New horizons in algebra for multiparameter persistence

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Centre for Topological Data Analysis

University of Oxford, UK

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Outline

Persistent homology over arbitrary posets

Input. Topological space X filtered by set Q of subspaces: $X_q \subseteq X$ for $q \in Q$

$\Rightarrow Q$ is a partially ordered set: $X_q \subseteq X_{q'} \Leftrightarrow q \preceq q'$

Def. $\{X_q\}_{q\in Q}$ has persistent homology $\{H_q = H(X_q; \Bbbk)\}_{q\in Q}$.

Def. Q-module over the poset Q

- Q-graded vector space $M = \bigoplus_{q\in Q} M_q$ over the field $\Bbbk$ with
- homomorphism $M_q \to M_{q'}$ whenever $q \prec q'$ in Q such that
- $M_q \to M_{q''}$ equals the composite $M_q \to M_{q'} \to M_{q''}$ whenever $q \prec q' \prec q''$

Essentially equivalent

- representation of Q [Nazarova–Roĭter 1972]
- functor from Q to the category of vector spaces (e.g., [Curry 2019])
- vector-space valued sheaf on Q (e.g., [Curry's thesis 2014])
- representation of incidence algebra of Q [Doubilet–Rota–Stanley 1972]
- module over directed acyclic graph Q [Chambers–Letscher 2018]
- representation of quiver Q with (commutative) relations (e.g., [Oudot 2015])
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Example. Encode fruit fly wing with 2-parameter persistence

- 1st parameter: distance from vertex set
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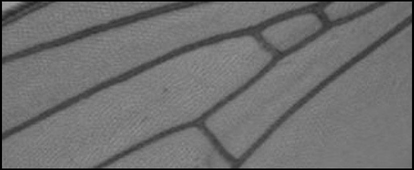

Sublevel set $W_{r,s}$ is near edges but far from vertices $\Rightarrow H_{r,s} = H_i(W_{r,s})$

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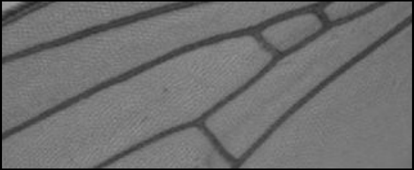

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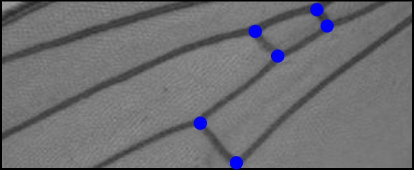

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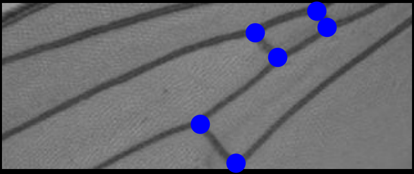

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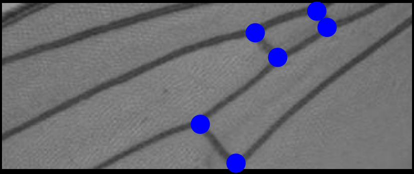

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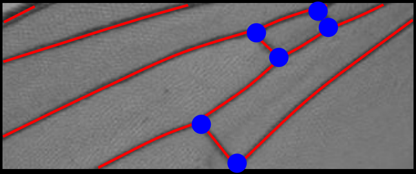

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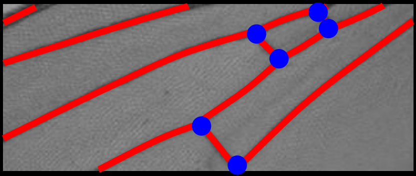

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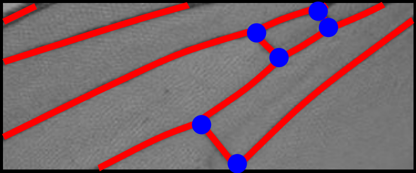

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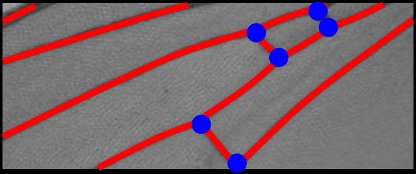

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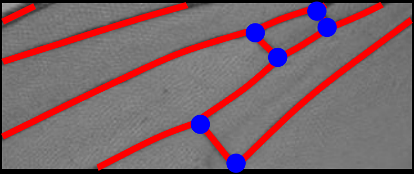

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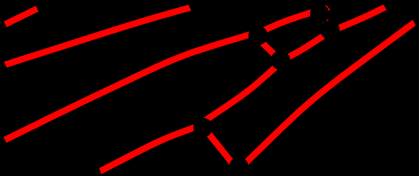

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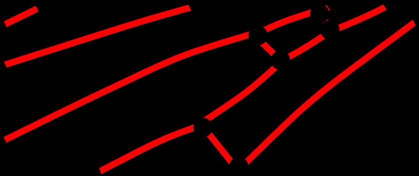

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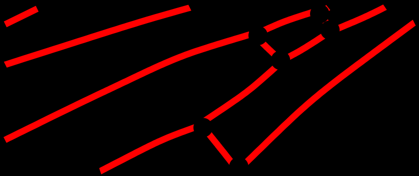

