Tutorial 4:
Convergence to Gaussian objects

Ezra Miller

Duke University, Department of Mathematics
and Department of Statistical Science

ezra@math.duke.edu

joint with Jonathan Mattingly (Duke)
Do Tran (Deutsche Bank (was: Göttingen))

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Interactions of Statistics and Geometry (ISAG) II

National University of Singapore

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Outline

1. Random tangent fields
2. Tangential collapse
3. Stratified Gaussians
4. Escape vectors
5. Distortion
6. Central limit theorems

Random tangent fields

Def. A random tangent field on $T_{\bar{\mu}}M$ is a stochastic process $f : \Omega \times S_{\bar{\mu}}M \to \mathbb{R}$, so $f(V) : \Omega \to \mathbb{R}$ for each $V \in S_{\bar{\mu}}M$.

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Tangential collapse

Def. Localized μ on smoothly stratified M has fluctuating cone

$$C_\mu = \{X \in T_{\bar\mu}M \mid \nabla_{\bar\mu}F(X) = 0 \text{ and } X \in \text{convex cone generated by } \operatorname{supp}(\mu \circ \log_{\bar\mu}^{-1})\}$$

pushforward of μ

Lemma. Adding mass to μ can only cause $\bar\mu$ to move into C_μ

Thm [Mattingly, M–, Tran 2023]. M smoothly stratified $\Rightarrow$ some sequence of limit log maps, followed by convex projection to the relevant smooth stratum, is a tangential collapse: a continuous map $\mathcal{L} : T_{\bar\mu}M \to \mathbb{R}^m$ that is

- injective on C_μ and
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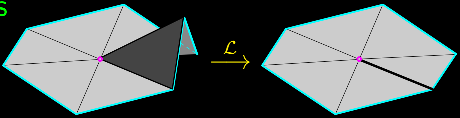
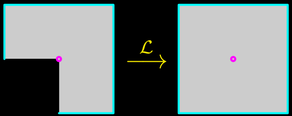
Examples

- kale: 

- nonconvex quadrants: 

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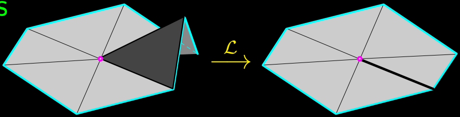
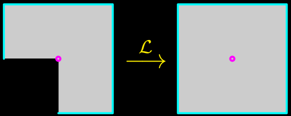
- kale: 

- nonconvex quadrants: 

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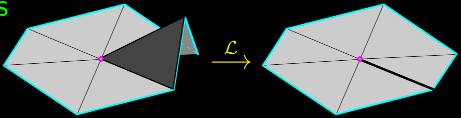

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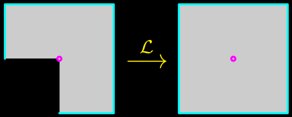
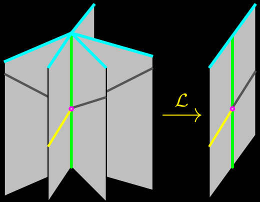


- open book:



Stratified Gaussians

Smooth M: $T_{\bar\mu}M \cong \mathbb{R}^m$ already

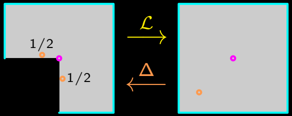
Singular M: use tangential collapse $T_{\bar\mu}M \xrightarrow{\mathcal{L}} \mathbb{R}^m$

Lemma. The map $\mathcal{L}$ has a measurable section over $\mathbb{R}^\ell = \operatorname{conv}(\operatorname{image}\mathcal{L})$,

$$\Delta : \mathbb{R}^\ell \to \text{discrete measures on } \mathbb{R}_+ \operatorname{supp}(\mu \circ \log_{\bar\mu}^{-1}) \subseteq T_{\bar\mu}M$$

with $\mathcal{L} \circ \Delta = \mathrm{id}_{\mathbb{R}^\ell}$, where $\mathcal{L}(\lambda_1\delta_{Y^1} + \cdots + \lambda_j\delta_{Y^j}) = \lambda_1\mathcal{L}(Y^1) + \cdots + \lambda_j\mathcal{L}(Y^j)$.

Example.



Def [Mattingly, M–, Tran 2023]. A Gaussian tangent mass Γ_μ is any measurable section of any $\mathbb{R}^\ell$-valued random variable $\mathcal{N} \sim N(0, \Sigma)$:

$$\Gamma_\mu = \Delta(\mathcal{N}).$$

Perspective shift: continuous variation in Gaussians can come from redistributing weights on unmoving points rather than from spatial variation

Thm [Mattingly, M–, Tran 2023]. $G(X) = \langle \Gamma_\mu, X \rangle_{\bar\mu}$ for all $X \in C_\mu$.

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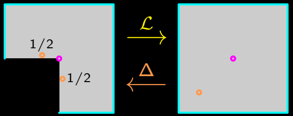

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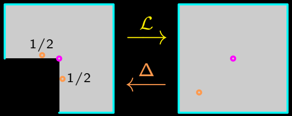
Example. 

