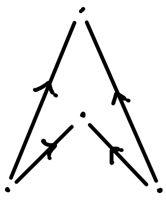
1. II. Finiteness conditions

$\boxed{\text{Fix field } \Bbbk}$

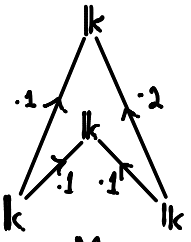
Def: If Q is a poset, a Q-module is

- $\{M_q\}_{q\in Q}$ or $\bigoplus_{q\in Q} M_q$ family of vector spaces
- $M_q \to M_{q'}$ for $q \prec q'$ linear maps
- $M_q \to M_{q''}$ is $M_q \to M_{q'} \to M_{q''}$ for any $q \prec q' \prec q''$ commutative

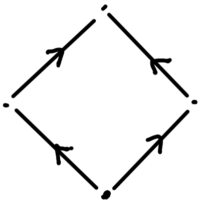
E.g. ①



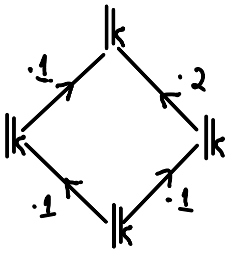
Q



M



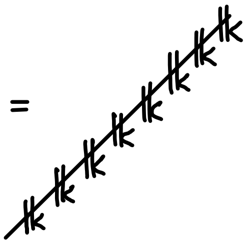
Q

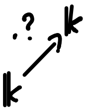


Q-module? no: quiver rep...

② $Q = \mathbb{R}^2$

M =

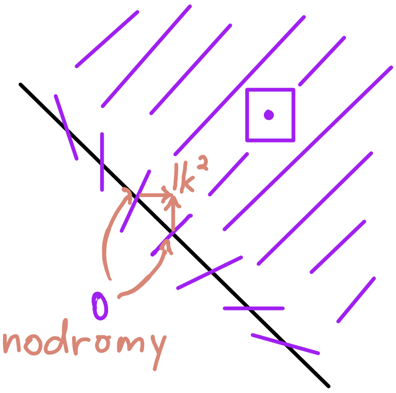




$? = 0$ forced

③ $Q = \mathbb{R}^2$

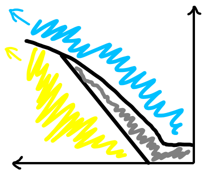
M =



no monodromy

Earlier today: M rarely noetherian (or even finitely generated)

Even over $\mathbb{Z}^n$: discretized



not finitely presented!

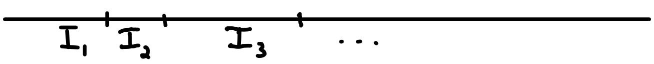
now: finiteness hypothesis?

goal: don't disallow; encompass!

Def: $\{M_\alpha\}_{\alpha\in\mathbb{R}}$ tame if

- $\dim_{\Bbbk} M_\alpha < \infty \quad \forall \alpha \in \mathbb{R}$
- $\mathbb{R} = I_1 \cup \cdots \cup I_m$ with $\{M_\alpha\}_{\alpha\in I_j}$ constant: $M_\alpha \xrightarrow{\sim} M_\beta$ for $\alpha \leq \beta$ in I_j.

$\Bbbk^{d_1} \xrightarrow{\sim} \Bbbk^{d_2} \xrightarrow{\sim} \Bbbk^{d_3} \xrightarrow{\sim} \cdots$



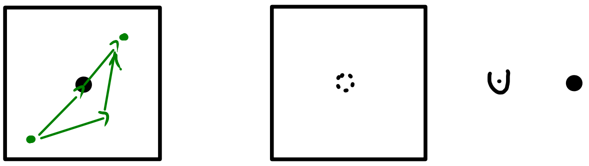
other posets?

Def: Q-module M is *tame* if $\dim_{\Bbbk} M_q < \infty$ $\forall q \in Q$ and M admits a finite *constant subdivision*: partition of Q into

- *constant regions* A, each with vector space $M_A \xrightarrow{\sim} M_a$ $\forall a \in A$ having
- *no monodromy*: all $a \preceq b$ with $a \in A$ and $b \in B$ induce same $M_A \xrightarrow{\sim} M_a \to M_b \xrightarrow{\sim} M_B$.

