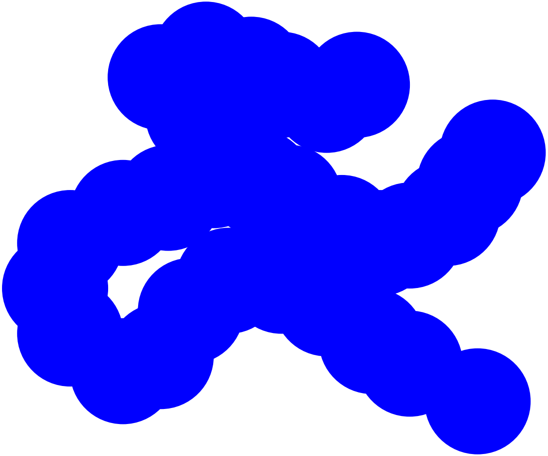

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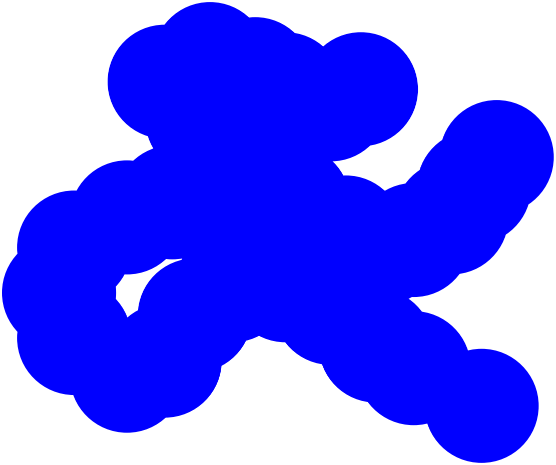

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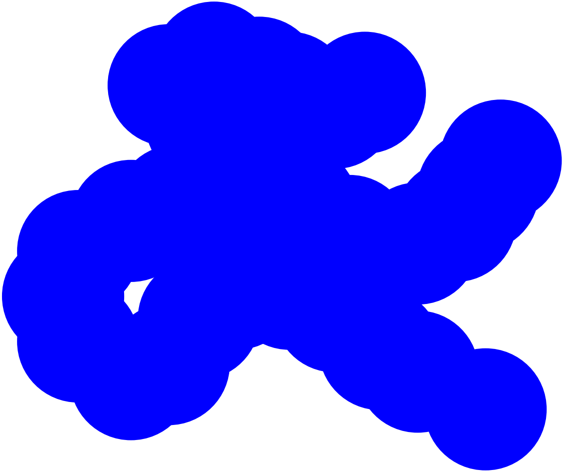

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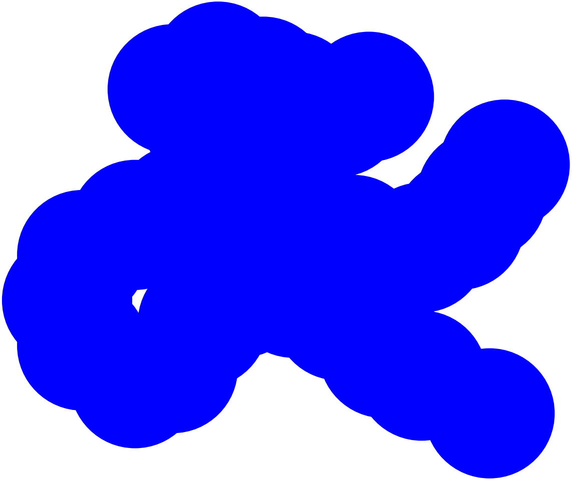

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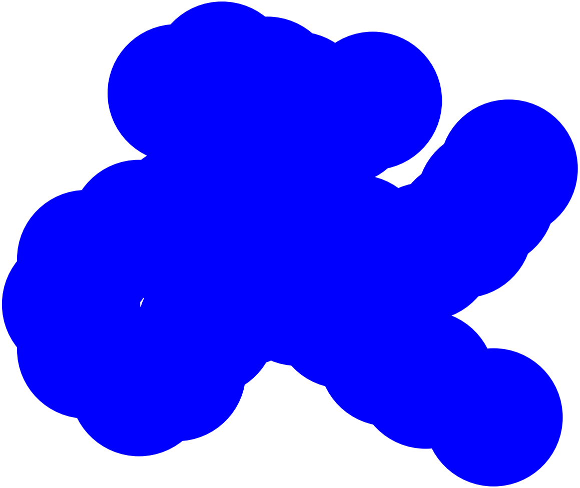

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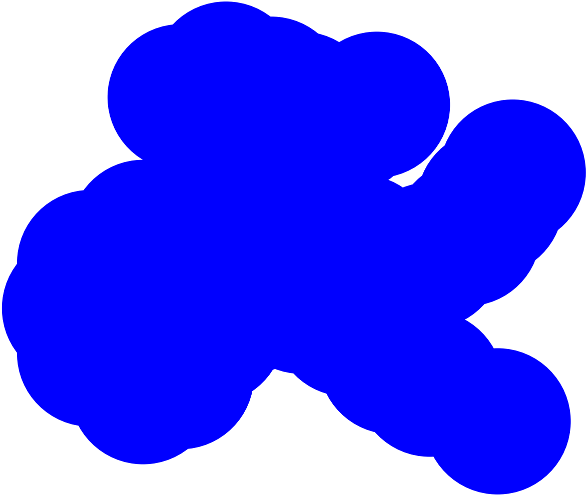

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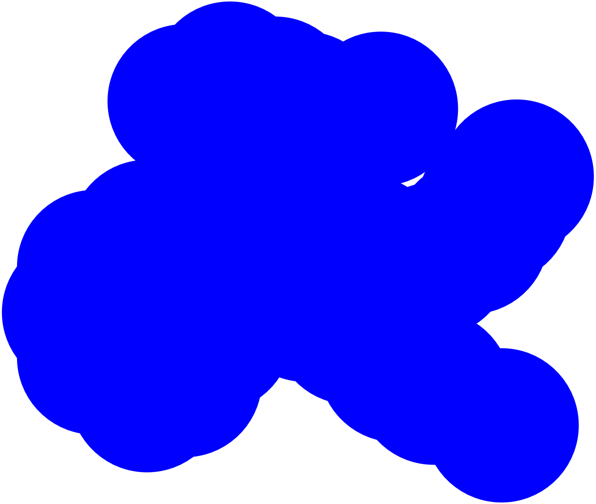

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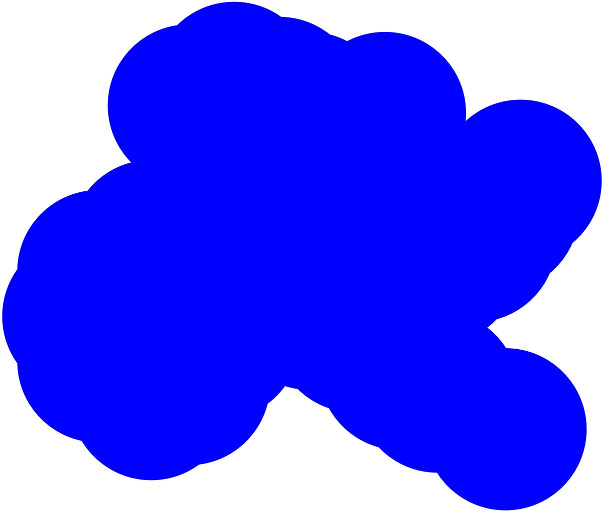

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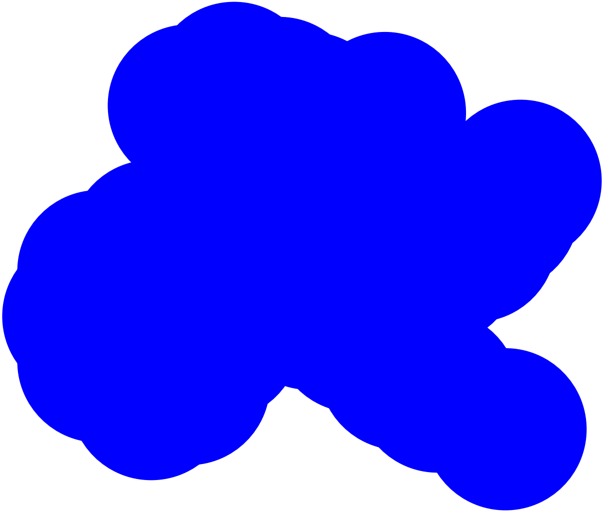

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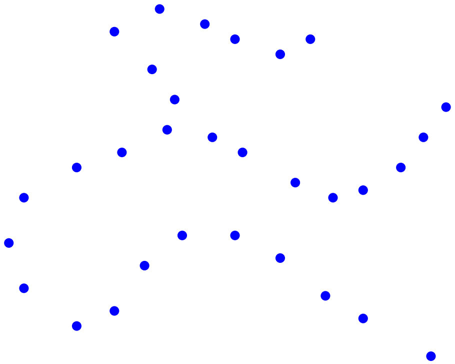

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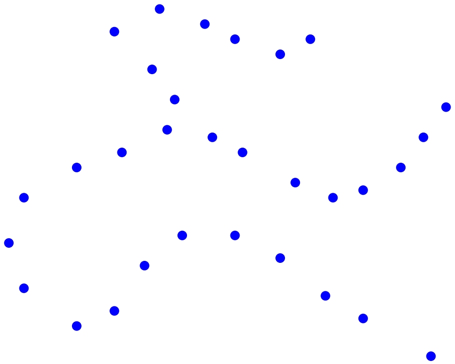
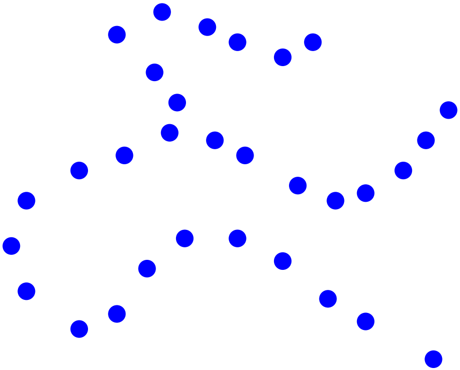


$\dim(H_0) = 31$

Example: expanding balls



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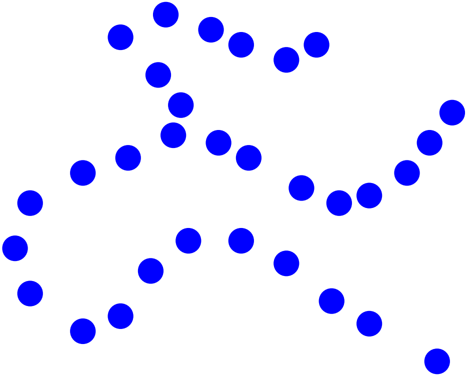