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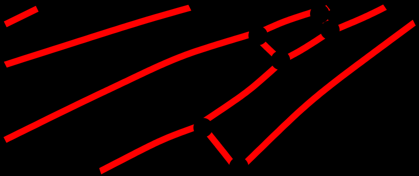

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$\mathbb{Z}^2$-module:

$$\begin{array}{ccccccc}
 & \uparrow & & \uparrow & & \uparrow & \\
\to & H_{r-\varepsilon,s+\delta} & \to & H_{r,s+\delta} & \to & H_{r+\varepsilon,s+\delta} & \to \\
 & \uparrow & & \uparrow & & \uparrow & \\
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 & \uparrow & & \uparrow & & \uparrow & \\
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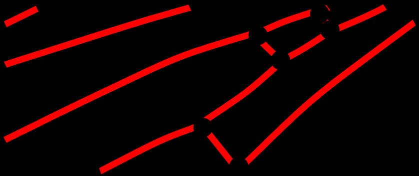

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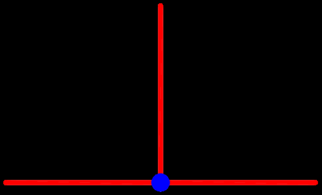
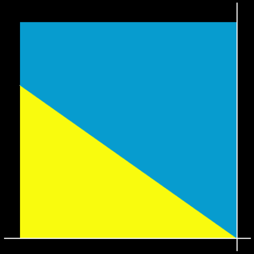


A piece of fly wing vein

$\rightsquigarrow$



The (r, s)-plane $\mathbb{R}^2$

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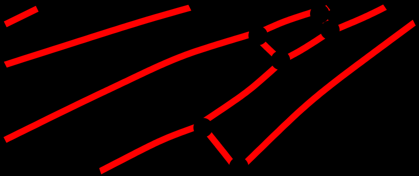

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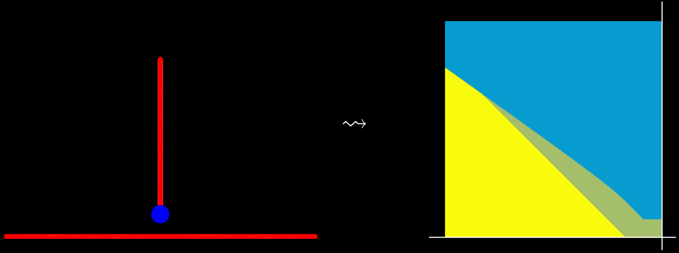

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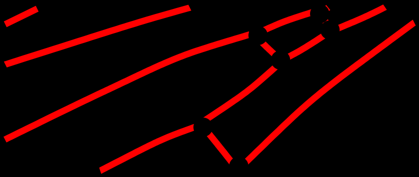

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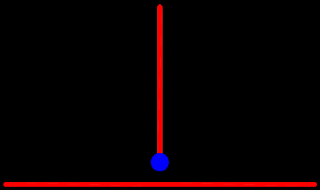

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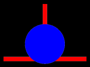

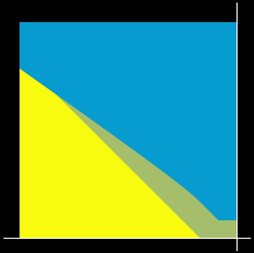

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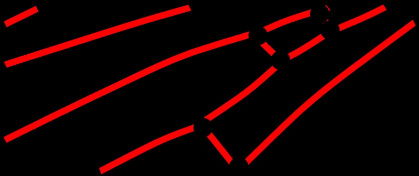

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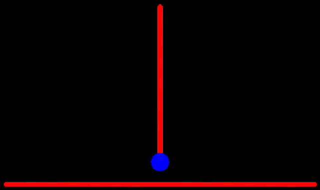

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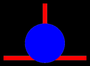

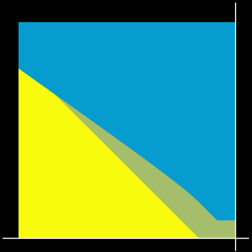

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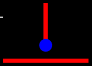

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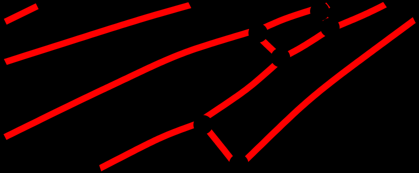

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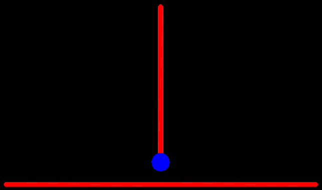

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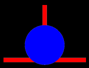

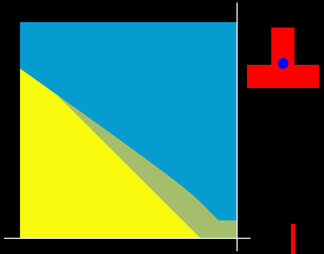

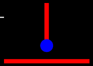

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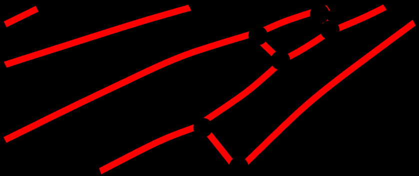

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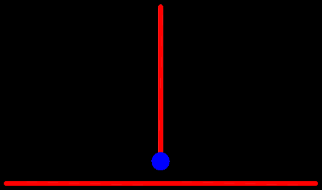