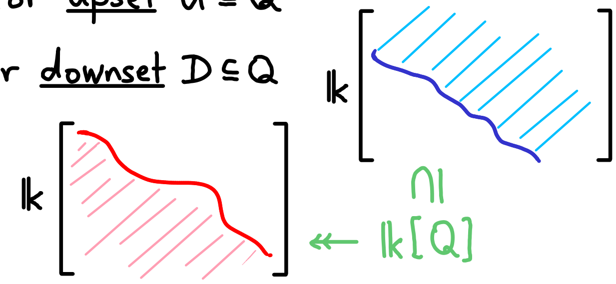
E.g. $Q = \mathbb{R}^2$

$M = \Bbbk_0 \oplus \Bbbk[\mathbb{R}^2]$ has constant subdivision

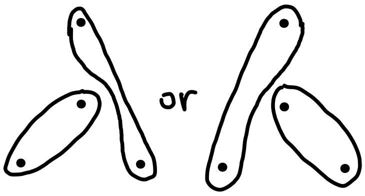


E.g. any Q $M = \Bbbk[U]$ for *upset* $U \subseteq Q$

or $\Bbbk[D]$ for *downset* $D \subseteq Q$



E.g. ① isn't a single constant region, but any partition refining

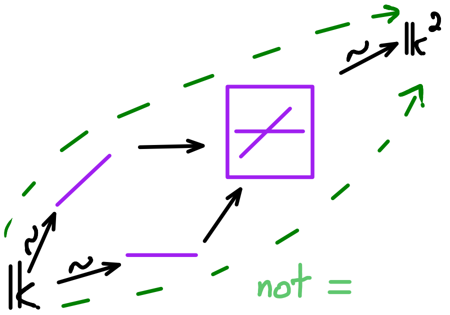


is a constant subdivision *subordinate* to M.

② isn't tame:

$$\begin{tikzcd}
& \Bbbk_{a'} \arrow[r, "\sim"] & \Bbbk_A \\
\Bbbk_A \arrow[r, "\sim"] \arrow[urr, bend right, "="'] & \Bbbk_a \arrow[u, "0"'] &
\end{tikzcd}$$

③ isn't tame: ⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯→



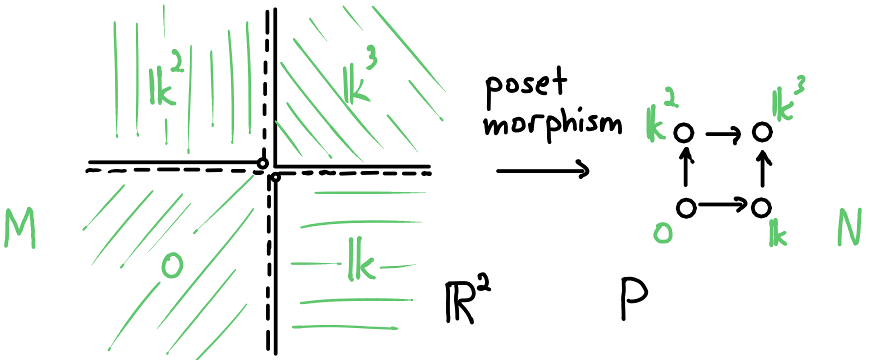
Problem: constant regions need not be partially ordered **E.g.** $Q = \mathbb{R}^2$

$M = \Bbbk_0 \oplus \Bbbk[\mathbb{R}^2]$

Def: Q-module M has *finite encoding* $\pi : Q \to P$ if

- π is a poset morphism: $q \preceq q' \Rightarrow \pi(q) \preceq \pi(q')$
- $M \cong \pi^* N = \{N_{\pi(q)}\}_{q \in Q}$ the *pullback* of some P-module N *along π*
- P is finite and $\dim_{\Bbbk} N < \infty$

E.g.



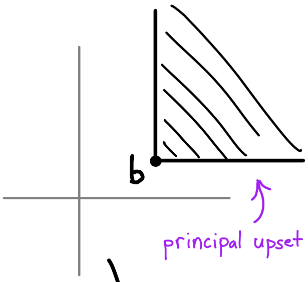
Thm: tame $\Leftrightarrow$ finitely encodable. Pf: more powerful; coming up!

Exercise: Does this poset morphism encode $\Bbbk_0 \oplus \Bbbk[\mathbb{R}^2]$?