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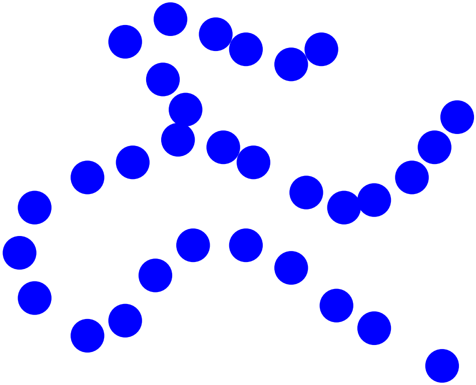

$\dim(H_0) = 26$

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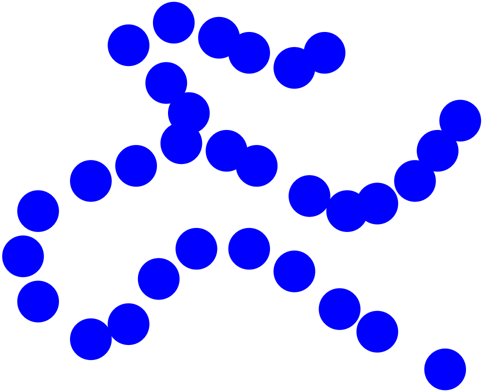
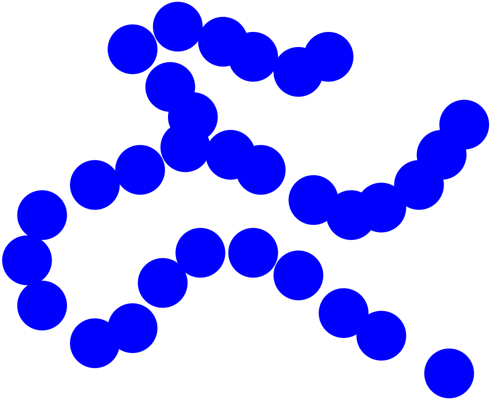


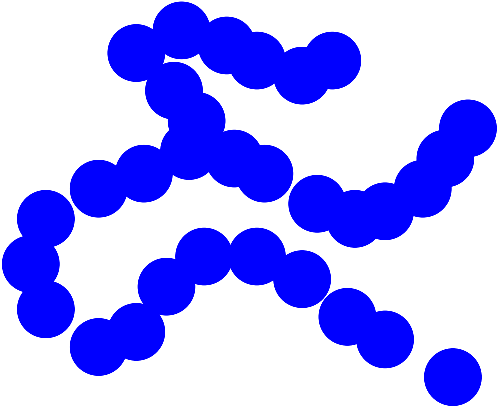
$\dim(H_0) = 21$

Example: expanding balls



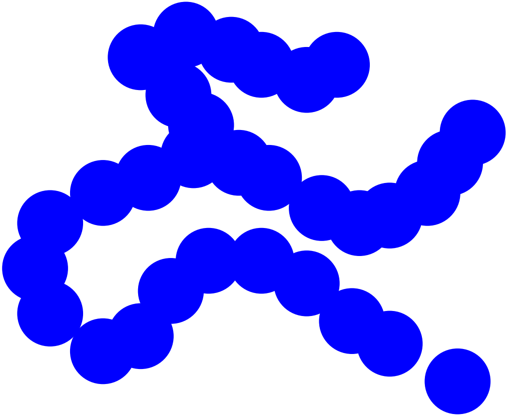
$\dim(H_0) = 12$

Example: expanding balls



$\dim(H_0) = 6$

Example: expanding balls



$\dim(H_0) = 2$

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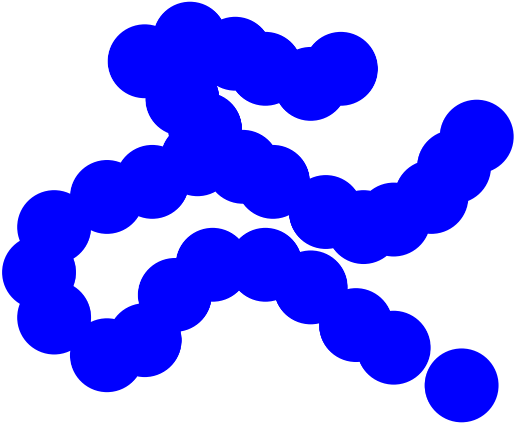

$\dim(H_0) = 2$

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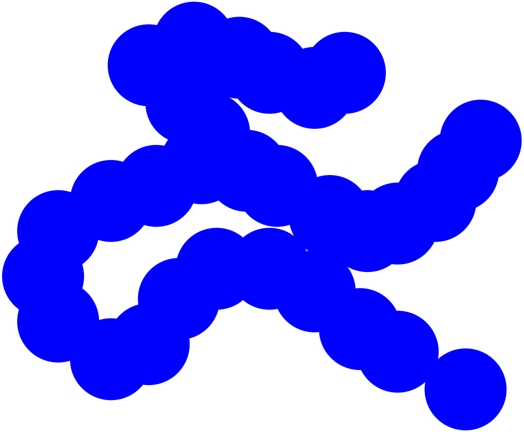

$\dim(H_0) = 1 \qquad \dim(H_1) = 2$

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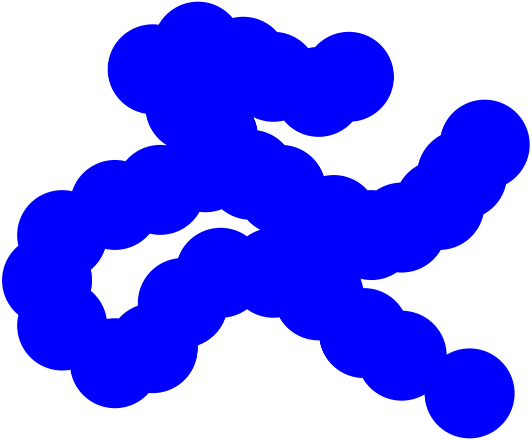

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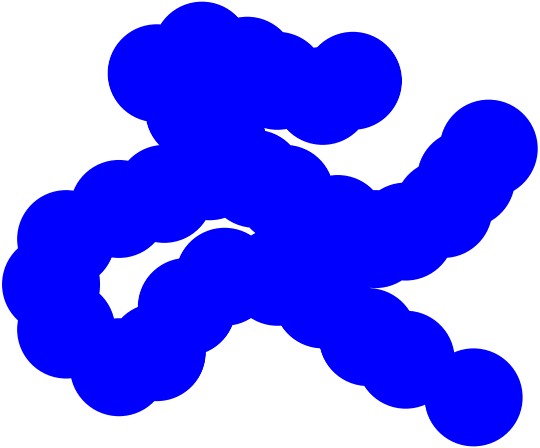

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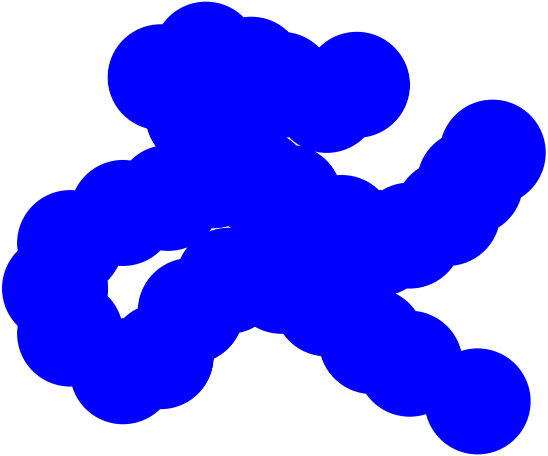

$\dim(H_0) = 1 \qquad \dim(H_1) = 3$

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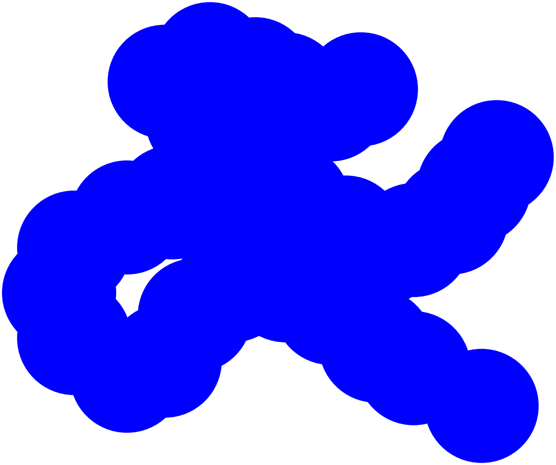

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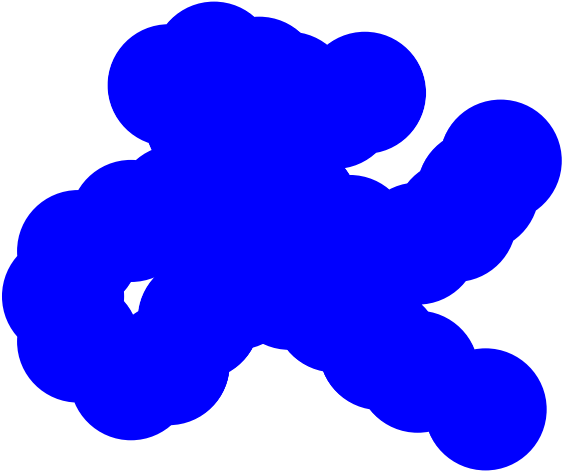

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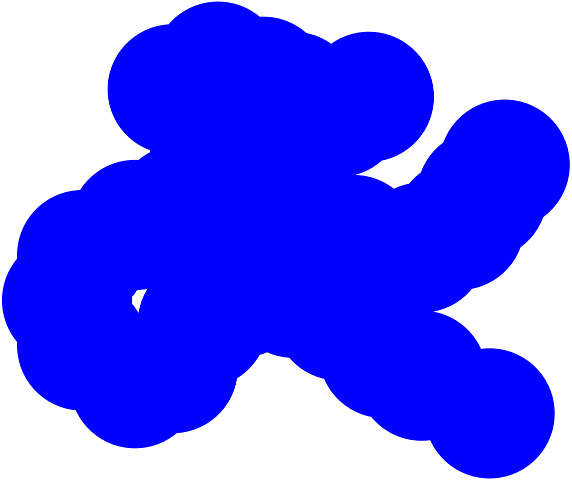