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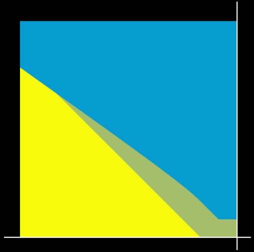

The (r, s)-plane $\mathbb{R}^2$

Wing vein persistence [w/Houle, et al., ongoing]

Example. Encode fruit fly wing with 2-parameter persistence

- 1st parameter: distance from vertex set (require distance $\geq -r$)
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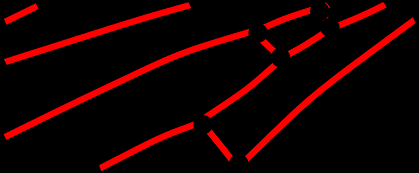
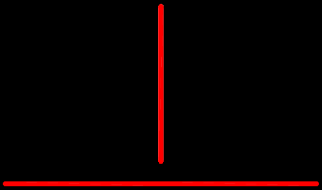


Sublevel set $W_{r,s}$ is near edges but far from vertices $\Rightarrow H_{r,s} = H_i(W_{r,s})$



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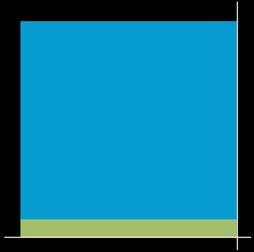

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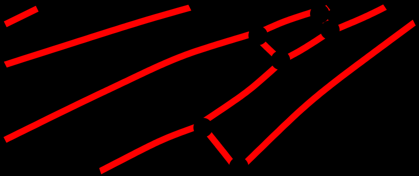

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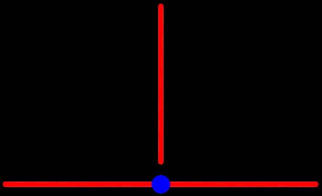

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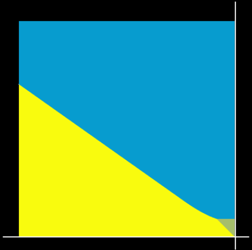

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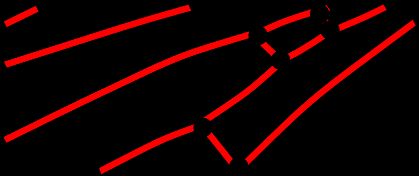

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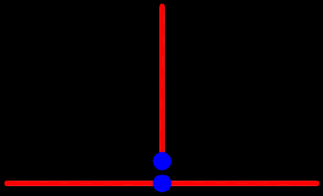

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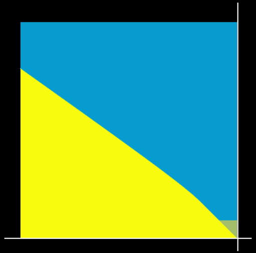

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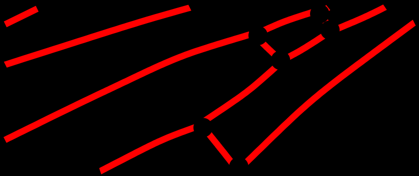

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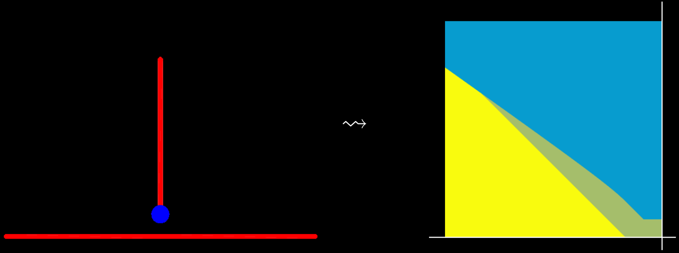

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Persistent homology over arbitrary posets

Input. Topological space X filtered by set Q of subspaces: $X_q \subseteq X$ for $q \in Q$

$\Rightarrow Q$ is a partially ordered set: $X_q \subseteq X_{q'} \Leftrightarrow q \preceq q'$

Def. $\{X_q\}_{q\in Q}$ has persistent homology $\{H_q = H(X_q; \Bbbk)\}_{q\in Q}$. This is a

Def. Q-module over the poset Q

- Q-graded vector space $M = \bigoplus_{q\in Q} M_q$ over the field $\Bbbk$ with
- homomorphism $M_q \to M_{q'}$ whenever $q \prec q'$ in Q such that
- $M_q \to M_{q''}$ equals the composite $M_q \to M_{q'} \to M_{q''}$ whenever $q \prec q' \prec q''$

Essentially equivalent

- representation of Q [Nazarova–Roĭter 1972]
- functor from Q to the category of vector spaces (e.g., [Curry 2019])
- vector-space valued sheaf on Q (e.g., [Yuzvinsky 1987], [Yanagawa 2001], [Curry 2014])
- representation of incidence algebra of Q [Doubilet–Rota–Stanley 1972]
- module over directed acyclic graph Q [Chambers–Letscher 2018]
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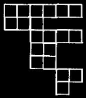
Old bar codes

It is convenient to represent λ^A as a "*diagram of boxes*", each row starting at i and ending at j stands for one indecomposable factor of type $E_{(i,j)}$.