2. III. Presentation and resolution

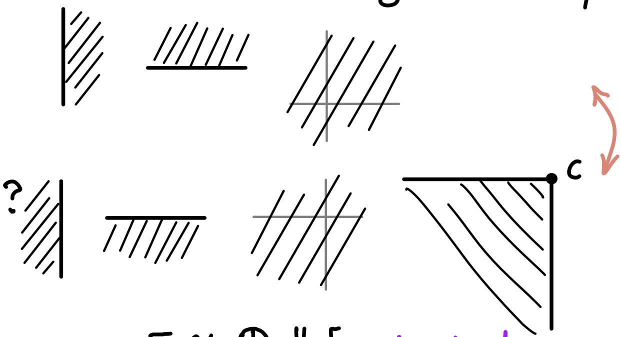
$Q = \mathbb{R}^n$ or $\mathbb{Z}^n$: What is a Q-graded free module?

- free over what? $\Bbbk[Q_+]$
- $\Leftrightarrow$ has Q-graded basis B
- $b \in B \Rightarrow \langle b \rangle$ = submodule generated by $b \cong \langle x^{\deg b} \rangle$



flat?

injective?



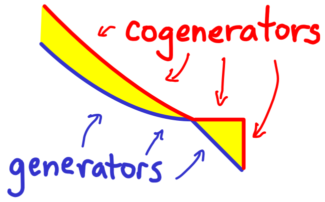
Matlis duality

c = cogenerator

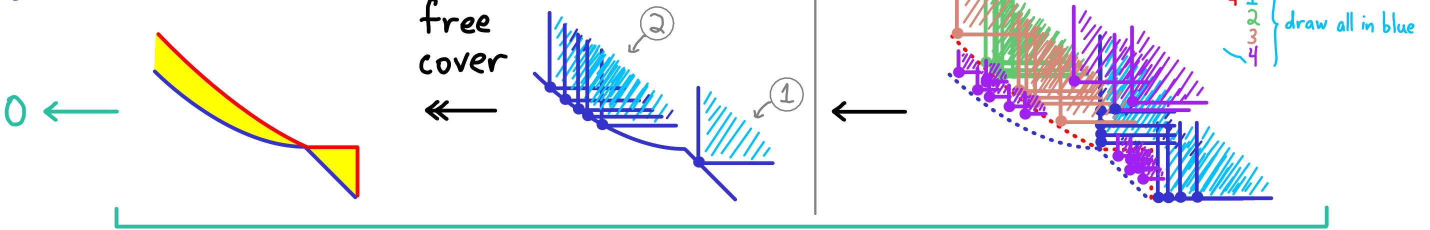
F free $\Leftrightarrow F \cong \bigoplus \Bbbk[$principal upset$]$

Def: $Q = \mathbb{R}^n$ or $\mathbb{Z}^n$: free cover is $F \twoheadrightarrow M$ with F free.

E.g.



$$M_q = \begin{cases} \Bbbk & \text{if colored} \\ 0 & \text{if not} \end{cases}$$



free presentation

Def: Q arbitrary: ________ presentation of M is a morphism $F_0 \xleftarrow{\varphi} F_1$ with

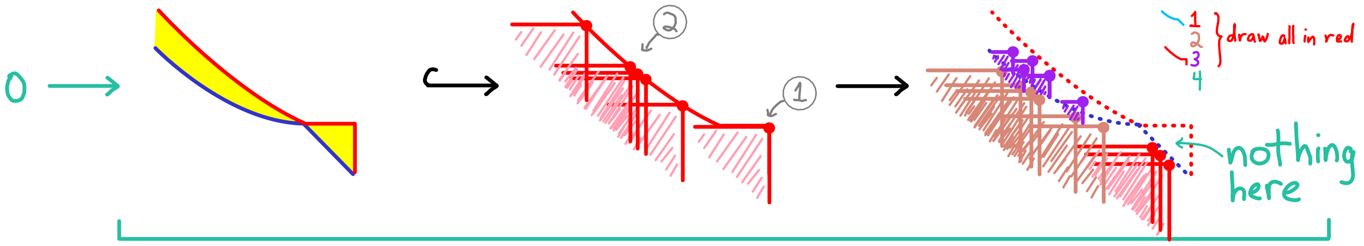
- $M \twoheadleftarrow F_0$ and $\ker \overset{\varphi}{\twoheadleftarrow} F_1$ equivalently: • $M \cong \operatorname{coker} \varphi = F_0/\operatorname{im}\varphi$
- all F_i are ________ • $0 \leftarrow M \leftarrow F_0 \leftarrow F_1$ exact

now go to free presentation

Def: Q arbitrary: ________ copresentation of M is a morphism $E^0 \xrightarrow{\varphi} E^1$ with

- $M \hookrightarrow E^0$ and $M \cong \ker\varphi$ equivalently: • $0 \to M \to E^0 \to E^1$ exact
- all E^i are ________

E.g.



injective copresentation

Why stop at index 1?