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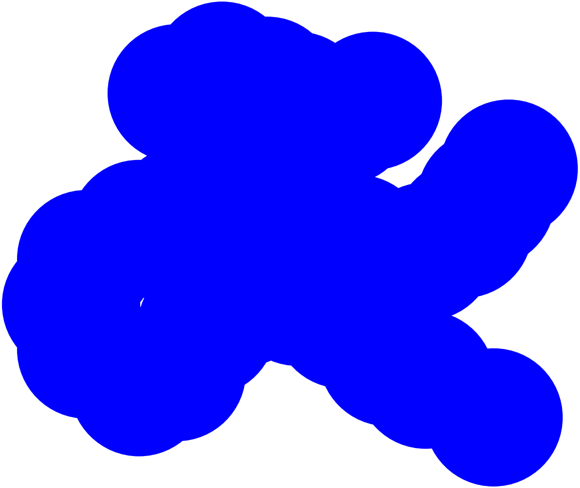

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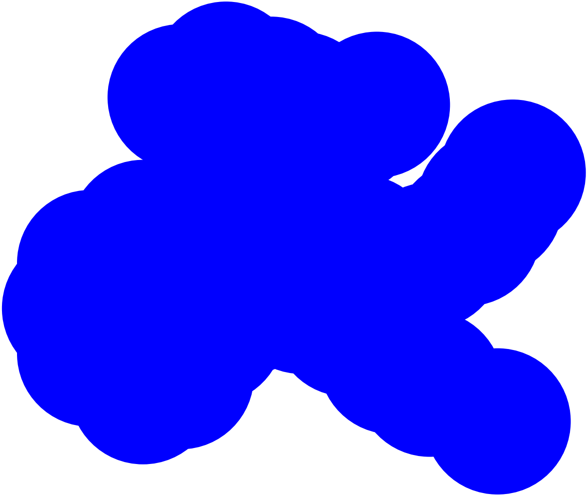

$\dim(H_0) = 1 \qquad \dim(H_1) = 0$

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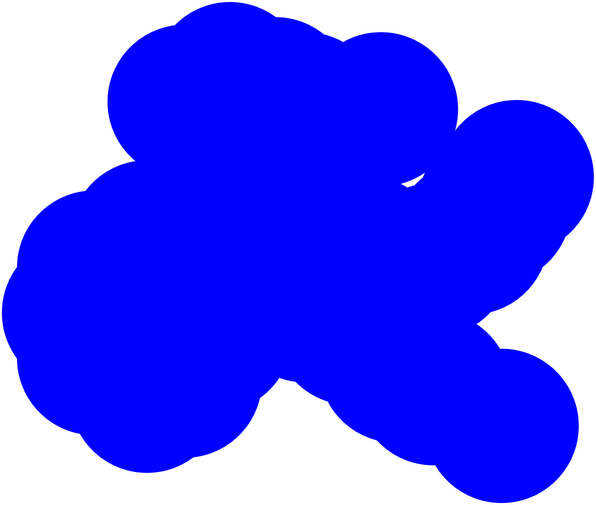

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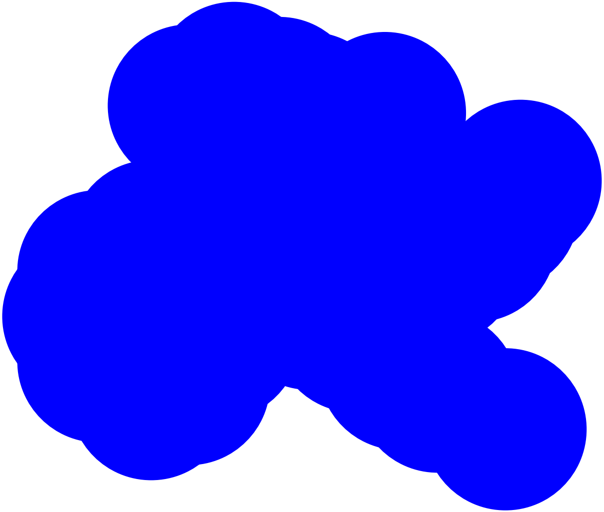

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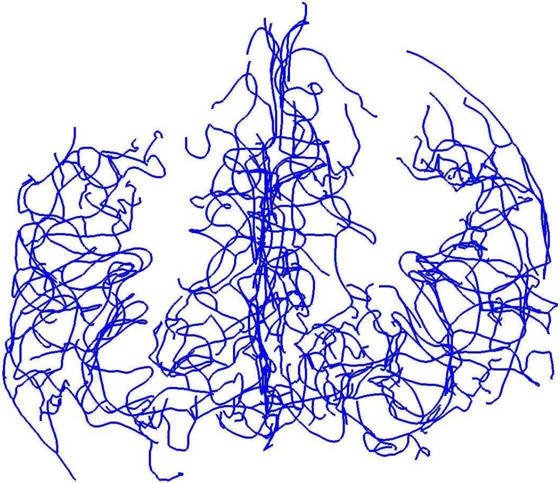

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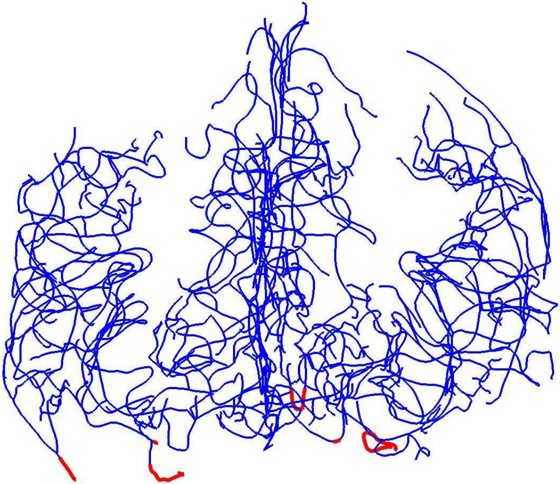

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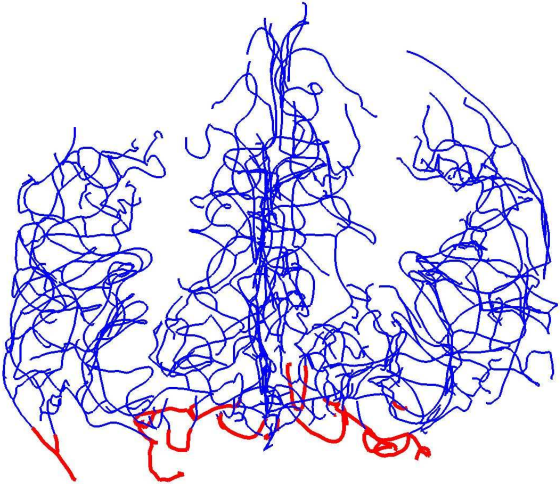

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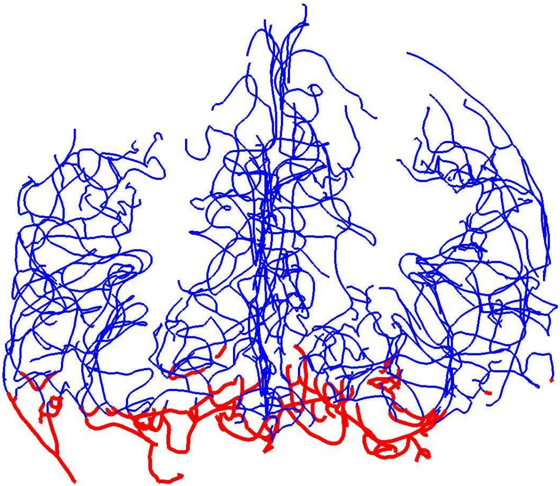

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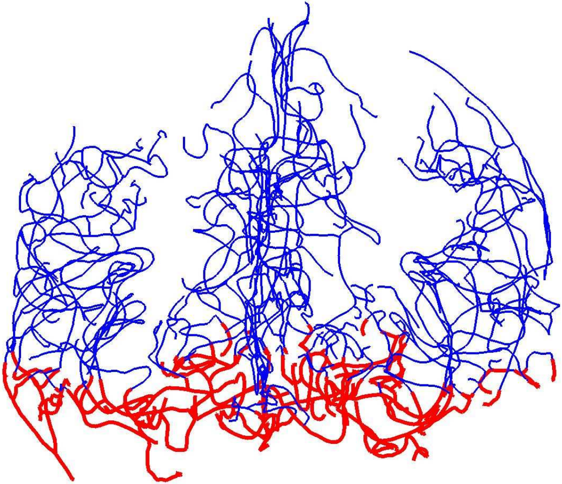

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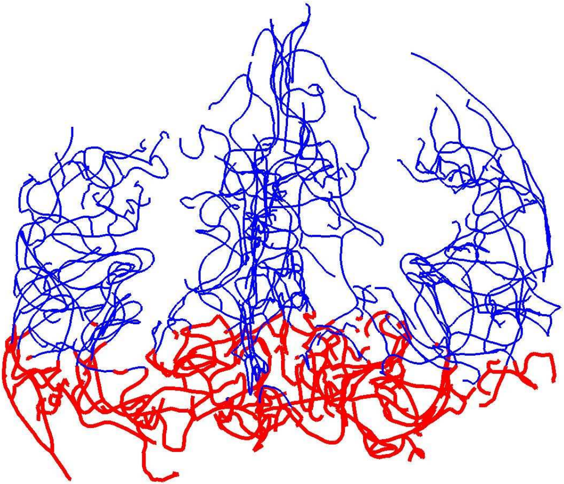

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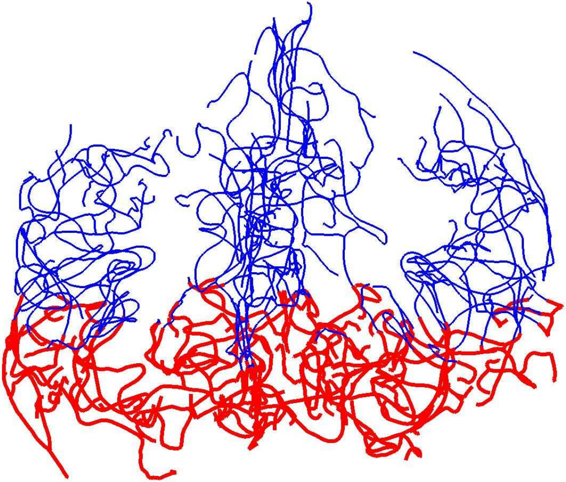

