Homological algebra and sheaf theory for multipersistence

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and Department of Statistical Science

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Representation Theory and Topological Data Analysis

Banff, AB

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Outline

Persistent homology over arbitrary posets

Input. Topological space X filtered by set Q of subspaces: $X_q \subseteq X$ for $q \in Q$

$\Rightarrow Q$ is a partially ordered set: $X_q \subseteq X_{q'} \Leftrightarrow q \preceq q'$

Def. $\{X_q\}_{q\in Q}$ has persistent homology $\{H_q = H(X_q; \Bbbk)\}_{q\in Q}$.

Def. Q-module over the poset Q (e.g., [M–, `arXiv:math.AT/2008.00063`]):

- Q-graded vector space $H = \bigoplus_{q\in Q} H_q$ over the field $\Bbbk$ with
- homomorphism $H_q \to H_{q'}$ whenever $q \prec q'$ in Q such that
- $H_q \to H_{q''}$ equals the composite $H_q \to H_{q'} \to H_{q''}$ whenever $q \prec q' \prec q''$

Essentially equivalent

- representation of Q [Nazarova–Roiter 1972]
- functor from Q to the category of vector spaces (e.g., [Curry 2019])
- vector-space valued sheaf on Q (e.g., [Curry's thesis 2014])
- representation of incidence algebra of Q [Doubilet–Rota–Stanley 1972]
- module over directed acyclic graph Q [Chambers–Letscher 2018]
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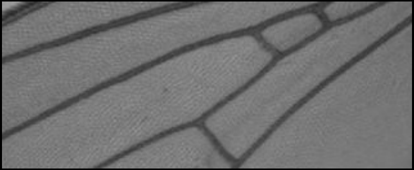
Wing vein persistence [w/Houle, et al., ongoing]

Example. Encode fruit fly wing with 2-parameter persistence

- 1st parameter: distance from vertex set
- 2nd parameter: distance from edge set



Sublevel set $W_{r,s}$ is near edges but far from vertices $\Rightarrow H_{r,s} = H_i(W_{r,s})$

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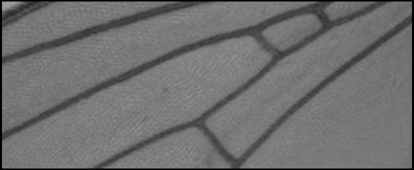

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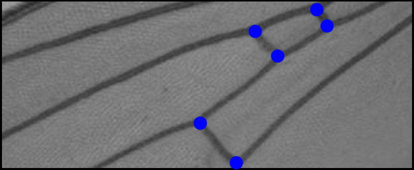

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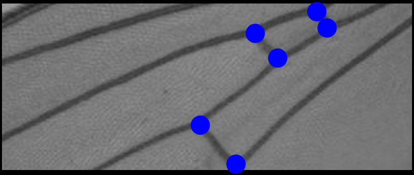

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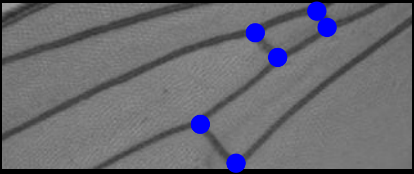

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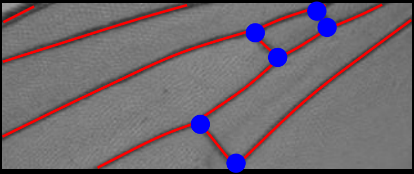

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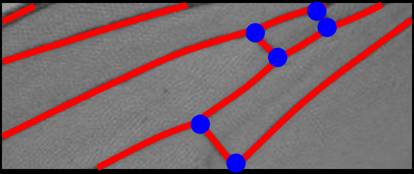

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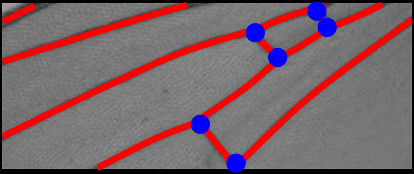

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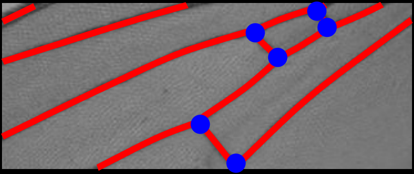

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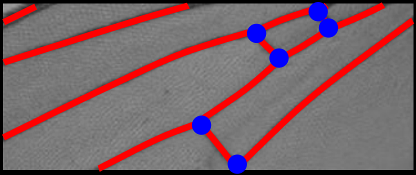

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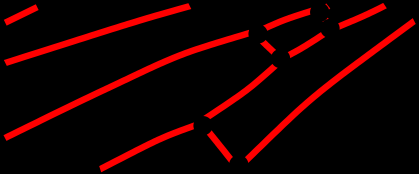