Def: A ____ cohomological resolution of M is an exact sequence

$F_0 \xleftarrow{\varphi_1} F_1 \leftarrow \cdots \leftarrow F_{i-1} \xleftarrow{\varphi_i} F_i \leftarrow \cdots$ with $M = \operatorname{coker} \varphi_1$ and all F_i are ____

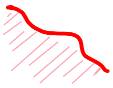
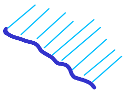
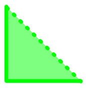
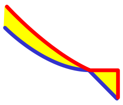
$E^0 \xrightarrow{\varphi^0} E^1 \to \cdots \to E^i \xrightarrow{\varphi^i} E^{i+1} \to \cdots$ $M = \ker \varphi^0$ and all E^i are ____

All well and good for $Q = \mathbb{R}^n$ or $\mathbb{Z}^n$. What about arbitrary poset Q?

Observe:
- free, flat ⇝ upsets   more generally, over arbitrary Q: projective
- injective ⇝ downsets
- (co)generators along $\underset{\text{hypersurfaces}}{\text{curves}}$ ⇒ very infinite data structures
  useless predictable syzygies

Suggestion: gather generators into finitely many upsets!
co down

Def: Fix any Q.
- An <u>upset module</u> is $\bigoplus_\alpha \Bbbk[U_\alpha]$ with $U_\alpha \subseteq Q$ upset $\forall \alpha$
- A <u>downset module</u> is $\bigoplus_\beta \Bbbk[D_\beta]$ with $D_\beta \subseteq Q$ downset $\forall \beta$
- An <u>interval module</u> is $\bigoplus_\gamma \Bbbk[I_\gamma]$ with $I_\gamma \subseteq Q$ <u>interval</u> $\forall \gamma$:
  - ⋆ $I = U \cap D$ for upset U and downset D
  - ⋆ I is <u>connected</u>: $a, b \in I \Rightarrow a \preceq i_1 \succeq j_1 \preceq i_2 \succeq j_2 \cdots \preceq i_m \succeq j_m \preceq b$

  <u>E.g.</u> Intervals in $\mathbb{R}^2$:  or  or  but not 

<u>E.g.</u> $\underbrace{\text{upset, downset, interval}}_{\text{indicator}}$
- covers
- presentations
- resolutions

Note: persistent homology usually in terms of birth and death
copresentation " " "cogenerators "corelations

Def: A <u>fringe presentation</u> of a Q-module M is a homomorphism

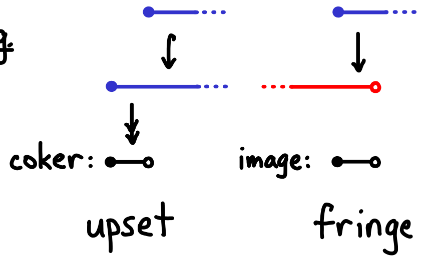
"fr-inj"

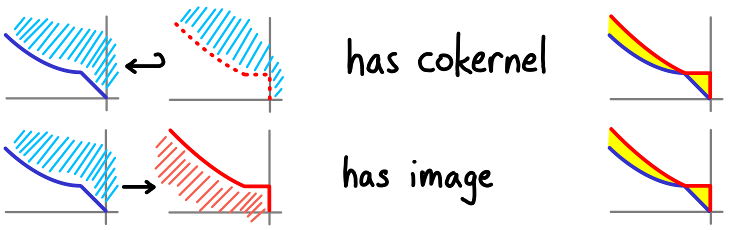
$$\bigoplus_\alpha \Bbbk[U_\alpha] = \underset{\text{upset module}}{F} \xrightarrow{\varphi} \underset{\text{downset module}}{E} = \bigoplus_\beta \Bbbk[D_\beta]$$

<u>finite</u> if $\overset{<\infty}{\bigoplus}$'s

with
- $\varphi(F) \cong M$
- $\varphi_{\alpha\beta}: \Bbbk[U_\alpha] \to \Bbbk[D_\beta]$ is connected   next lecture (don't worry: makes life easy, not hard)

<u>E.g.</u>



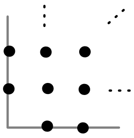
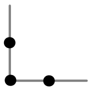


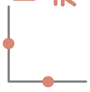
<u>Syzygy thm</u>: tame ⇔ finitely encodable ⇔ finite fringe presentation ⇔ finite $\overset{\text{indicator}}{\text{resolutions}}$