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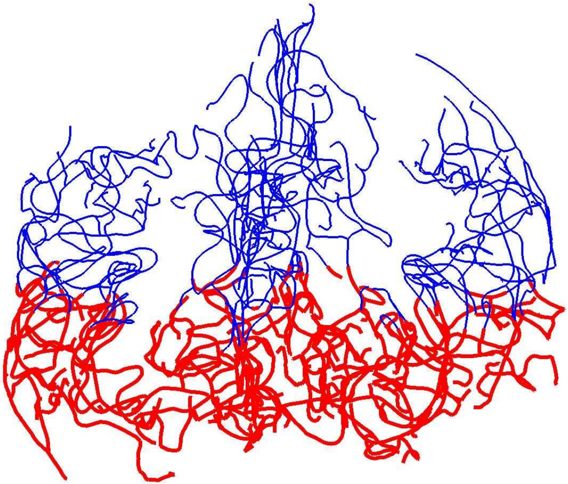

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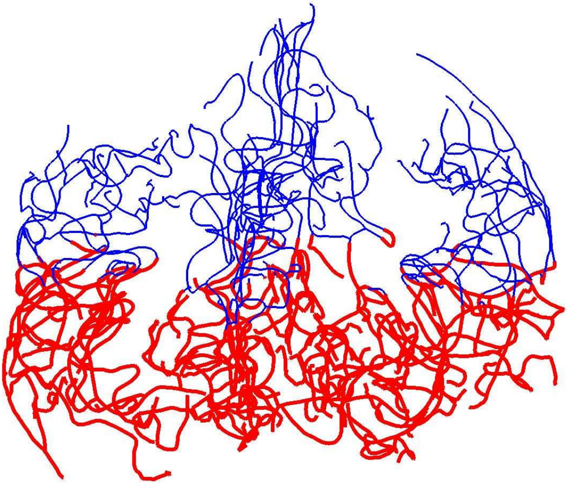

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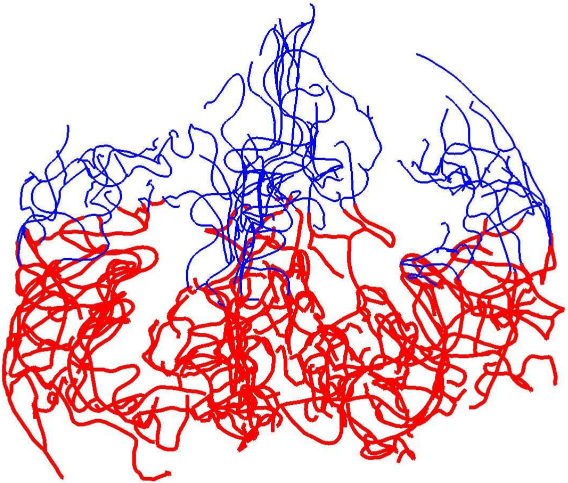

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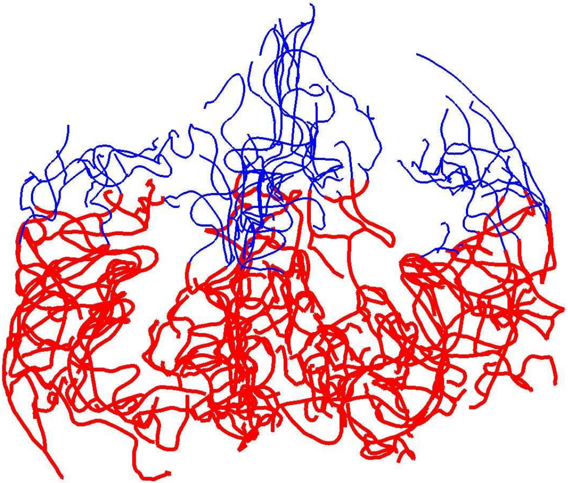

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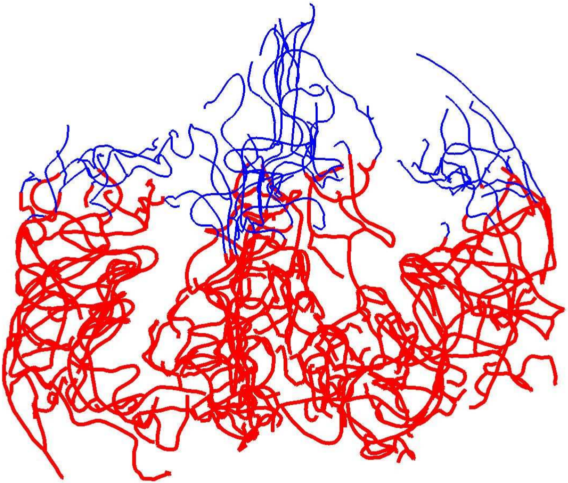

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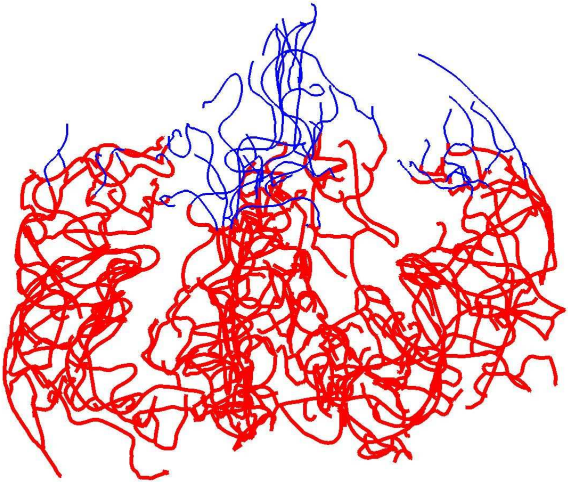

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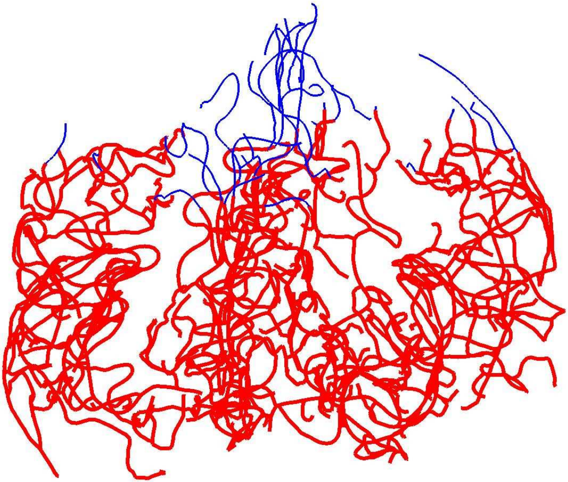

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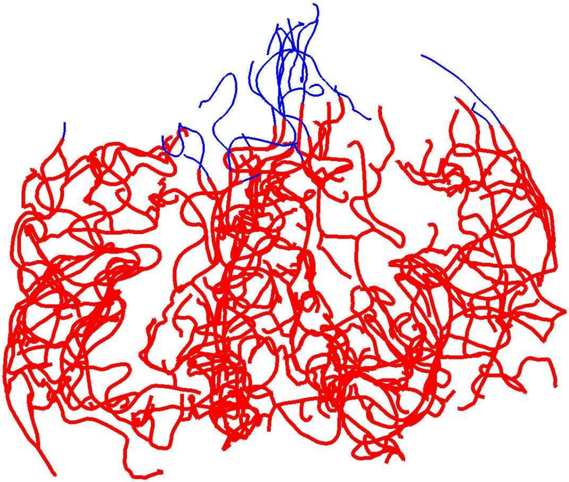

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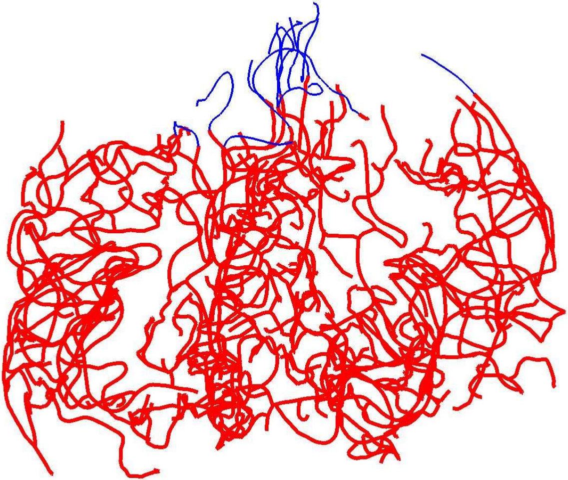

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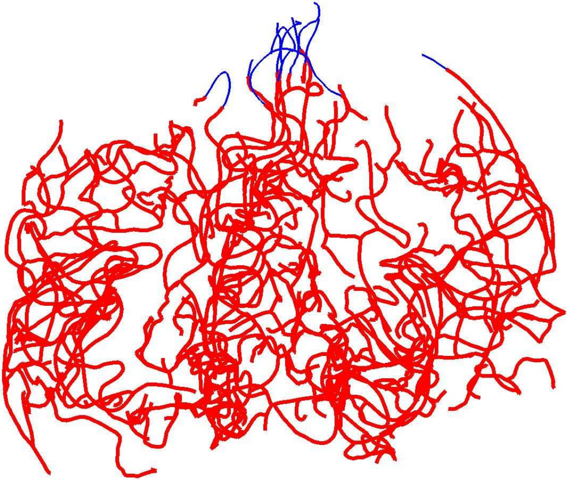

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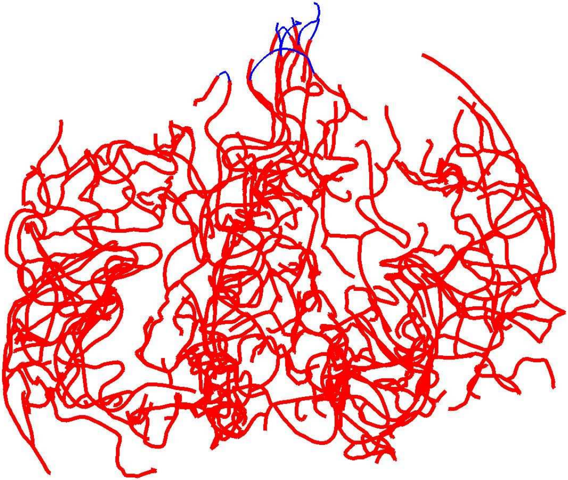