E.g. the following diagram represents λ^A for A isomorphic to

$$E_{(1,6)} \oplus E_{(1,3)} \oplus E_{(3,6)} \oplus E_{(3,4)} \oplus E_{(3,4)} \oplus E_{(5,6)} \oplus E_{(5,5)}:$$



2.4. Conversely any indexed set $\lambda = (\lambda_{(i,j)})_{1 \leqq i \leqq j \leqq m}$ of natural numbers determines an orbit in $L(V_1, V_2, \ldots, V_m)$ provided $\dim V_i = \hat{\lambda}_i := \sum_{r \leqq i \leqq s} \lambda_{(r,s)}$ (= # boxes in the i^{th} column of λ). We will shortly call such an indexed set a *diagram*, define [...]

Let us introduce now the set of non-negative integers $n^A = \{n^A_{rs}\}_{1 \leqslant r \leqslant s \leqslant m}$ associated to A and defined by

$$n^A_{rs} := \sum_{p \leqslant r \leqslant s \leqslant q} e^A_{pq}. \tag{2.3}$$

n^A_{rs} is the number of the segments of the diagram of $|A|$ which contain the integers r, s. It follows that we have

$$e^A_{pq} = n^A_{pq} - n^A_{p-1,q} - n^A_{p,q+1} + n^A_{p-1,q+1} \tag{2.4}$$

where we set $n^A_{rs} = 0$ if $r < 0$ or $s > m + 1$.

[Abeasis–Del Fra–Kraft 1981, Abeasis–Del Fra 1985]

Old bar codes

Example 1.5. Consider the rank array $\mathbf{r} = (r_{ij})$, its lace array $\mathbf{s} = (s_{ij})$, and its rectangle array $\mathbf{R} = (R_{ij})$, which we depict as follows.

$$\mathbf{r} = \begin{array}{cccc|c} 3 & 2 & 1 & 0 & i \diagup j \\ \hline & & & 2 & 0 \\ & & 3 & 2 & 1 \\ & 4 & 2 & 1 & 2 \\ 3 & 2 & 1 & 0 & 3 \end{array} \qquad \mathbf{s} = \begin{array}{cccc|c} 3 & 2 & 1 & 0 & i \diagup j \\ \hline & & & 0 & 0 \\ & & 0 & 1 & 1 \\ & 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 3 \end{array} \qquad \mathbf{R} = \begin{array}{ccc|c} 2 & 1 & 0 & i \diagup j \\ \hline & & & 1 \\ & \boxed{} & \boxed{} & 2 \\ \boxed{\substack{\\ }} & \boxed{} & \boxed{} & 3 \end{array}$$

The relation (1.2) says that an entry of $\mathbf{r}$ is the sum of the entries in $\mathbf{s}$ that are weakly southeast of the corresponding location. The height of R_{ij} is obtained by subtracting the entry r_{ij} from the one above it, while the width of R_{ij} is obtained by subtracting the entry r_{ij} from the one to its left.

It follows from the definition of R_{ij} that

$$\sum_{k \geq j} \operatorname{height}(R_{ik}) = r_{i,j-1} - r_{i,n} \leq r_{i,j-1} \qquad \text{for all } i \tag{1.3}$$

$$\sum_{\ell \leq i} \operatorname{width}(R_{\ell j}) = r_{i+1,j} - r_{0,j} \leq r_{i+1,j} \qquad \text{for all } j. \tag{1.4}$$

(This will be applied in Proposition 8.12.) The relation (1.2) can be inverted to obtain

$$s_{ij} = r_{ij} - r_{i-1,j} - r_{i,j+1} + r_{i-1,j+1} \tag{1.5}$$

[Knutson–M.–Shimozono 2005]

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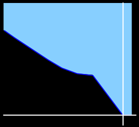
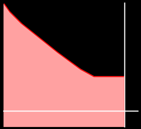
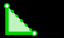
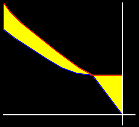
 or  or  but not 

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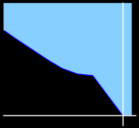
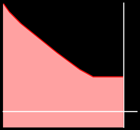
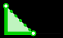
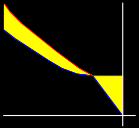
 or  or  but not 

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Essentially unique: multiset $\{M_\alpha\}_{\alpha \in A}$ of isomorphism classes is invariant.

Remark. Interval modules are indecomposable (easy), but they can't tell the full decomposition story, even over $\mathbb{Z}^n$ [Carlsson–Zomorodian 2009].

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Thm [Botnan–Crawley-Boevey 2020], cf. [Gabriel–Roĭter 1992]. Over arbitrary poset Q, M has indecomposable decomposition: $M \cong \bigoplus_{\alpha \in A} M_\alpha$ with M_α indecomposable.

Essentially unique: multiset $\{M_\alpha\}_{\alpha \in A}$ of isomorphism classes is invariant.

Remark. Interval modules are indecomposable (easy), but they can't tell the full decomposition story, even over $\mathbb{Z}^n$ [Carlsson–Zomorodian 2009].

Thm [Buchet–Escolar 2020], [Moore 2022]. General indecomposable $\mathbb{Z}^n$-modules are big.