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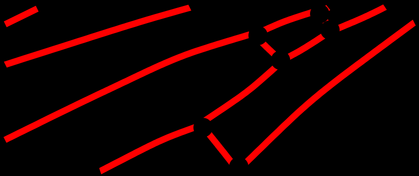

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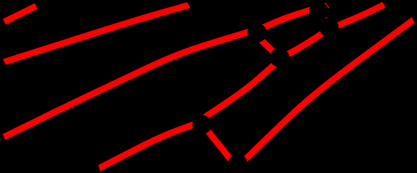

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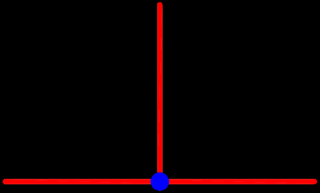

A piece of fly wing vein

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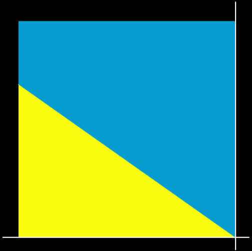

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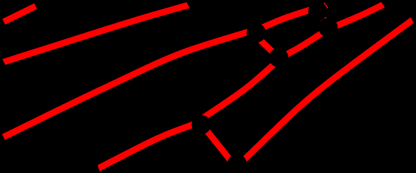

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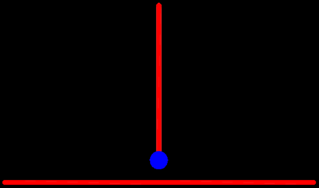

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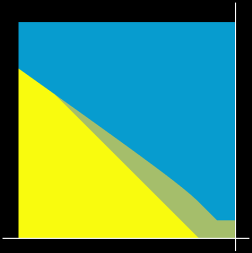

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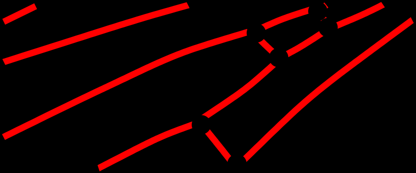

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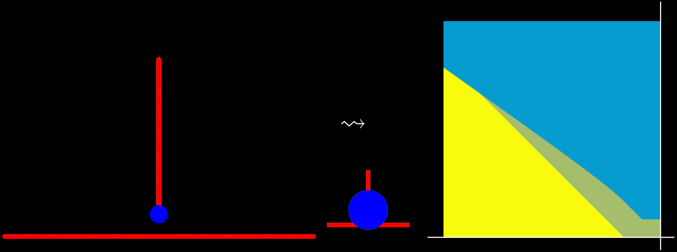

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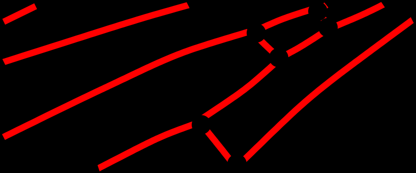

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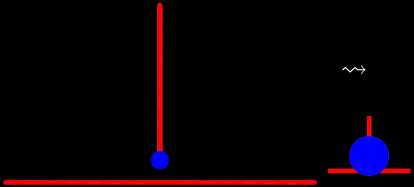

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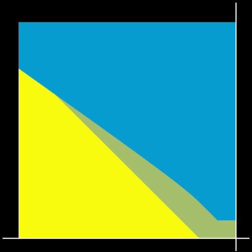

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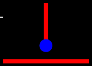

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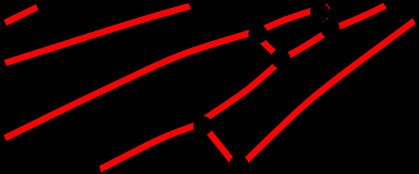

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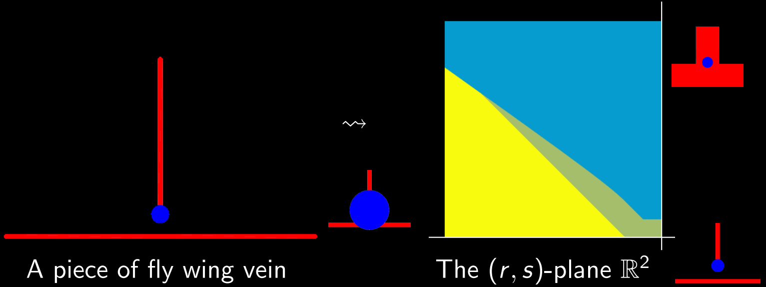

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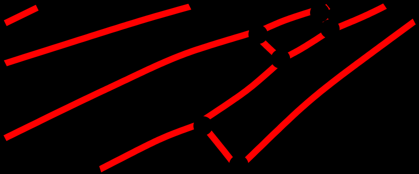