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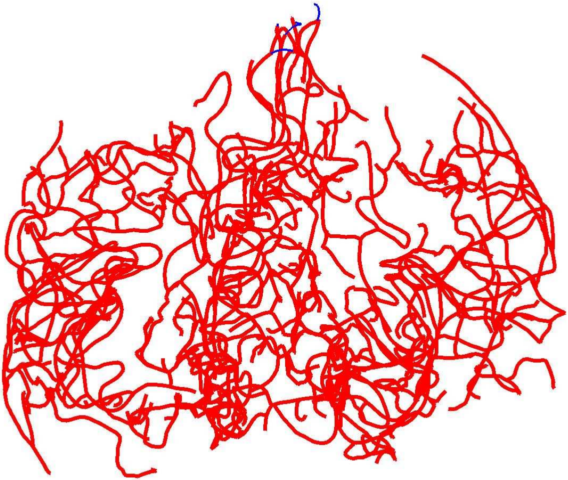

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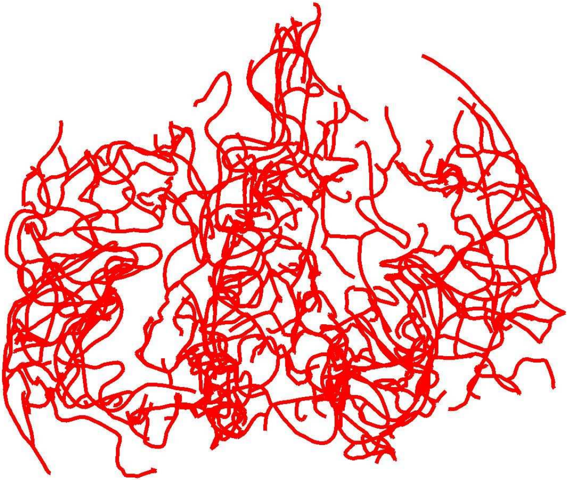

Persistent homology

Build X step by step

- measure evolving topology.

Def. Suppose X is a filtered space, meaning X is a union of an increasing sequence of subspaces: $\varnothing = X_0 \subset X_1 \subset \cdots \subset X_m = X$.

- The persistent homology of this filtration is $H_i X_1 \to H_i X_2 \to \cdots \to H_i X_m$, a sequence of vector space homomorphisms.
- A feature persists from j to k if it appears first in $H_i X_j$ and last in $H_i X_k$.

Examples:

1. Given a function $f : X \to \mathbb{R}$, let $X_t = \{x \in X \mid f(x) \leq t\}$. Good choice of $t_0, \ldots, t_m \in \mathbb{R}$: the values of t across which $H_i X_t$ changes.
2. Any simplicial complex: build it simplex by simplex in some order.

History. invented by [Frosini, Landi 1999], [Robins 1999];
[Edelsbrunner, Letscher, Zomorodian 2002]: includes efficient computation;
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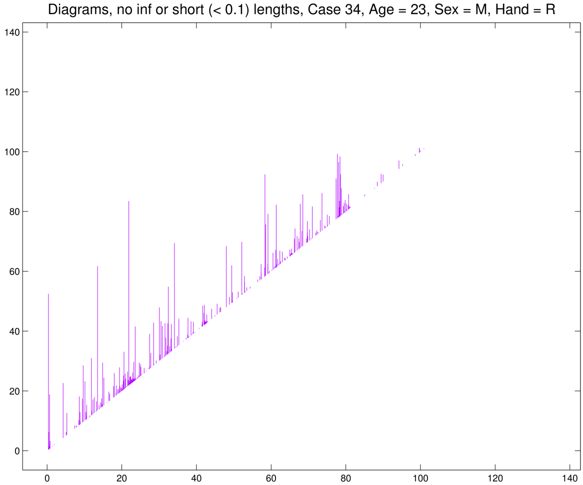
Bar codes

Data structure: 3D tree ⇝ bar code / lace array / persistence diagram:



- multiset of (vertical) line segments $[t, t']$ (plotted at x-coordinate t)
- one for each class with birth time t and death time t'.

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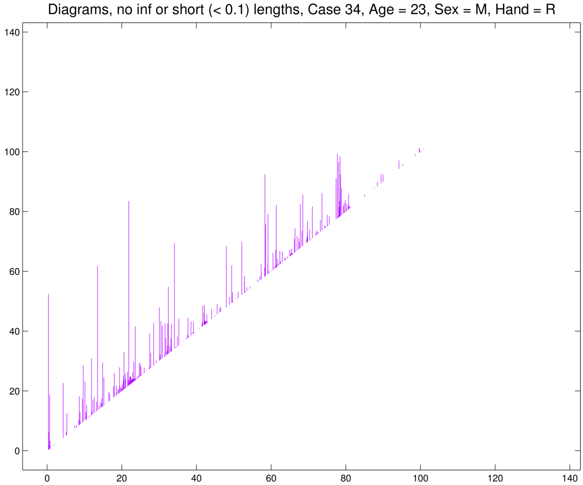

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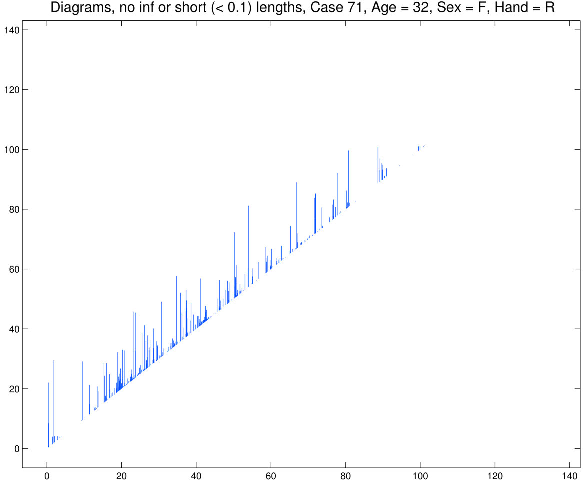

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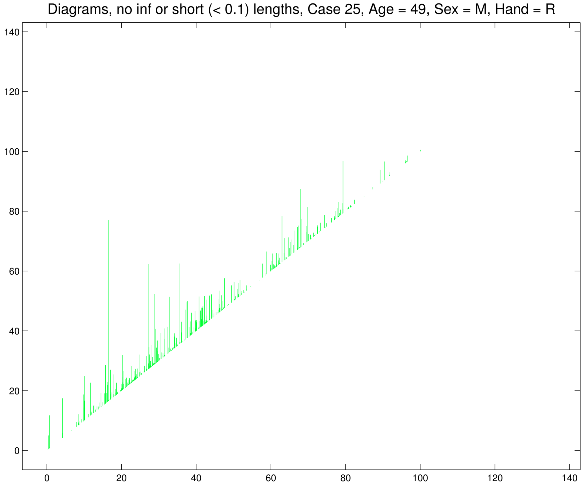

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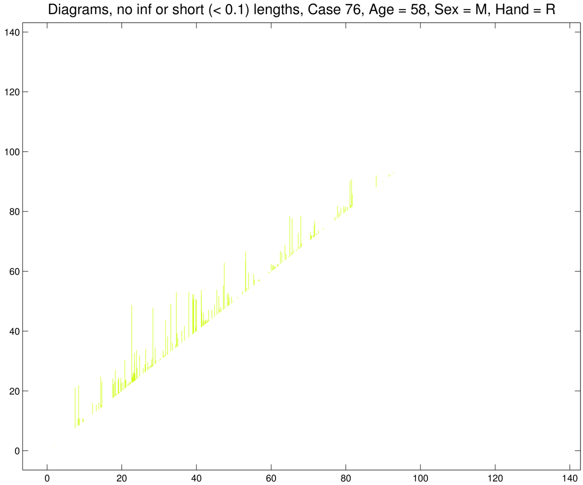

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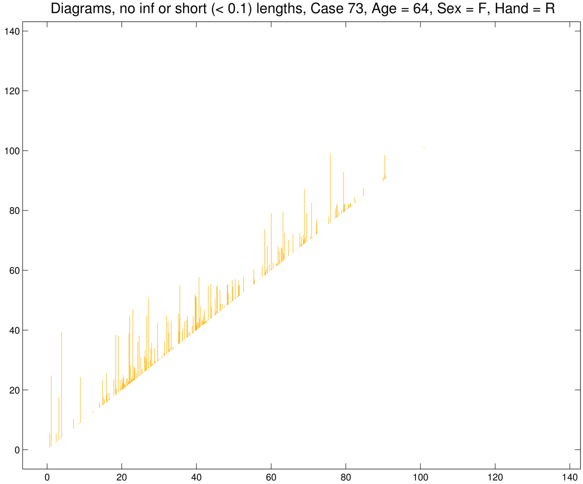

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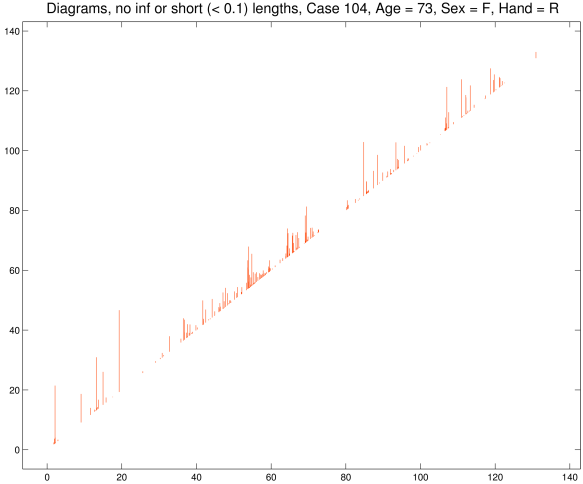

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Sweep filtration

Goal: statistical analysis taking into account

- 3D structure, in particular
- “bendiness”, or “tortuosity”

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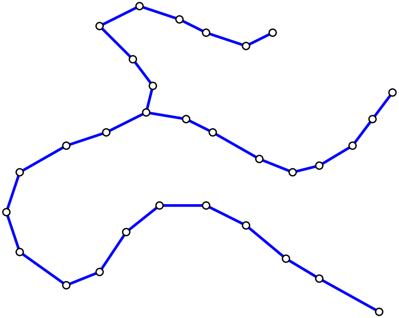

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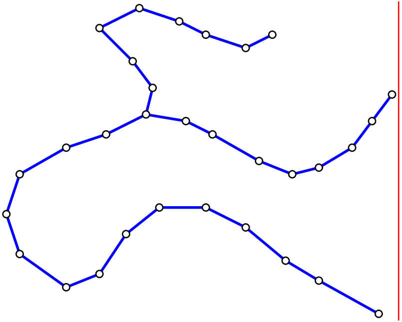