3. IV. Finite data structures

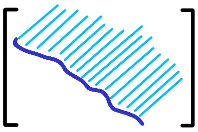
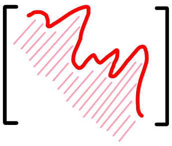
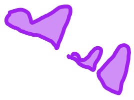
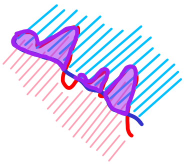
Fix upset U and downset D in poset Q

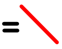
1. $\mathrm{Hom}(\Bbbk[U],\Bbbk[D]) = ?$

E.g. $Q = \mathbb{N}^2$ $\quad U = \mathbb{N}^2 \setminus \{0\} =$  $\quad D =$  $\Rightarrow \mathrm{Hom}(\Bbbk[U],\Bbbk[D]) \cong \Bbbk^2$

$\Bbbk[U] = \mathfrak{m}$ $\qquad \Bbbk[U] = \Bbbk[x,y]/\mathfrak{m}^2$ $\qquad$ Why? $U \cap D =$  antichain of size 2

General: $\mathrm{Hom} = \Bbbk^{\pi_0(U\cap D)}$.

E.g. $\mathrm{Hom}\Big(\Bbbk\Big[$  $\Big], \Bbbk\Big[$  $\Big]\Big) = \Bbbk^3$ since $U \cap D =$  $=$ 

E.g. $\mathrm{Hom}(\Bbbk[$  $], \Bbbk[$  $]) = \cancel{?}$ if  $=$  $\Bbbk^{\mathbb{R}}$ ← really 

Def: Fix intervals I and J in Q. $\varphi : \Bbbk[I] \to \Bbbk[J]$ is <u>connected</u> if

$$\varphi_q : \Bbbk[I]_q \to \Bbbk[J]_q \quad \text{is} \quad \Bbbk \xrightarrow{\cdot\lambda} \Bbbk$$

for some fixed $\lambda \in \Bbbk$. independent of $q \in Q$

2. $\mathrm{Hom}(\Bbbk[\underset{D'}{U}],\Bbbk[\underset{D}{U'}]) = \Bbbk^{\{S \in \pi_0 U \mid S \subseteq U'\}}$ for upsets U, U'
3. downsets D, D'

Cor: $Q = \mathbb{R}^n$ or $\mathbb{Z}^n \Rightarrow \mathrm{Hom}(\Bbbk[\underset{D'}{U}],\Bbbk[\underset{D}{U'}]) = \begin{cases} \Bbbk & \text{if } \underset{D}{U} \subseteq \underset{D'}{U'} \\ 0 & \text{if not} \end{cases}$

Pf: $|\pi_0 \underset{D}{U}| = 1$. $\square$

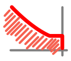
Data structure: <u>monomial matrix</u>

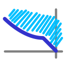
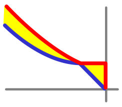
$$\begin{array}{c|ccc} & D_1 & \cdots & D_\ell \\ \hline U_1 & \varphi_{11} & \cdots & \varphi_{1\ell} \\ \vdots & \vdots & \ddots & \vdots \\ U_k & \varphi_{k1} & \cdots & \varphi_{k\ell} \end{array}$$

death downsets

scalar entries $\in \Bbbk$

birth upsets

 represent $\Bbbk[U_1] \oplus \cdots \oplus \Bbbk[U_k] \longrightarrow \Bbbk[D_1] \oplus \cdots \oplus \Bbbk[D_\ell]$

E.g.  $\begin{bmatrix} 1 \end{bmatrix}$ represents the fringe presentation of  from last time

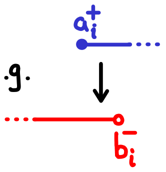
Prop: $Q = \mathbb{R}$ or $\mathbb{Z}$ $\Rightarrow$ M has tame fringe presentation

$(\Leftrightarrow M \cong \bigoplus \text{intervals})$

$$\begin{array}{c|ccc} & b_1^{\pm} & \cdots & b_k^{\pm} \\ \hline a_1^{\pm} & 1 & & \\ \vdots & & \ddots & \\ a_k^{\pm} & & & 1 \end{array}$$

where for $\mathbb{R}$,

$+$: closed and $-$: open

e.g. 