1. Every finitely supported $\mathbb{Z}^n$-module is a hyperplane section of some indecomposable finitely supported $\mathbb{Z}^{n+1}$-module.
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Thm [Bauer–Scoccola 2022]. Over $\mathbb{Z}^n$,

1. Indecomposables are dense in interleaving distance.
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Past avenues

Divergence of interval vs. indecomposable $\Rightarrow$ two avenues:

1. study indecomposables and decomposition into direct sums thereof
2. relate indicator modules to arbitrary modules in some way
   - homologically (using resolutions, complexes or invariants from there)
   - virtually (using signed expressions for positive integers, like dim or rank)

Avenue 1. substantially developed

- algorithms [QPA 2022, Dey–Xin 2022]
- bottleneck distances from ε-matching between decompositions

Avenue 2. So many options! E.g.,

- Betti numbers from free resolutions [Carlsson–Zomorodian 2009], [Cerri–DiFabio–Ferri–Frosini–Landi 2013], [Oudot–Scoccola 2023]
- modules $\rightsquigarrow$ resolutions or complexes [Kashiwara–Schapira 2017–19], [M.– 2017–], [Scolamiero–Chachólski–Lundman–Ramanujam–Öberg 2017]
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Current and future avenue: filtration

Positivity. $M = \bigoplus_{\alpha \in A} M_\alpha$ expresses M positively in term of the M_α. Choose:

1. retain positivity or
2. retain description in terms of intervals.

Question. Can both be achieved? Look for advice from elsewhere.

Examples. groups or vector bundles

Solution. Answer filtration with filtration! filtered space $X \rightsquigarrow$ filtered module

- $M = M_\ell \supseteq M_{\ell-1} \supseteq \cdots \supseteq M_1 \supseteq M_0 = 0$
- with all M_i / M_{i-1} "nice"

Pipeline

$$\begin{array}{ccccc} \text{data} \rightsquigarrow \text{filtered topological spaces} \rightsquigarrow & \text{algebraic objects} & & & \\ & \Big\downarrow & & & \\ \text{“nice”} & \text{algebraic objects} & \rightsquigarrow \text{invariants} & \rightsquigarrow & \text{statistics} \end{array}$$

Need

- Filtration method
- Stability

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$\wr\!\downarrow$ old: decompose

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Example: Harder–Narasimhan filtration

Idea [Fersztand–Jacquard–Nanda–Tillmann 2023], [Fersztand 2024]. Copy vector bundle theory:

- vector bundle is “nice” if semistable for group action on moduli space
- semistable $\Leftrightarrow$ simple numerical criterion on subbundles: deg/rank decreases
- no $\bigoplus$ decomposition into semistables, but
- $\exists$ filtration with semistable quotients

Input. M and central charge: additive group morphism $Z : K(Q\text{-mods}) \to \mathbb{C}$

- think: $\mathrm{Im} Z \leftrightarrow$ deg and $\mathrm{Re} Z \leftrightarrow$ dim
- M is semistable if slopes decrease: $\frac{\mathrm{Im} Z(M)}{\mathrm{Re} Z(M)} \geq \frac{\mathrm{Im} Z(M')}{\mathrm{Re} Z(M')}$ whenever $M \supseteq M'$

Output. *unique* filtration

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Central charge. Choose $\mathrm{Re} Z = 1$ and $\mathrm{Im} Z$ to be, e.g.,

- dim at some fixed vertex x — finer than rank invariant!
- more generally: anything factoring through $\dim : K(Q\text{-mods}) \to \mathbb{Z}^{\mathrm{vertices}(Q)}$

Thm [Fersztand 2024]. HN filtration $\rightsquigarrow$ HN filtered rank functions.
Under erosion distance, these are interleaving-stable.

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Idea [Fersztand–Jacquard–Nanda–Tillmann 2023], [Fersztand 2024]. Copy vector bundle theory:

- vector bundle is "nice" if semistable for group action on moduli space
- semistable $\Leftrightarrow$ simple numerical criterion on subbundles: deg/rank decreases
- no $\bigoplus$ decomposition into semistables, but
- $\exists$ filtration with semistable quotients

Input. M and central charge: additive group morphism $Z : K(Q\text{-mods}) \to \mathbb{C}$

- think: $\mathrm{Im} Z \leftrightarrow$ deg and $\mathrm{Re} Z \leftrightarrow$ dim
- M is semistable if slopes decrease: $\frac{\mathrm{Im} Z(M)}{\mathrm{Re} Z(M)} \geq \frac{\mathrm{Im} Z(M')}{\mathrm{Re} Z(M')}$ whenever $M \supseteq M'$

Output. *unique* filtration ← key point

- $F_\bullet : M = M_\ell \supseteq M_{\ell-1} \supseteq \cdots \supseteq M_1 \supseteq M_0 = 0$
- with all $\mathrm{gr}_i M = M_i/M_{i-1}$ semistable and decreasing slopes

Central charge. Choose $\mathrm{Re} Z = 1$ and $\mathrm{Im} Z$ to be, e.g.,

- dim at some fixed vertex x — finer than rank invariant! ← key point
- more generally: anything factoring through $\dim : K(Q\text{-mods}) \to \mathbb{Z}^{\text{vertices}(Q)}$

Thm [Fersztand 2024]. HN filtration $\rightsquigarrow$ HN filtered rank functions.
Under erosion distance, these are interleaving-stable.