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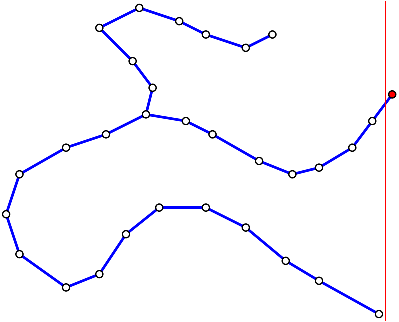

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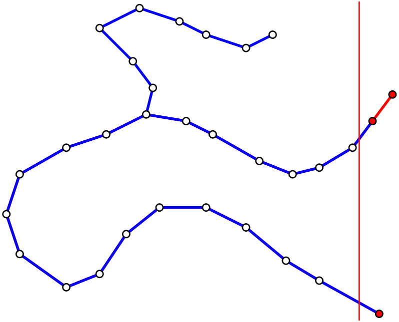

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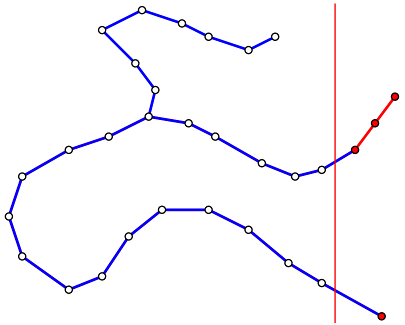

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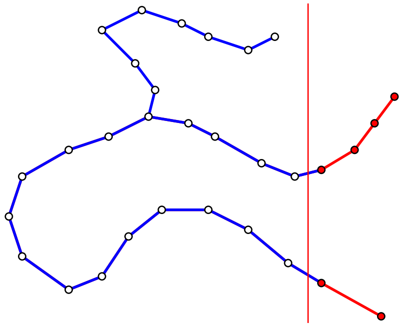

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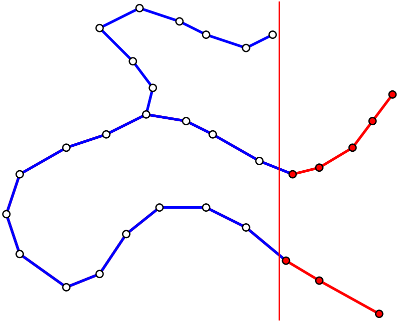

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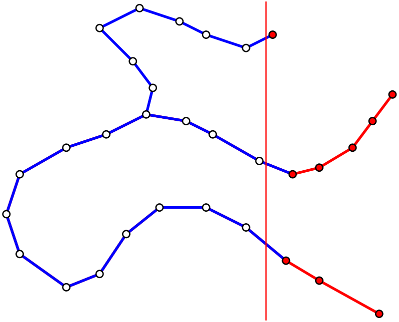

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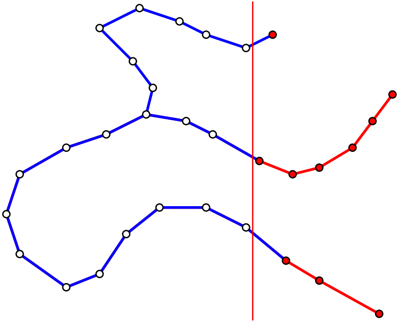

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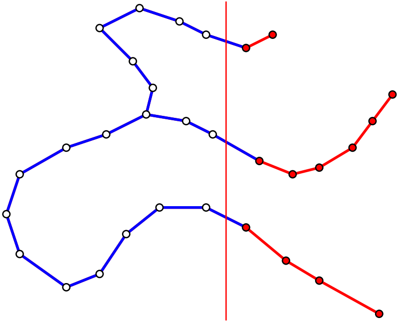

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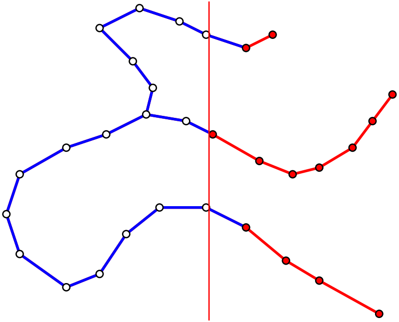

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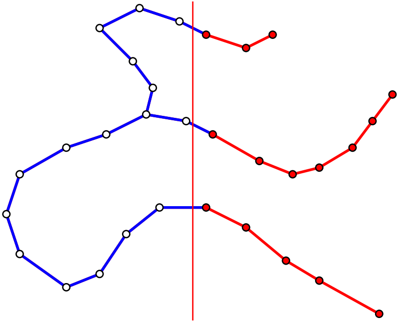

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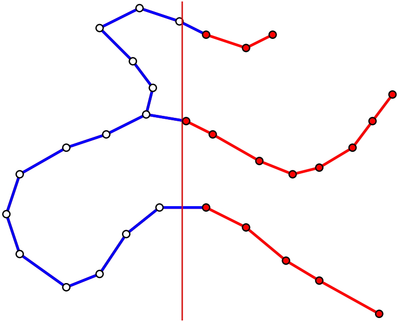

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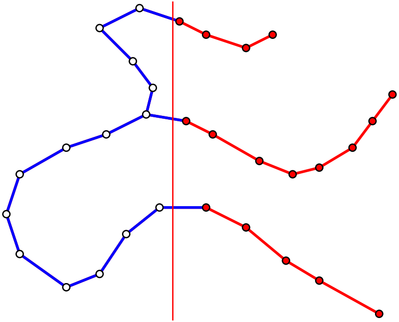

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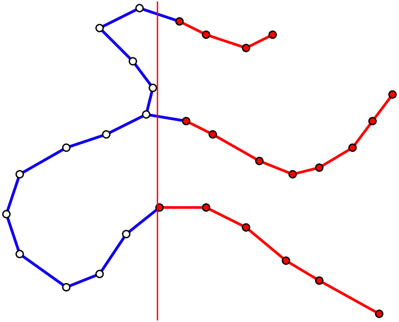

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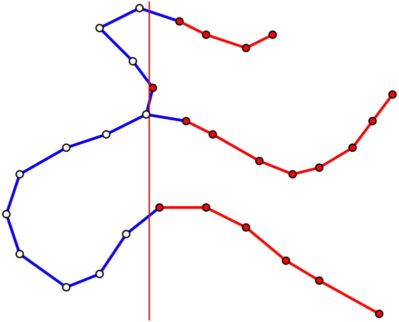

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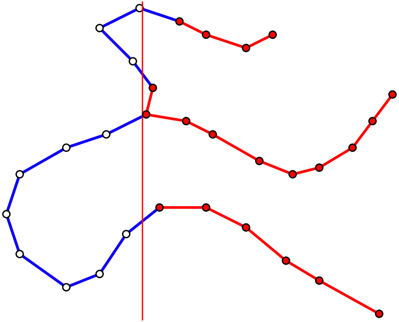

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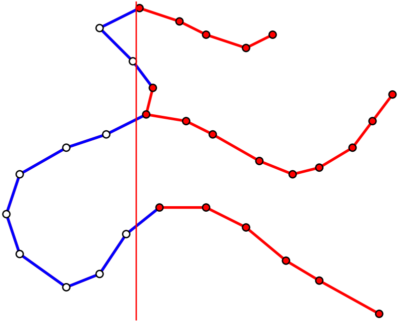

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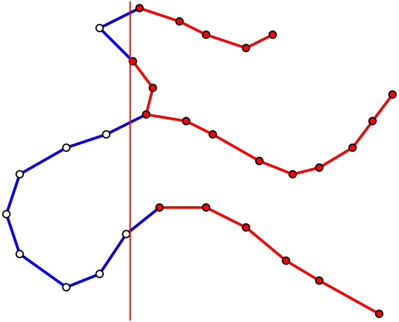

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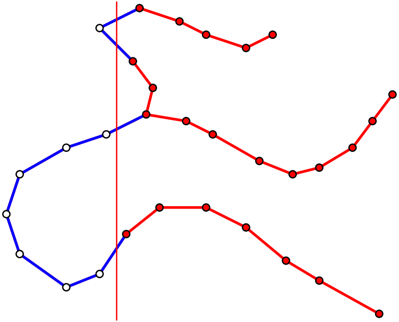

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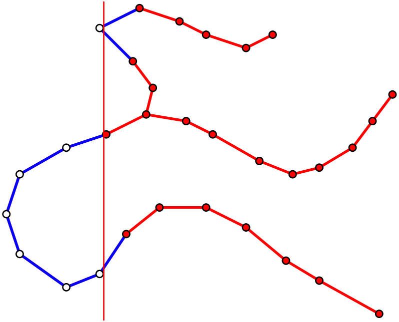

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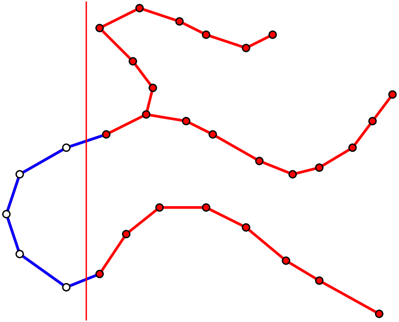

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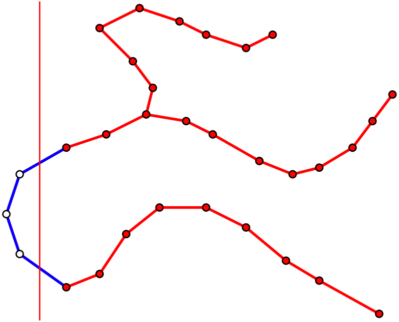

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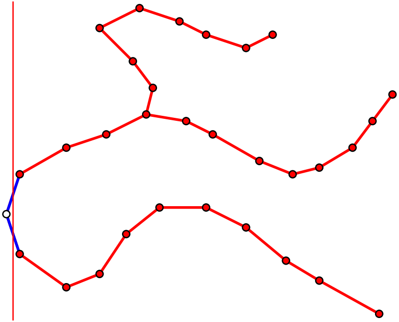

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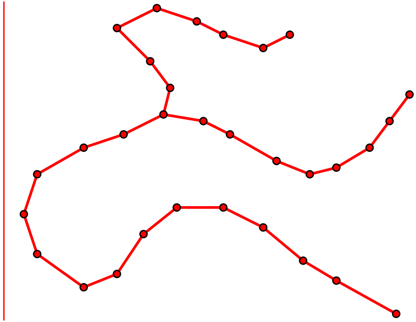