Note: monomial matrix works as well for upset → upset

or downset → downset

$\Rightarrow$ finite data structures for upset and downset presentations and resolutions

How to compute? Reduce to finite poset setting

Lemma: pullback is functorial: if

- $\pi : Q \to P$ poset morphism and
- $N \to N'$ morphism of P-modules

then get morphism $\pi^* N \to \pi^* N'$

which is, in deg q: $N_{\pi(q)} \to N'_{\pi(q)}$.

Prop: π^*(monomial matrix) = monomial matrix of same type

$$\begin{array}{c|c|} & J_\beta \\ \hline I_\alpha & \varphi_{\alpha\beta} \end{array} \mapsto \begin{array}{c|c|} & \pi^{-1}(J_\beta) \\ \hline \pi^{-1}(I_\alpha) & \varphi_{\alpha\beta} \end{array}$$

Pf: $\pi(q) \in I$ or $J \Leftrightarrow q \in \pi^{-1}(I$ or $J)$.

I $\underset{\text{down}}{\text{upset}} \Rightarrow \pi^{-1}(I)$ $\underset{\text{down}}{\text{upset}}$ by def of poset morphism. □

Syzygy thm: tame $\overset{1}{\Leftrightarrow}$ finitely encodable

$\overset{2}{\Leftrightarrow}$ finite fringe presentation

$\overset{3}{\Leftrightarrow}$ finite indicator resolutions

Pf: 1. Explicitly construct poset encoding from constant subdivision

3. Finite P has finite order dimension:

$$P \hookrightarrow \mathbb{Z}^d \quad \text{for } d = \text{⤴}.$$

P-module N is $H|_P$ for tame $\mathbb{Z}^d$-module H. not hard to construct using $\varinjlim$

H has finite free and injective resolutions by Hilbert syzygy thm.

Pull back!

2. $\underset{\text{free}}{F_\cdot} \twoheadrightarrow H \hookrightarrow \underset{\text{injective}}{E_\cdot}$ $\quad \Rightarrow F_0 \to E_0$ has image H

⇒ upset! ⇒ downset! ⇒ fringe presentation

Pull back!

Fringe presentation ⇝ constant subdivision by

common refinement of $U_1, \ldots, U_k, D_1, \ldots, D_\ell$. □

Open problem: upper bound on indicator dim: min length of upset or downset resolution

Bonus thm [Geist–M. '23]: $\operatorname{gl.dim} \Bbbk[\mathbb{R}^n_+] = n+1$.

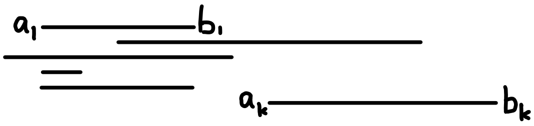
4. V. Measures of size and distance between modules

crucial for stats

assume ∘—∘ unless told otherwise
and now stop drawing fat endpoints

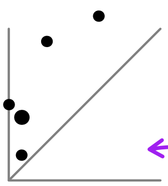
Recall: $Q = \mathbb{R}$ or $\mathbb{Z}$ $\Rightarrow$ tame $M \cong \bigoplus$ intervals

E.g.



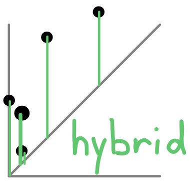
bar code: multiset of intervals

Another way to record:



persistence diagram

$\{(a,b) \mid [a,b) \in \text{bar code}\}$



note: lace array in CCA
(literally: it's Def 17.6)

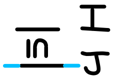
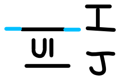
Size: based on lengths of bars

Distance: • How far apart are two bars?

• Then deal with multisets of bars.

• Surely $d(M,N) = 0$ if $M \cong N$. $d(M,N) \le \varepsilon$ if ... ?

Combinatorics • match bars of M to bars of N if they're "close":

$I \subseteq$ ε-fattening of J  and vice versa: 

• ignore remaining bars of length $\le \varepsilon$ they're within ε of 0 anyway

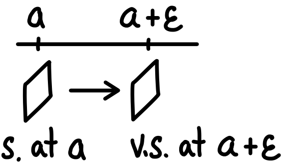
Def: bottleneck distance $d_B = \inf\{\varepsilon \mid \exists\ \varepsilon\text{-matching}\}$.