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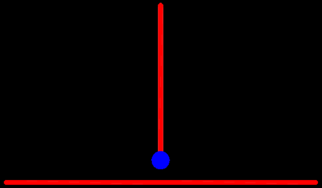

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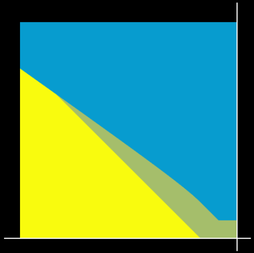

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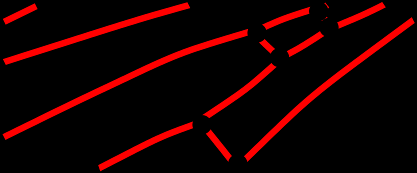

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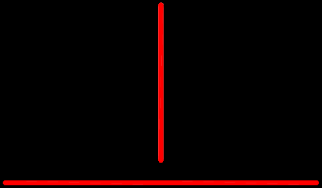

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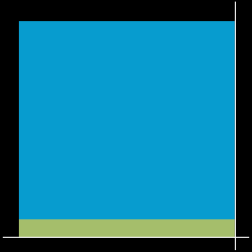

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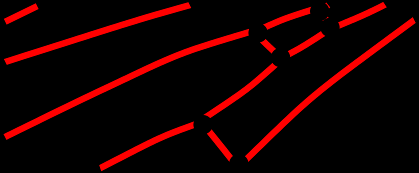

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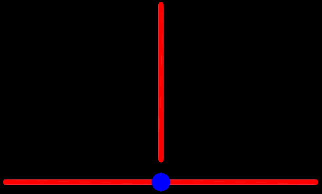

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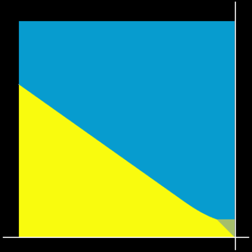

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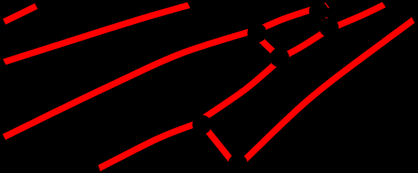
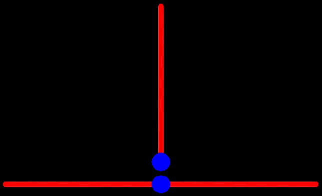
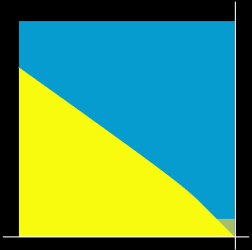


Sublevel set $W_{r,s}$ is near edges but far from vertices $\Rightarrow H_{r,s} = H_i(W_{r,s})$



A piece of fly wing vein

$\rightsquigarrow$



The (r, s)-plane $\mathbb{R}^2$

Wing vein persistence [w/Houle, et al., ongoing]

Example. Encode fruit fly wing with 2-parameter persistence

- 1st parameter: distance from vertex set (require distance $\geq -r$)
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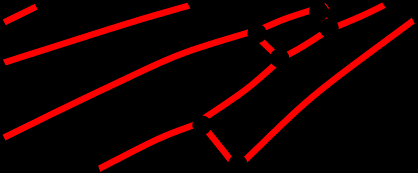

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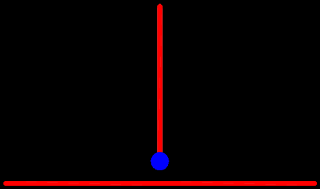

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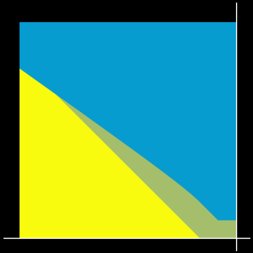

The (r, s)-plane $\mathbb{R}^2$

Persistent homology over arbitrary posets

Input. Topological space X filtered by set Q of subspaces: $X_q \subseteq X$ for $q \in Q$
$\Rightarrow Q$ is a partially ordered set: $X_q \subseteq X_{q'} \Leftrightarrow q \preceq q'$

Def. $\{X_q\}_{q \in Q}$ has persistent homology $\{H_q = H(X_q; \Bbbk)\}_{q \in Q}$. This is a

Def. Q-module over the poset Q (e.g., [M–, `arXiv:math.AT/2008.00063`]):

- Q-graded vector space $H = \bigoplus_{q \in Q} H_q$ over the field $\Bbbk$ with
- homomorphism $H_q \to H_{q'}$ whenever $q \preceq q'$ in Q such that
- $H_q \to H_{q''}$ equals the composite $H_q \to H_{q'} \to H_{q''}$ whenever $q \preceq q' \preceq q''$