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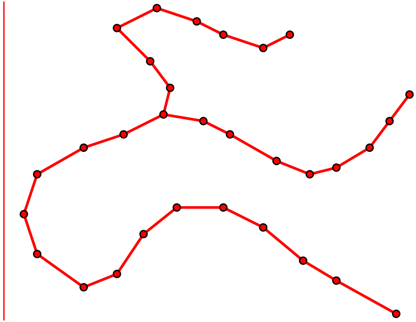

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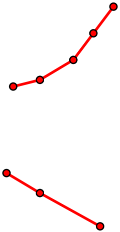

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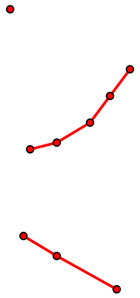

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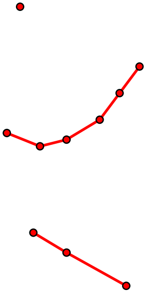

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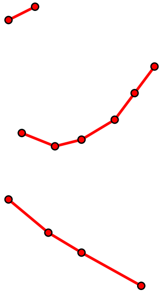

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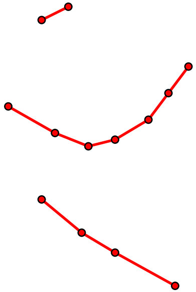

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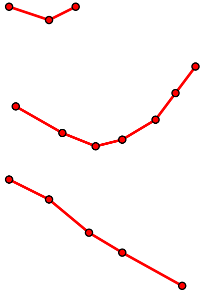

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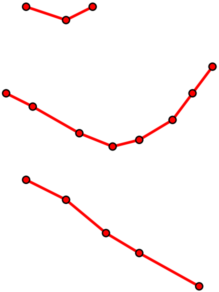

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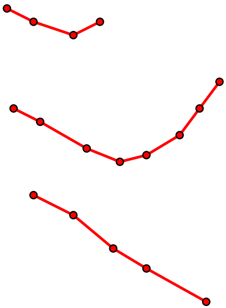

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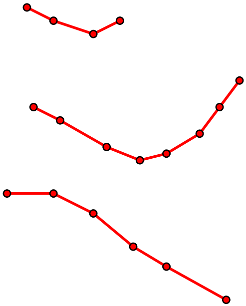

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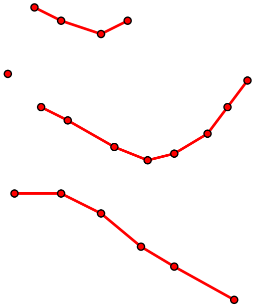

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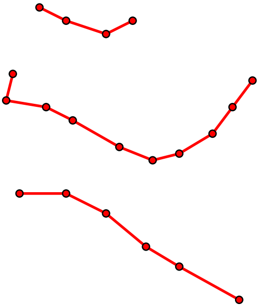

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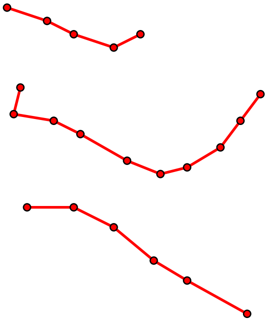

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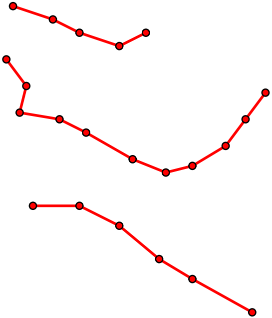

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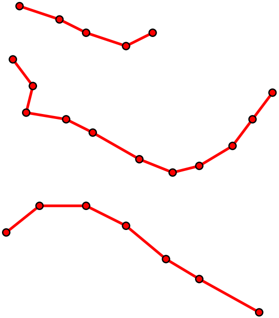

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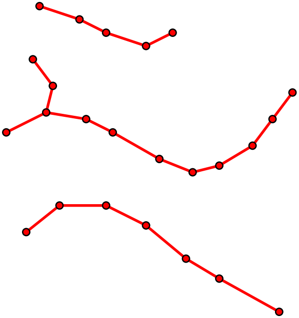

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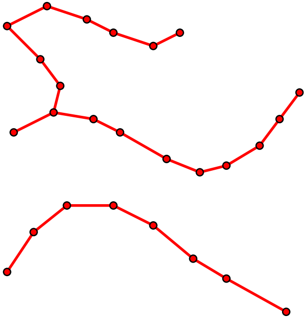

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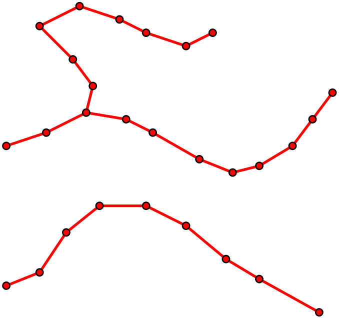

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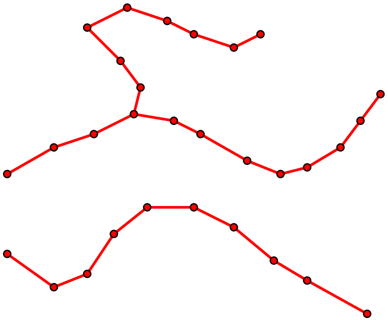

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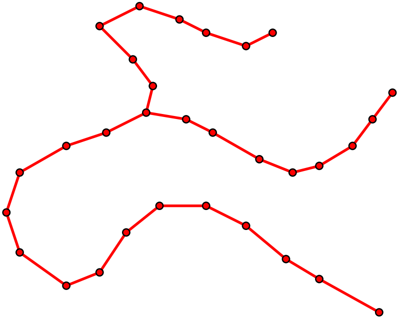

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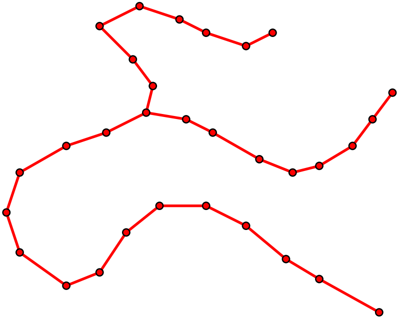

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Easily computable (if dim X is low; ambient space dim irrelevant).

Statistical analysis

Reduce to linear methods. 3D tree $\rightsquigarrow$ bar code $\rightsquigarrow$ vector in $\mathbb{R}^{100}$:

- top 100 bar lengths, in decreasing order, log scale
- correlate first principal component score vs. age

Conclusions. [Bendich, Marron, M.—, Pieloch, Skwerer 2014]
Longest bars in older brains tend to be shorter and later.

- Pearson correlation 0.52663
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Remarks. Results essentially unchanged after

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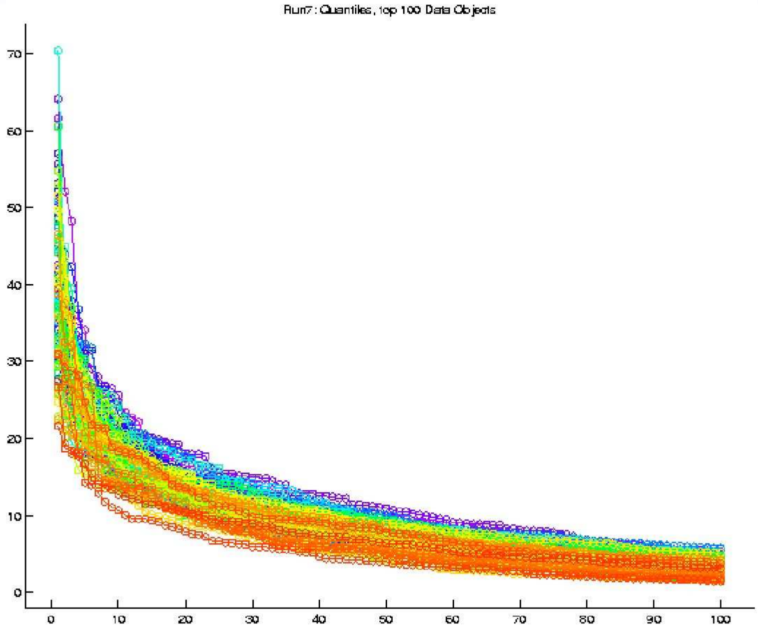
Moral. Persistent homology can topologically detect statistically significant geometric motifs.

Top 100 bars


Run7 : Quantiles, top 100 Data Objects
70
60
50
40
30
20
10
0
0
10
20
30
40
50
60
70
80
90
100

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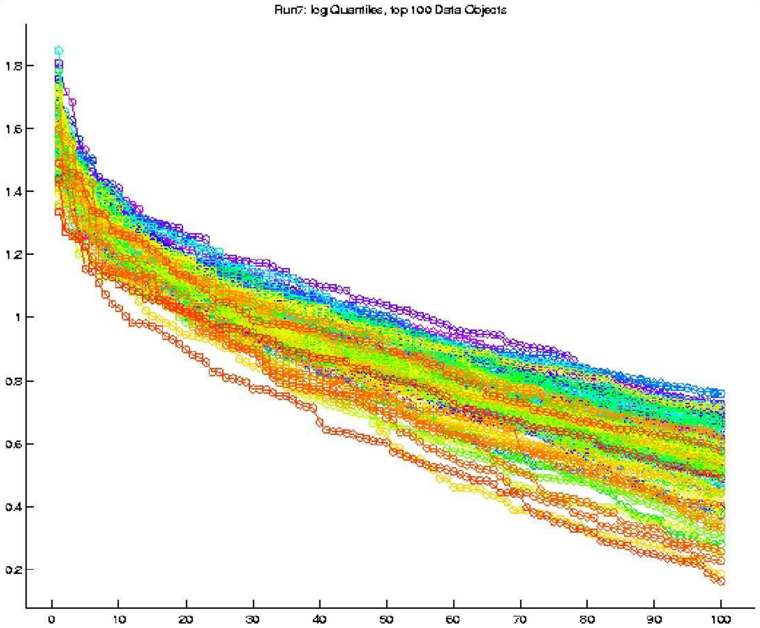
Top 100 bars: log scale