Algebra

Def: $M(\varepsilon)$ is M shifted down (left) by ε. So maybe $M \not\to N$, but $M \to N(\varepsilon)$

E.g. $M \xrightarrow[\iota_\varepsilon^M]{} M(\varepsilon)$ since $M_a \to M_{a+\varepsilon}$ and $N \not\to M$, but $N \to M(\varepsilon)$.

$$M_{a+\varepsilon} = M(\varepsilon)_a$$



Def: An ε-interleaving between M and N is

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[rr, bend right, "\iota_{2\varepsilon}^M"'] & N(\varepsilon) \arrow[r, "g(\varepsilon)"] & M(2\varepsilon) \end{tikzcd}$$

and similarly with M and N swapped:

$$\begin{tikzcd} N \arrow[r, "g"] \arrow[rr, bend right, "\iota_{2\varepsilon}^N"'] & M(\varepsilon) \arrow[r, "f(\varepsilon)"] & N(2\varepsilon) \end{tikzcd}.$$

Def: interleaving distance $d_I = \inf\{\varepsilon \mid \exists\ \varepsilon\text{-interleaving}\}$.

Lemma: Agrees with d_B on bars

Thm: " " " " modules — that is, on $\bigoplus$ bars

Arbitrary Q? How about just $\mathbb{R}^n$ or $\mathbb{Z}^n$?

Fails: • tame $M \Leftrightarrow \bigoplus$ intervals even with interval = connected $U \cap D$

• indecomposables ⟷ intervals

can be arbitrarily complicated in a precise sense

• discrete invariant ⟷ ≅ class

continuous moduli

What does still work?

- tame $M \cong \bigoplus$ indecomposables over any Q
- d_I interleaving distance
  - * $\mathbb{R}^n$ or $\mathbb{Z}^n$: shift by $\varepsilon = (\varepsilon,\ldots,\varepsilon)$
  - * more general Q: define shift (ε)
- ⇒ • d_B bottleneck distance

  d_I(indecomposables) + ε-matching

But: indecomposable decomp is

- rarely informative: indecomposables are dense and essentially open
- unstable: all modules nearby M can have radically different indecomps from M

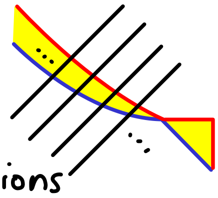
  crucially bad for data analysis

  extremely important and pervasive

  - * to use invariant of M as statistical summary
  - * need: wiggle input ⇒ wiggle summary

So what kinds of invariants / summaries / distances do we use?

- Hilbert functions
$$\begin{aligned} H_M : Q &\longrightarrow \mathbb{Z} \\ q &\longmapsto \dim_{\Bbbk} M_q \end{aligned}$$
- rank functions
$$\begin{aligned} r_M : Q \times Q &\longrightarrow \mathbb{Z} \\ (q \preceq q') &\longmapsto \operatorname{rank}(M_q \longrightarrow M_{q'}) \end{aligned}$$
- approximation by $\bigoplus$ intervals
  - * E.g. $F \twoheadrightarrow M$ compares M to $\bigoplus$ principal upset modules!
  - continuing this thread: * resolve by indicator modules
    - ↙ K-theoretic invariants
      alternating sums of intervals
    - ↘ derived categorical constructions
      constructible sheaves, …
  - * reduce to $\mathbb{R}$ or $\mathbb{Z}$ by slicing along rays



Challenges for us

- compute fringe presentations, indicator resolutions

  if (say) constant regions are semialgebraic or polyhedral
- devise effective summaries for statistical purposes
  - ↙ useful statistically
  - ↘ algorithmically computable

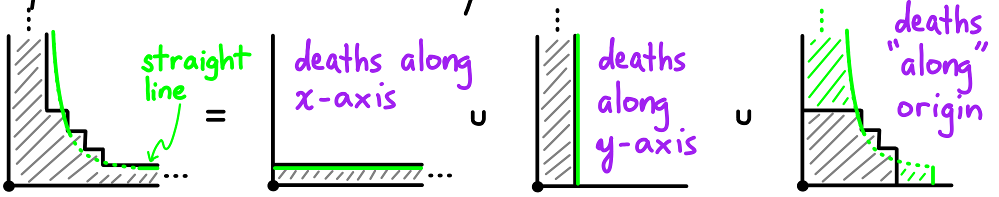
TDA people tend to be less aware of algebra literature, methods and ways of thinking