Essentially equivalent

- representation of Q [Nazarova–Roiter 1972]
- functor from Q to the category of vector spaces (e.g., [Curry 2019])
- vector-space valued sheaf on Q (e.g., [Curry's thesis 2014])
- representation of incidence algebra of Q [Doubilet–Rota–Stanley 1972]
- module over directed acyclic graph Q [Chambers–Letscher 2018]
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Upsets and downsets

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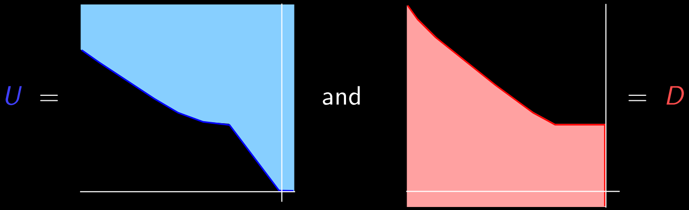

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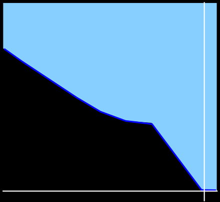
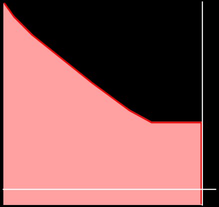


and



$= D$

- In $\mathbb{R}^3$:

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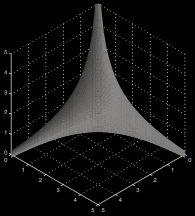
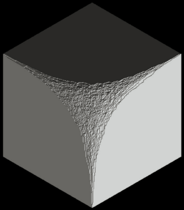


semialgebraic

or



piecewise linear

$= D$

[Andrei Okounkov, *Limit shapes, real and imagined*, Bulletin of the AMS **53** (2016), no. 2, 187–216]

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Def. subgroup $Q \subseteq$ partially ordered real vector space has positive cone

$$Q_+ = \{q \in Q \mid q \succeq \mathbf{0}\}$$

Examples

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4. (derived) sheaves with microsupport contained in negative polar cone $V_+^\vee$
5. (derived) sheaves in conic topology

Intervals in posets

Def. An interval I in a poset Q is a convex connected subset: $a, b \in I \Rightarrow$

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Examples In $\mathbb{R}^2$, intervals can look like

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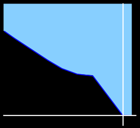

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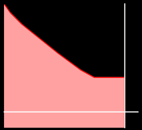

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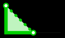

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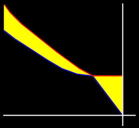

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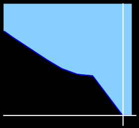

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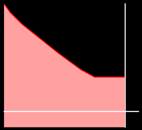

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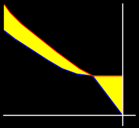

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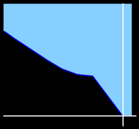

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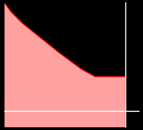

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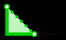

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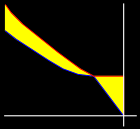

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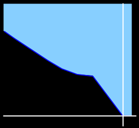

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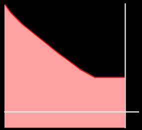

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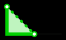

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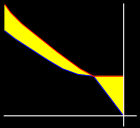

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How to define sheaves "constructed from interval sheaves" on V^{ord}...

Def. A partition of a subset of a vector space V into strata $S_\alpha \subseteq V$ is

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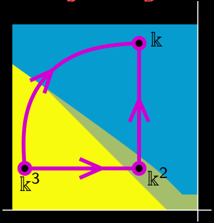

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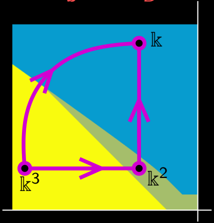

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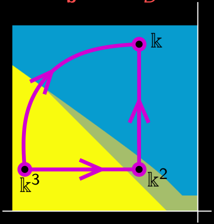

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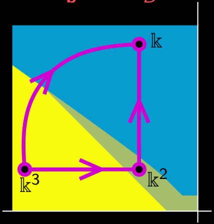

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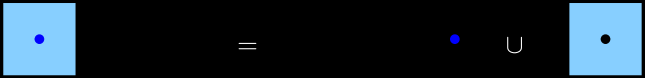

Encoding persistence modules

Def. A complex $M^\bullet$ of modules over a poset Q has finite encoding $\pi : Q \to P$ if

- P is a finite poset,
- π is a poset morphism, and
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The encoding is subanalytic or PL if its fibers are.