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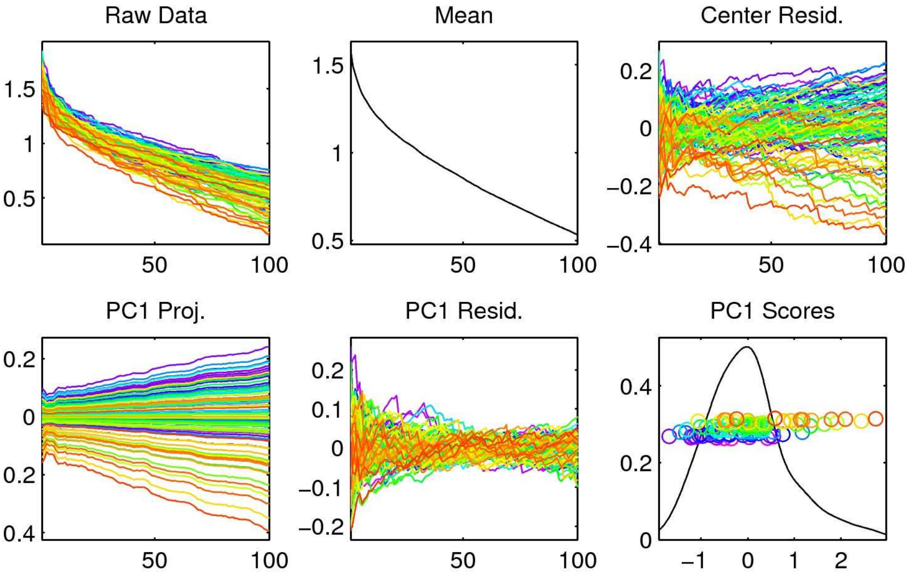
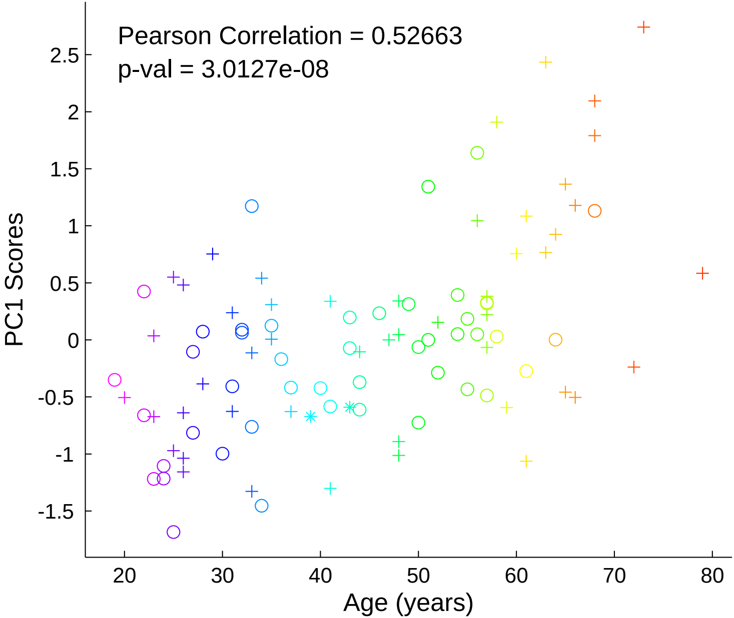
Age vs. PC1


Raw Data
1.5
1
0.5
50
100
Mean
1.5
1
0.5
50
100
Center Resid.
0.2
0
−0.2
−0.4
50
100
PC1 Proj.
0.2
0
0.2
0.4
50
100
PC1 Resid.
0.2
0.1
0
−0.1
−0.2
50
100
PC1 Scores
0.4
0.2
0
−1
0
1
2

Age vs. PC1


Pearson Correlation = 0.52663
p-val = 3.0127e-08
PC1 Scores
Age (years)
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
20
30
40
50
60
70
80

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Reflections on persistent homology

Where did the best correlation occur?

- How did we choose top 100 bar lengths?
- What choices yield the best correlation? Why?

Persistent homology mantra: most significant features

- are “biggest”
- live “far from the diagonal” in bar codes.

For brain artery trees.

- Not surprising that very short bars $\leftrightarrow$ noise, although in future studies they might not.
- While biggest features are important,
- they hinder strength of correlation.

Morals.

- Importance $\not\Rightarrow$ significance for geometric features.
- Persistent homology can detect significant features lying between important and noise.

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Persistent homology mantra: most significant features

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- live “far from the diagonal” in bar codes.

For brain artery trees.

- Not surprising that very short bars $\leftrightarrow$ noise, although in future studies they might not.
- While biggest features are important,
- they hinder strength of correlation.

Morals.

- Importance $\not\Rightarrow$ significance for geometric features.
- Persistent homology can detect significant features lying between important and noise.

Reflections on persistent homology

Where did the best correlation occur?

- How did we choose top 100 bar lengths?
- What choices yield the best correlation? Why?

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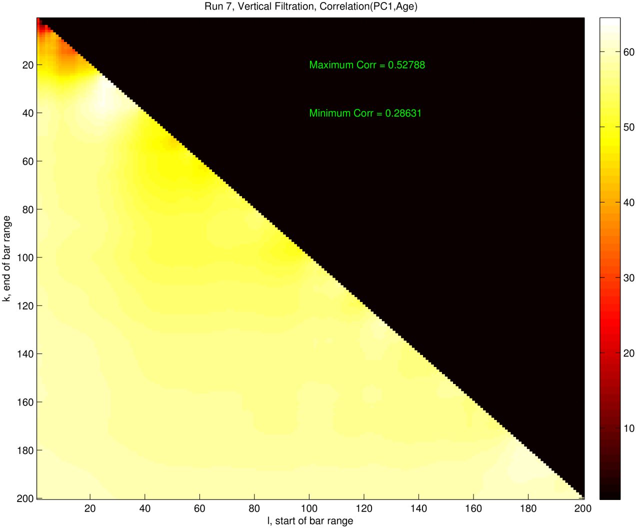
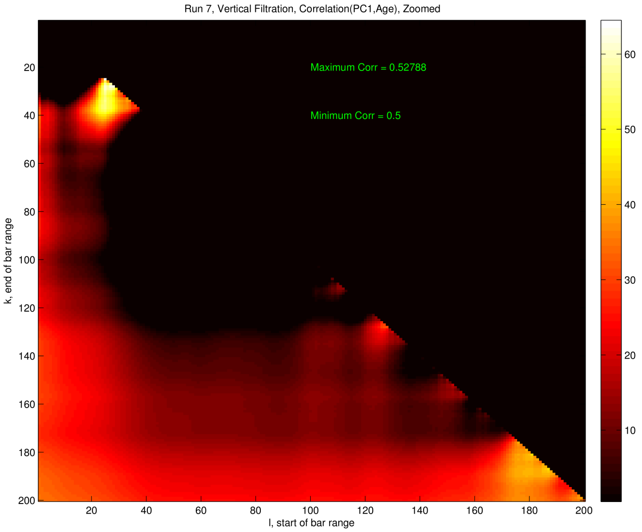
Top 100 bars


Run 7, Vertical Filtration, Correlation(PC1,Age)
Maximum Corr = 0.52788
Minimum Corr = 0.28631
k, end of bar range
l, start of bar range
20
40
60
80
100
120
140
160
180
200
10
30
50

Top 100 bars


Run 7, Vertical Filtration, Correlation(PC1,Age), Zoomed
Maximum Corr = 0.52788
Minimum Corr = 0.5
k, end of bar range
l, start of bar range
20
40
60
80
100
120
140
160
180
200
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Lesson for students. Integration of biology, math, stat, and computation in research and application.

Future directions

- fruit fly wings (with Houle, Thomas, Curry, Beriwal)

- lung arteries (with McLean et al., Marron)

- fMRI (with Lazar et al.)

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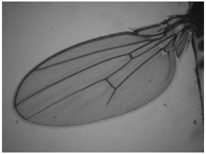
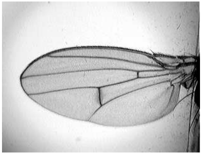
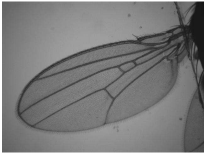

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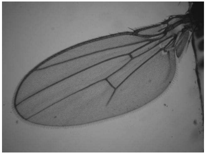
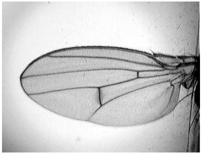
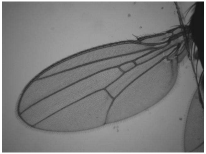

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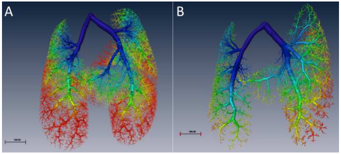

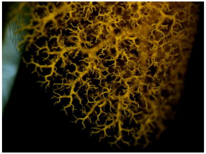

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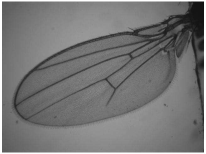
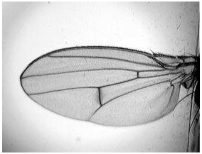
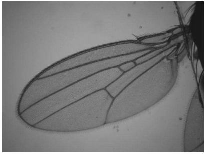

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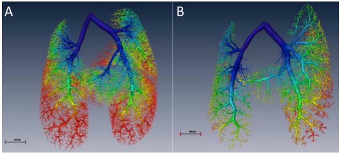

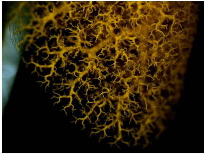

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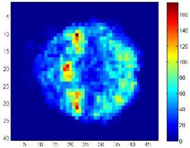
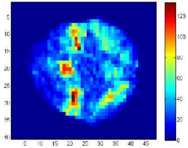
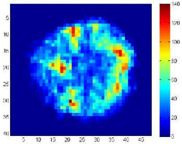

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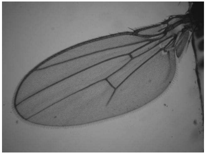

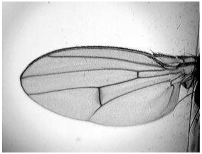

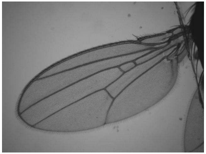

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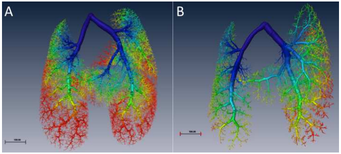

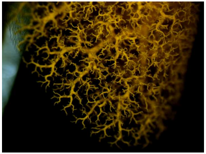

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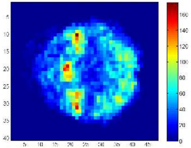

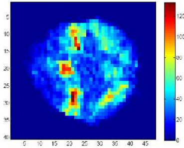

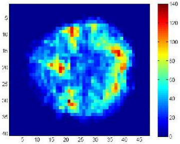

Thank You