Lifetime filtration

Idea [M.–Zhang 2024]. Filter using indicator modules $\Bbbk[S]$:

- find a "maximally persistent" element $x \in M$
- L = lifetime of $x \Rightarrow$ lifetime submodule $\Bbbk[L] \subseteq M$
- replace M with $M/\Bbbk[L]$
- iterate: view M as "stack of lifetimes"

Motivation. What could "top 100 bar lengths" mean in multipersistence?

Input. Q-module M for arbitrary poset Q

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Prop. M tame $\Rightarrow$ M has finite lifetime filtration.

Thm [M.–Zhang 2024]. Lifetime filtrations $\rightsquigarrow$ lifetime-bottleneck distance $\mathbf{d}_L$, which is interleaving-stable: tame M and N admit lifetime filtrations verifying

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Tameness

Def [M.– 2017]. A module M over an arbitrary poset Q admits a constant subdivision if Q is partitioned into

- constant regions A, each with vector space $M_A \xrightarrow{\sim} M_{\mathbf{a}}$ for all $\mathbf{a} \in A$, having
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M is tame if it admits a finite constant subdivision and $\dim_{\Bbbk} M_q < \infty$ for all q.

Example. $\Bbbk_{\mathbf{0}} \oplus \Bbbk[\mathbb{R}^2]$ admits constant regions $\{\mathbf{0}\}$ and $\mathbb{R}^2 \smallsetminus \{\mathbf{0}\}$

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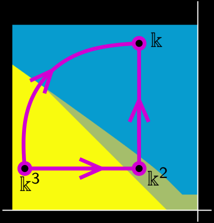

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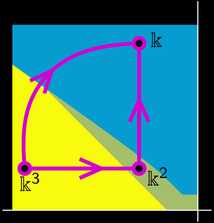

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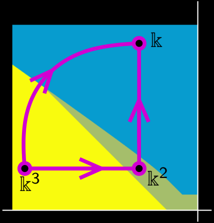

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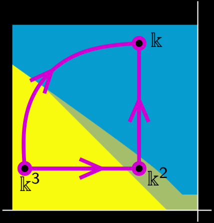

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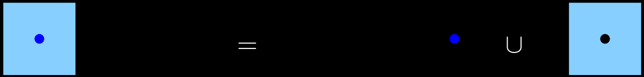

Encoding persistence modules

Def. Q-module M has finite encoding $\pi : Q \to P$ if

- P is a finite poset,
- π is a poset morphism, and
- $M \cong \pi^* N = \bigoplus_{q \in Q} N_{\pi(q)}$, the pullback of some P-module N.

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Cor. Much can be done over finite Q or $\mathbb{Z}^n$ (e.g., lifetime filtration existence)

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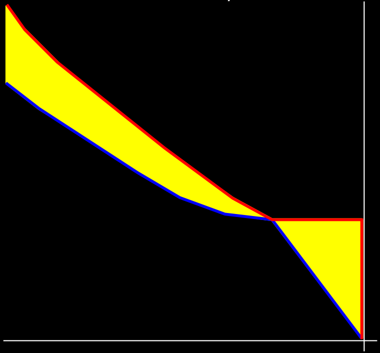

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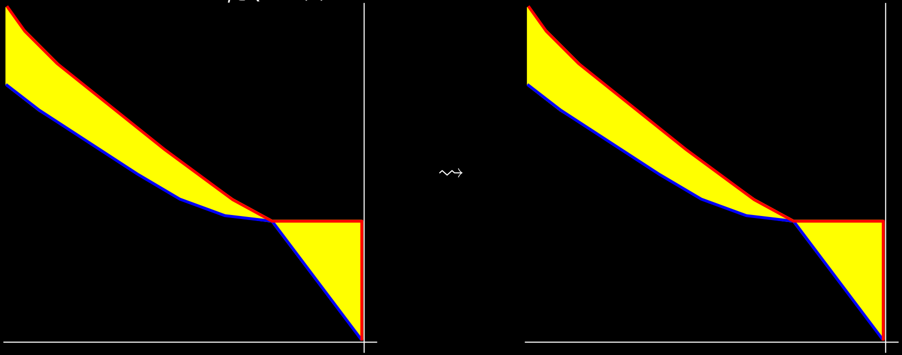

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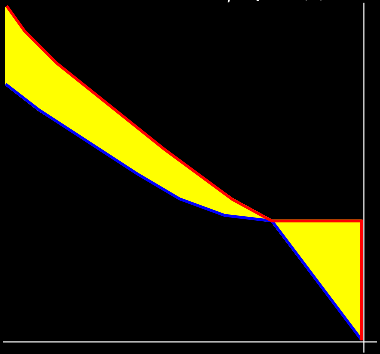

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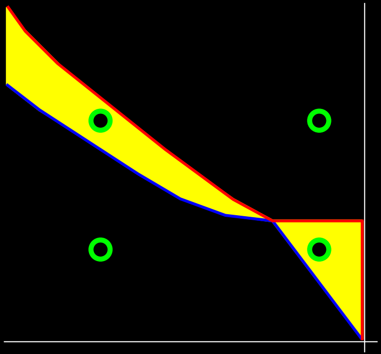