Thm [M– 2017, see `arXiv:math.AT/2008.00063`]. tame $\Leftrightarrow$ finitely encodable

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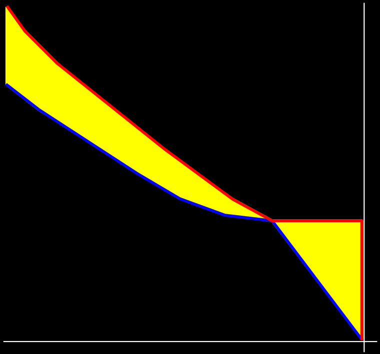

An $\mathbb{R}^2$-module

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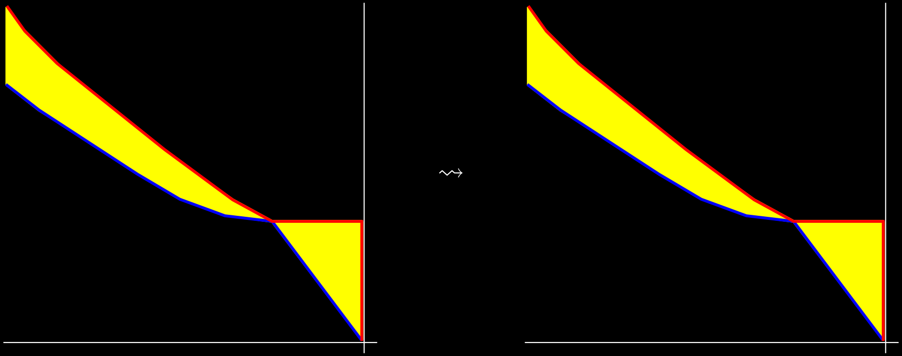

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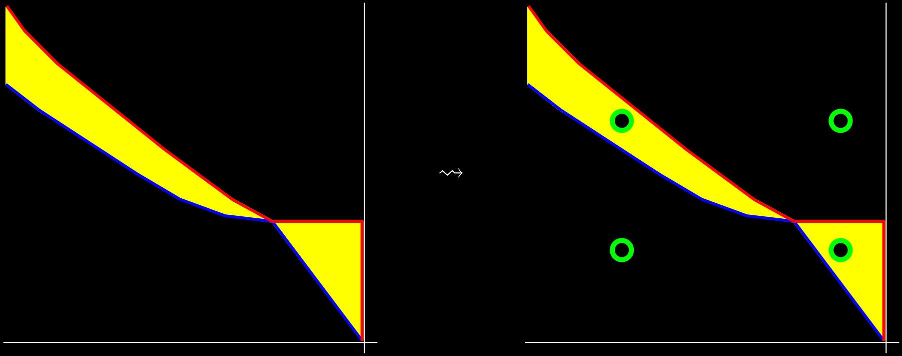

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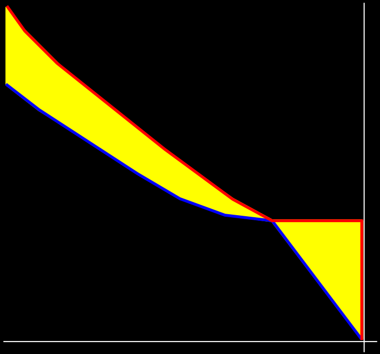

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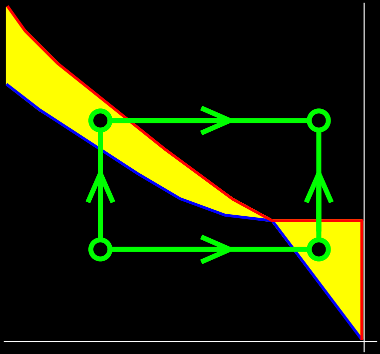

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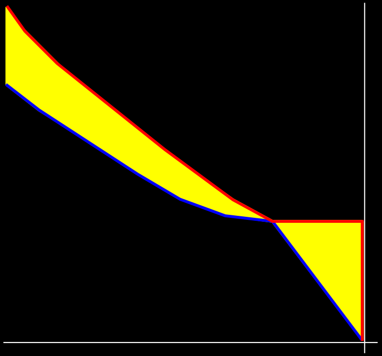

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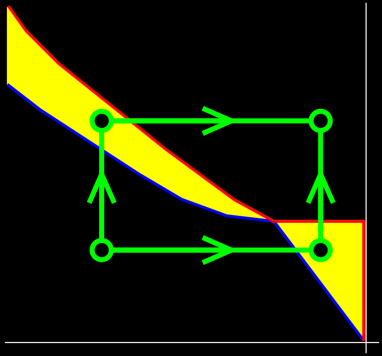

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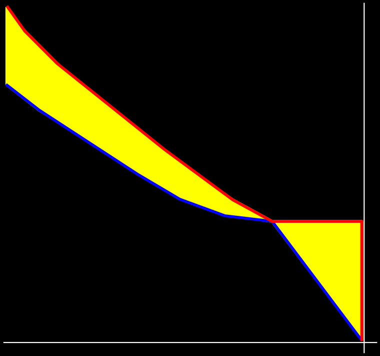