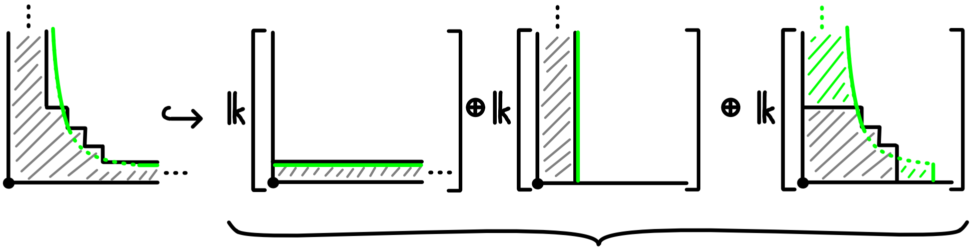
5. VI. Primary decomposition

Def: hull of M: injection $M \hookrightarrow E$ also called envelope

usually: means minimal in some way

E.g. $/\mathbb{Z}$





$/\mathbb{R}$

downset module E records deaths of various types

A rule of life: there are lots of ways to die.

Let's make this more precise. (But not completely: I won't define socles.)

Def: Fix $\tau \subseteq \{1, \ldots, n\}$,

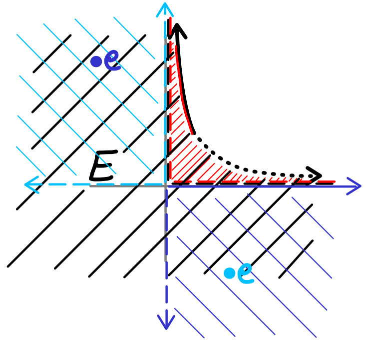
$Q = \mathbb{R}^n$ or $\mathbb{Z}^n$, and

Q-module E. An element $e \in E$ is coprimary if

τ-persistent • e lives when pushed up along any combination of τ-axes;

τ-transient • e dies when pushed sufficiently up along any other axis.

E.g.



Where are the coprimary elements?

$\tau = \emptyset$

$\tau =$ x-axis

$\tau =$ y-axis

$\tau =$ xy-plane (none!)

Def: E is τ-coprimary if every nonzero element in E_q divides a τ-coprimary element $\forall\, q \in Q$

$$\begin{tikzcd} e' \arrow[r, phantom, "\in"] & E_{q'} \\ e \arrow[u] \arrow[r, phantom, "\in"] & E_q \arrow[u] \end{tikzcd} \Rightarrow e \mid e'$$

E.g. Is E coprimary? τ-coprimary for some τ

Yes: $\tau = \emptyset$

but not $\tau =$ x-axis •e

or $\tau =$ y-axis •e

Interesting exercises: 1. E coprimary ⇒ E is τ-coprimary for unique τ

2. When $Q = \mathbb{Z}^n$, E coprimary ⇒ every element is coprimary

<u>Thm</u>: $Q = \mathbb{R}^n$ or $\mathbb{Z}^n$ and M tame

$\Rightarrow$ M has <u>primary decomposition</u> $M \hookrightarrow \bigoplus_{\tau \subseteq \{1,\dots,n\}} M_\tau$ with τ-coprimary M_τ

M is glued together (more or less as a union)

from components along coordinate subspaces

<u>Compare</u>: usual primary decomposition for modules over rings:

$M \hookrightarrow \bigoplus_{j=1}^{r} M_\tau$ where M_j $\underbrace{\text{has only one associated prime}}_{\text{coprimary}}$

Why is primary decomposition useful?

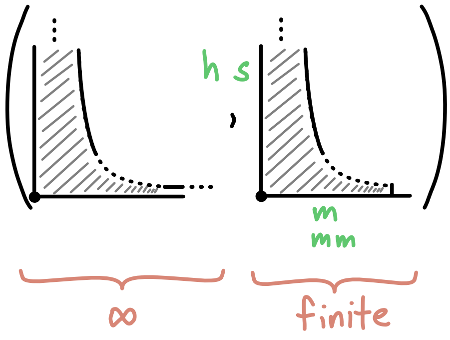
- parameters don't all mean the same thing
- not measured on same scale

$\Rightarrow$ can wreak havoc on distances between modules

$\Rightarrow$ important to tease apart persistence behaviors

$$d(M,N) = \sum_\tau w_\tau d(M_\tau, N_\tau)$$

<u>E.g.</u> d (



) = ?

<u>Challenges</u>

- compute
- represent   data structures
- apply   invent / prove statistical methods
- interpret   meaningful summary for domain scientists

<u>Connections to other areas of math</u>

see Lec I slide:

	<u>e.g.</u>
topology	sheaves)
categories	exit paths
representation theory	quivers
combinatorics	posets
homological algebra	relative
algebraic geometry	quantum noncommutative toric varieties
probability	random modules null distribution?