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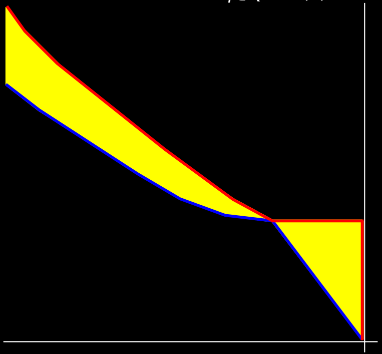

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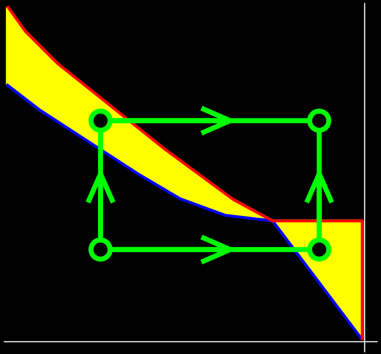

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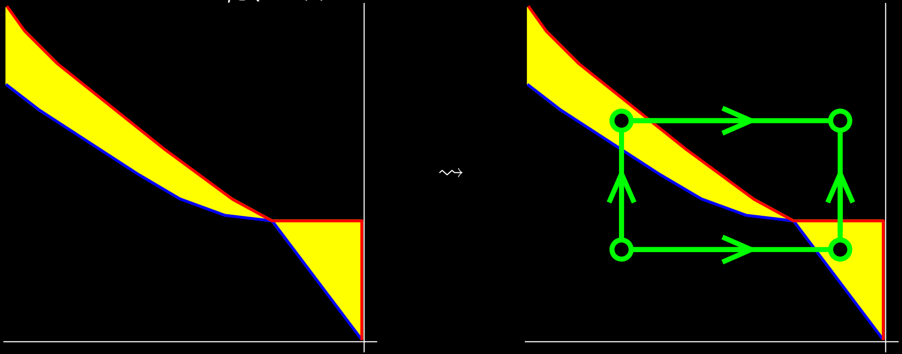

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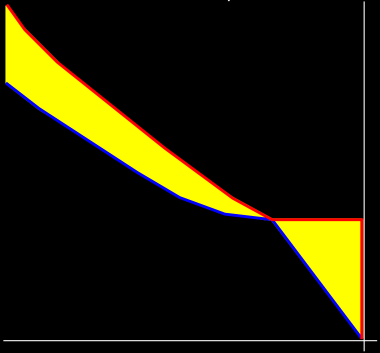

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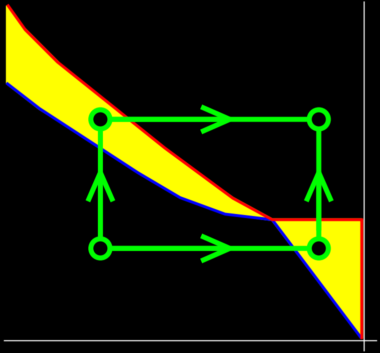

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Example. fringe presentation of a lifetime module:

In general, lifetime $=$ antichain of intervals

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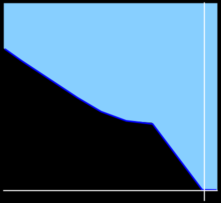

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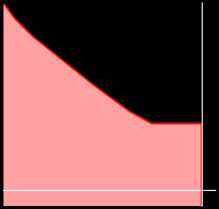

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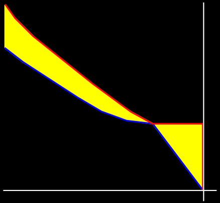

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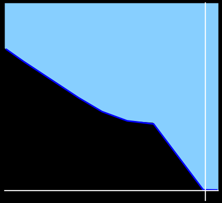

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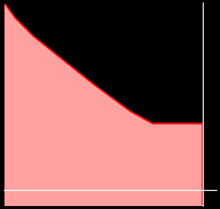

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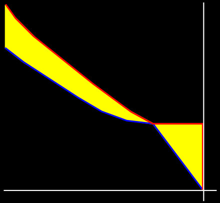

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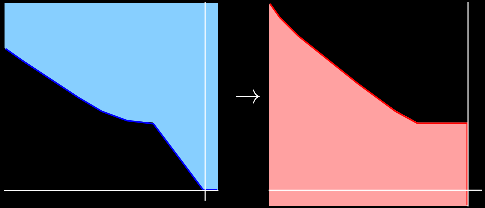

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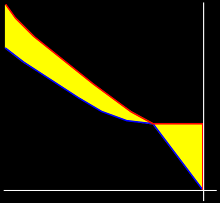

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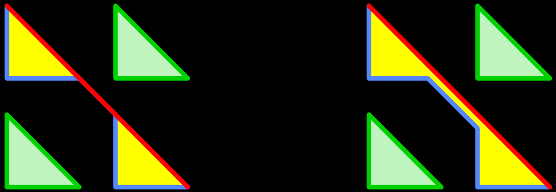

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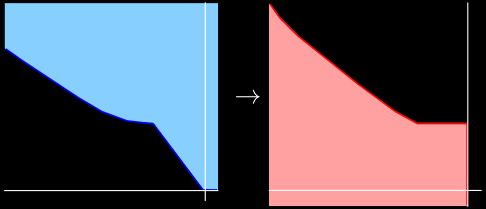