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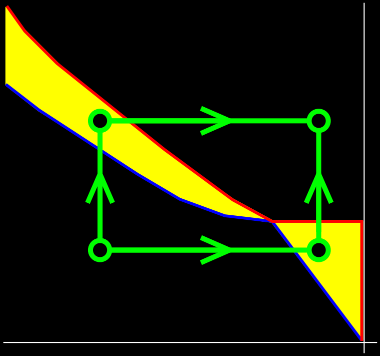

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Thm [M– 2017, see `arXiv:math.AT/2008.00063`]. tame ⇔ finitely encodable, and same for subanalytic or PL

Constructibility see [Kashiwara–Schapira 1990]

Def. Fix a real analytic manifold X and (derived) sheaf $\mathcal{F}$ of $\Bbbk$-vector spaces.

1. subanalytic triangulation of subanalytic $Y \subseteq X$ is
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3 types of stratification into intervals

- conic stratification
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- conic stratification	Relations among them?
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Can constructibility be detected without refining Q^{con} to Q^{ord}? Hint: yes!

Presentation and resolution

Default

- free presentation
- injective copresentation

Def [M– 2017, see `arXiv:math.AT/2008.00063`]. Fix a module M over an arbitrary poset Q.

- An upset presentation of M is a homomorphism
$$\Bbbk[U_1{}'] \oplus \cdots \oplus \Bbbk[U_\ell{}'] \to \Bbbk[U_1] \oplus \cdots \oplus \Bbbk[U_k]$$
with cokernel $\cong M$. Dually for downset copresentation.
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Data structure: monomial matrix

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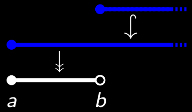

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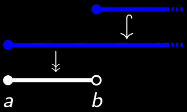

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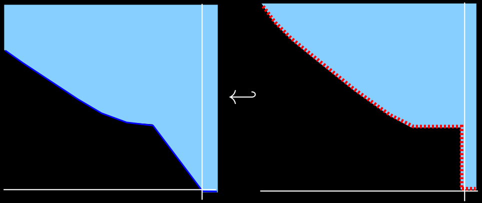

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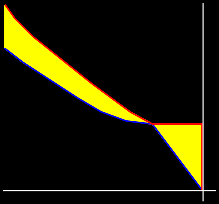

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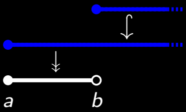

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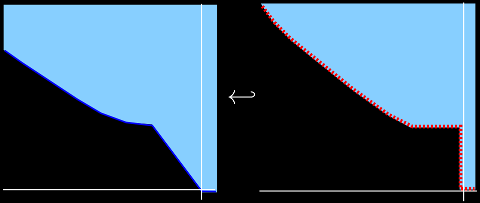

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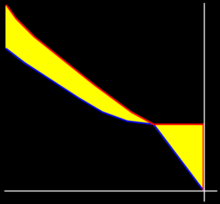

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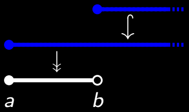

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presentation: upset

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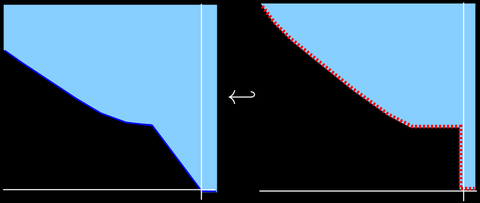
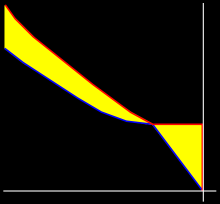


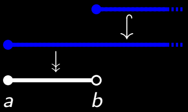
upset presentation

has cokernel



Presentation and resolution

In $\mathbb{R}$:



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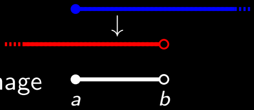
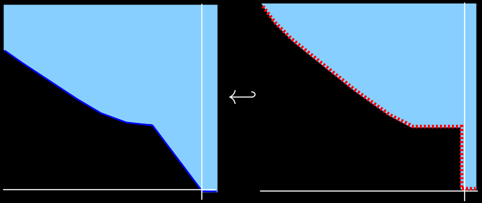


has image

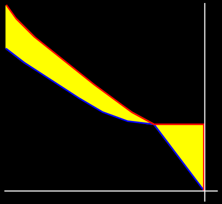
presentation: upset fringe

In $\mathbb{R}^2$:



upset presentation

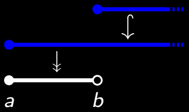
has cokernel



Presentation and resolution

In $\mathbb{R}$:

has cokernel



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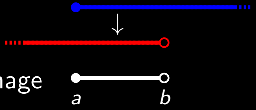

presentation: upset fringe

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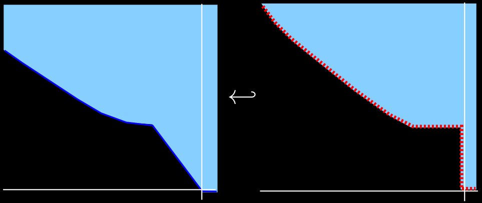
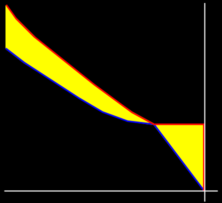
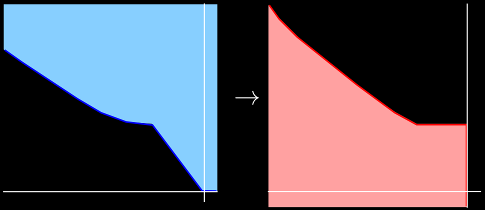


upset presentation

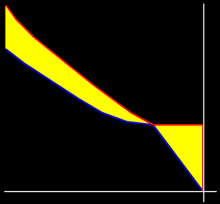
has cokernel





fringe presentation

has image



Presentation and resolution

Default

- free presentation
- injective copresentation

Notation: $\Bbbk[S] = \bigoplus_{s \in S} \Bbbk_s$ when $S \subseteq Q$

Def [M– 2017, see `arXiv:math.AT/2008.00063`]. Fix a module M over an arbitrary poset Q.

- An upset presentation of M is a homomorphism
$$\Bbbk[U_1{}'] \oplus \cdots \oplus \Bbbk[U_\ell{}'] \to \Bbbk[U_1] \oplus \cdots \oplus \Bbbk[U_k]$$
with cokernel $\cong M$. Dually for downset copresentation.
- A fringe presentation of M is a homomorphism
$$\Bbbk[U_1] \oplus \cdots \oplus \Bbbk[U_k] \to \Bbbk[D_1] \oplus \cdots \oplus \Bbbk[D_\ell]$$
with image $\cong M$.

Data structure: monomial matrix

$$\begin{array}{c|ccc} & D_1 & \cdots & D_\ell \\ \hline U_1 & \varphi_{11} & \cdots & \varphi_{1\ell} \\ \vdots & \vdots & \ddots & \vdots \\ U_k & \varphi_{k1} & \cdots & \varphi_{k\ell} \end{array}$$

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birth upsets →

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$$\text{birth upsets} \to \quad \begin{array}{c|ccc|} & D_1 & \cdots & D_\ell \\ U_1 & \varphi_{11} & \cdots & \varphi_{1\ell} \\ \vdots & \vdots & \ddots & \vdots \\ U_k & \varphi_{k1} & \cdots & \varphi_{k\ell} \end{array} \quad \leftarrow \text{death downsets}$$

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Presentation and resolution [M– 2017, see `arXiv:math.AT/2008.00063`]

Def. A homomorphism $\varphi : M \to N$ of modules over any poset Q is tame if

- M and N share a finite constant subdivision such that for each region I,
- $M_I \to M_{\mathbf{i}} \to N_{\mathbf{i}} \to N_I$ does not depend on $\mathbf{i} \in I$;

the subdivision is subordinate to φ, which is subanalytic or PL if the subdivision is.

Def. Fix a complex $M^\bullet$ of modules over a poset Q.

1. $M^\bullet$ is tame if its morphisms are tame.
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Syzygy theorem [M– 2017, see `arXiv:math.AT/2008.00063`]

Syzygy thm. A module M or bounded complex $M^\bullet$ of modules over a poset Q is tame if and only if it admits one, and hence all, of the following: a finite

1. upset resolution
2. downset resolution
3. fringe presentation
4. constant subdivision subordinate to any given one of items 1–3
5. encoding subordinate to any given one of items 1–4.

Remains true with

- "subanalytic" in place of "tame" and "finite", if $M^\bullet$ has compact support
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Any tame or subanalytic or PL morphism $M^\bullet \to N^\bullet$ lifts to a similarly well behaved morphism of resolutions as in parts 1 and 2.

Summary. tame $\Leftrightarrow$ stratified by intervals
$\Leftrightarrow$ finitely encodable
$\Leftrightarrow$ has finite resolution by intervals
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- "subanalytic" in place of "tame" and "finite", if $M^\bullet$ has compact support
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Any tame or subanalytic or PL morphism $M^\bullet \to N^\bullet$ lifts to a similarly well behaved morphism of resolutions as in parts 1 and 2.