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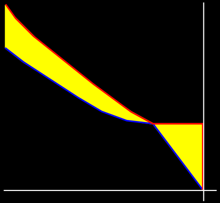

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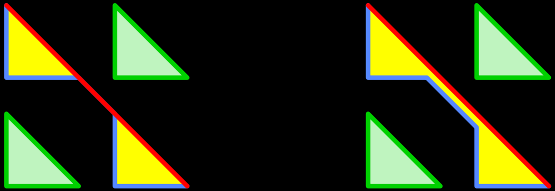

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Distances

Def [Lesnick 2015]. An ε-interleaving between $\mathbb{R}^n$-modules M and N is a pair of homomorphisms $f : M \to N(\varepsilon)$ and $g : N \to M(\varepsilon)$ whose composites induce
$g(\varepsilon) \circ f : M_q \to M_{q+2\varepsilon}$ and
$f(\varepsilon) \circ g : N_q \to N_{q+2\varepsilon}$, natural maps $\forall q \in \mathbb{R}^n$. The interleaving distance is

$$\mathbf{d}_I(M, N) = \inf\{\varepsilon \mid M \text{ and } N \text{ are } \varepsilon\text{-interleaved}\}$$

Def. Direct sums $\bigoplus_{\alpha \in A} M_\alpha$ and $\bigoplus_{\alpha \in A} N_\alpha$ are ε-matched if M_α and N_α are ε-interleaved for all $\alpha \in A$.

Def. Let $\mathcal{D}$: Q-mods $\to$ families of finitely decomposed Q-modules with ordered summands, so each element of $\mathcal{D}(M)$ is a direct sum $K = K_1 \oplus \cdots \oplus K_\ell$. The bottleneck distance determined by $\mathcal{D}$ is

$$\mathbf{d}_{\mathcal{D}}(M, N) = \inf\{\varepsilon \mid K \text{ and } L \text{ are } \varepsilon\text{-matched with } K \in \mathcal{D}(M) \text{ and } L \in \mathcal{D}(N)\}.$$

Note. Summands can be 0, and $L \in \mathcal{D}(M) \not\Rightarrow L \cong M$.

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Stability of lifetime distance

Prop. If M is a tame Q-module then M admits a finite lifetime filtration.

Proof sketch.

1. Reduce to $Q = \mathbb{Z}^n$ by finite encoding.
2. Use primary decomposition.

Lemma. Any homomorphism $f : M \to N$ takes any lifetime filtration of M, say $F_\bullet M : M_\ell \supseteq \cdots \supseteq M_0$, to a lifetime filtration $F_\bullet N$, where $N_i = F_i N = f(M_i)$.

Proof sketch. Apply Lemma on images and subquotients of lifetime modules.

Thm [M.–Zhang 2024]. Fix tame $\mathbb{R}^n$-modules M and N plus a lifetime filtration of M. Any ε-interleaving morphisms $f : M \to N(\varepsilon)$ and $g : N \to M(\varepsilon)$ induce a lifetime filtration of N such that $\mathrm{gr} M$ and $\mathrm{gr} N$ are ε-matched.

Proof sketch. The filtration of N is the shift up by ε of the image in $N(\varepsilon)$, under the interleaving map $f : M \to N(\varepsilon)$, of the filtration of M. That is,

$$F_\bullet N = f(F_\bullet M)(-\varepsilon).$$

Cor. $\mathbf{d}_L(M, N) \leq \mathbf{d}_I(M, N)$.

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Example: erosion neighborhoods

Moral. Filtrations play well with interleaving because they push and pull.

Idea [Bjerkevik 2023]. If you insist on decomposing, then allow perturbation first. How to perturb? Filter! More precisely, take a big filtered piece by pruning:

- $M \supseteq M^{\varepsilon} \supseteq M_{\varepsilon} \supseteq 0$
- with small successive quotients $M_{\varepsilon}/0$ and M/M^{ε}
- take subquotients before decomposing and comparing

Rephrase.

- Two modules are close if they have big filtered pieces that are close.
- Big filtered pieces can decompose more readily than the given modules.

Compare.

1. one parameter: ignore short bars by summing only the bigger ones
2. multiple parameters: ignore "short bars" by filtering them away

Crucially, filtration also dissolves threads that bind indecomposables into clumps.

Thm [Bjerkevik 2023]. Pruning of M is stable, in the sense that it detects all indecomposables appearing in modules within ε of M.

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Looking forward

Question. What could "top 100 bar lengths" mean in multipersistence?

- Locate "maximally persistent" elements
- $\Rightarrow$ Conj: $\mathbf{d}_L \geq c\mathbf{d}_I$ for some constant c, independent of # parameters
- What is meant by "maximally persistent"?
  - length, width, area, volume
  - "size" is crucial when parameters have incomparable scientific meanings
  - primary distances: separate classes according to birth and death types
  - note: primary decomposition is really another filtration!

Compare Bjerkevik's pruning distance stability/Lipschitz conjecture

- Must an indecomposable possess a big individual element?
- Is every indecomposable close to interval decomposing? If not, how likely is it?
- How likely is M to break into interpretable small pieces by perturbation?

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- Locate maximally persistent elements algorithmically
- Certify lower bounds for approximating $\mathbf{d}_L$

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- Locate maximally persistent elements algorithmically
- Certify lower bounds for approximating $\mathbf{d}_L$

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Thank You