Summary. tame $\Leftrightarrow$ stratified by intervals
$\Leftrightarrow$ finitely encodable
$\Leftrightarrow$ has finite resolution by intervals
$\Leftrightarrow$ has finite data structure by monomial matrices

Syzygy theorem [M– 2017, see `arXiv:math.AT/2008.00063`]

Syzygy thm. A module M or bounded complex $M^\bullet$ of modules over a poset Q is tame if and only if it admits one, and hence all, of the following: a finite

1. upset resolution
2. downset resolution
3. fringe presentation
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(and that's how the proof goes)

Resolving sheaves

Def (indicator sheaves). Fix a complex $\mathcal{F}^\bullet$ of sheaves on V^{ord}. A subanalytic

1. upset sheaf on V is the extension by zero of the rank 1 constant sheaf on an open subanalytic upset in V^{ord}
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Either type of indicator resolution is

- finite if finitely many summands across all homological degrees
- PL if V_+ is polyhedral and the upsets or downsets are PL

Application of syzygy thm. Fix a real vector space V partially ordered with V_+ closed, subanalytic, and of full dimension. If $\mathcal{F}^\bullet$ is

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Resolving sheaves

Cor [M– 2020, `arXiv:math.AT/2008.00091`]. The following are equivalent for any bounded, compactly supported derived sheaf $\mathcal{F}^\bullet$ on the conical topology V^{con}.

1. $\mathcal{F}^\bullet$ is subanalytically constructible after pulling back to V^{ord}
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Implications 2 $\Rightarrow$1 and 3 $\Rightarrow$1 do not require compact support for $\mathcal{F}^\bullet$.
V_+ polyhedral and $\mathcal{F}^\bullet$ PL $\Rightarrow$ claims all hold with "PL" in place of "subanalytic".

Proof. Compact support + constructible $\Rightarrow$ finite constant subdivision $\Rightarrow$ tame.

Cor 1 [Kashiwara–Schapira 2017, Conj. 3.17].
$\mathcal{F}^\bullet$ constructible $\Rightarrow$ $\operatorname{supp}\mathcal{F}$ has a subordinate conic stratification.

Cor 2 [Kashiwara–Schapira 2021, Conj. 3.20]. For polyhedral V_+ the following are equivalent.

- $\mathcal{F}^\bullet$ is a PL object in the bounded derived category of compactly supported constructible conic sheaves.
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$\mathcal{F}^\bullet$ constructible $\Rightarrow$ $\operatorname{supp} \mathcal{F}$ has a subordinate conic stratification.

Cor 2 [Kashiwara–Schapira 2021, Conj. 3.20]. For polyhedral V_+ the following are equivalent.

- $\mathcal{F}^\bullet$ is a PL object in the bounded derived category of compactly supported constructible conic sheaves.
- The isomorphism class of $\mathcal{F}^\bullet$ is represented by a complex that is a finite direct sum of constant sheaves on bounded V^{con}-locally closed polyhedra

At issue. strong PL version of $D^b(\text{constructible}) \xrightarrow{\sim}$ constructible D^b [Kashiwara 1984]

Resolving sheaves

Cor [M– 2020, `arXiv:math.AT/2008.00091`]. The following are equivalent for any bounded, compactly supported derived sheaf $\mathcal{F}^\bullet$ on the conical topology V^{con}.

1. $\mathcal{F}^\bullet$ is subanalytically constructible after pulling back to V^{ord}
2. $\mathcal{F}^\bullet$ has a finite subanalytic upset resolution
3. $\mathcal{F}^\bullet$ has a finite subanalytic downset resolution

$\left.\right\}$ indicator resolutions

Implications 2 ⇒1 and 3 ⇒1 do not require compact support for $\mathcal{F}^\bullet$.
V_+ polyhedral and $\mathcal{F}^\bullet$ PL ⇒ claims all hold with "PL" in place of "subanalytic".

Proof. Compact support + constructible ⇒ finite constant subdivision ⇒ tame.

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Looking forward

Implementation

- single preprocessing step for many multiPH computations; e.g., fly wings
- Lebesgue distance computations: no sampling for Riemann integration

Invariants

- E.g., what could "top 100 bar lengths" mean in multipersistence?
- E.g., boundaries of up- or downsets $\rightsquigarrow$ "highly persistent" elements

Real L^p distances [Bubenik–Scott–Stanley], [Skraba–Turner], [Bjerkevik–Lesnick]

- integer parameters: match pairs of generators
- real parameters: sums $\to \infty$ with finer discrete approximation
- instead: use L^p distances between boundaries of up- and downsets. . .
- . . . from corresponding associated primes (same history or mortality type)

Relative homological algebra

- resolve using upsets and/or downsets
- Conj: $\mathbb{R}^n$-modules have upset resolutions of length at most $n-1$.
- Compare [Geist–M– 2023]: $\mathbb{k}[\mathbb{R}^n_+]$ has global dimension $n+1$.

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