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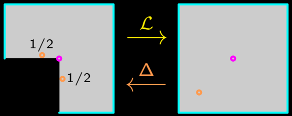

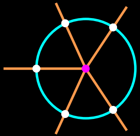

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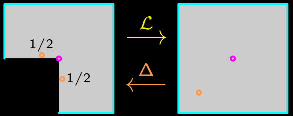

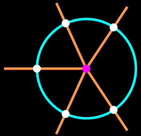

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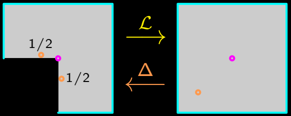

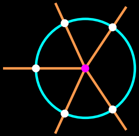

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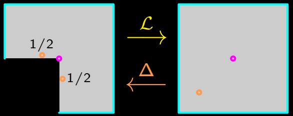

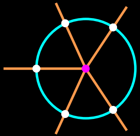

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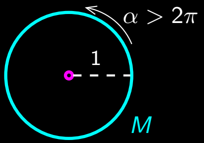

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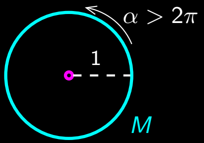
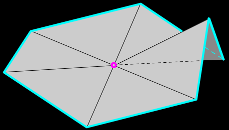


embedded in $\mathbb{R}^3$: 

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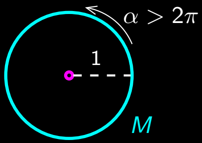

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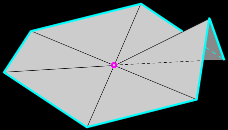
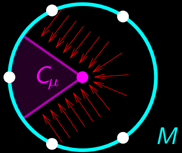
embedded in $\mathbb{R}^3$: 



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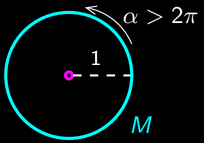

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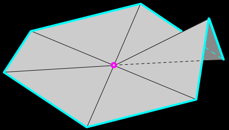

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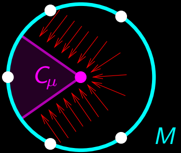

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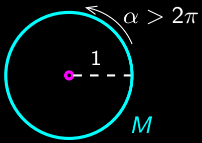

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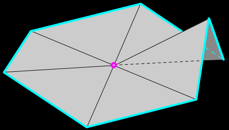

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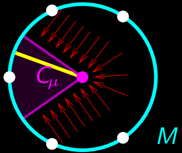

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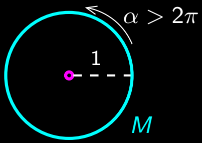
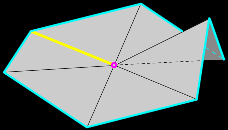
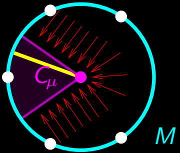


- Isolated hyperbolic planar singularity: angle sum at apex is $\alpha > 2\pi$ (that is, circumference at radius 1 is α)

embedded in $\mathbb{R}^3$:



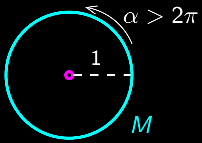


- $\mathscr{E}$ is convex projection to the fluctuating cone $C_\mu = \{V \in \text{hull}\,\mu \mid \nabla_{\bar{\mu}} F(V) = 0\}$

Escape vectors

Def. Fix $\Delta = \lambda_1\delta_{Y^1} + \cdots + \lambda_j\delta_{Y^j}$, a discrete measure on $T_{\bar\mu}M$. If $\delta = \lambda_1\delta_{y^1} + \cdots + \lambda_j\delta_{y^j}$ with $Y^i = \log_{\bar\mu} y^i$ then Δ has escape vector

$$\mathscr{E}(\Delta) = \lim_{t\to 0} \frac{1}{t}\log_{\bar\mu}(\overline{\mu + t\delta})$$



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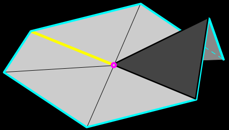

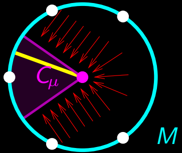

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Distortion

Perturbative CLT [Mattingly, M–, Tran 2023]. $\lim_{n\to\infty} \sqrt{n} \log_{\bar{\mu}} \bar{\mu}_n \xrightarrow{d} \mathscr{E}(\Gamma_\mu)$

Def. The distortion map is

$$\mathcal{H} = \mathscr{E} \circ \Delta : \mathbb{R}^\ell \to T_{\bar{\mu}} M$$

Prop. Distortion $\mathcal{H}$ does not depend on choice of section Δ

Geometric CLT [Mattingly, M–, Tran 2023]. $\lim_{n\to\infty} \sqrt{n} \log_{\bar{\mu}} \bar{\mu}_n \xrightarrow{d} \mathcal{H}_\sharp N(0, \Sigma)$

Cor. Smooth CLT [Bhattacharya and Patrangenaru 2003, 2005], etc., where

$$\mathcal{H} = (\nabla\nabla_{\bar{\mu}} F_\mu)^{-1}$$

is the inverse Hessian of the Fréchet function

Note. Hessian not defined in singular settings, but inverse Hessian is

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Variational CLT in a space of measures

CLT 3 [Mattingly, M–, Tran 2023]. $\lim\limits_{n\to\infty} \sqrt{n} \log_{\bar\mu} \bar\mu_n = \nabla_\mu \mathfrak{b}(\Gamma_\mu)$,

- the directional derivative, in the space $\mathcal{P}_2 M$ of L^2 measures on M,
- of the barycenter map $\mathfrak{b} : \mathcal{P}_2 M \to M$ sending $\mu \mapsto \bar\mu$
- at μ
- along any Gaussian tangent mass Γ_μ

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CLT 4 [Mattingly, M–, Tran 2023]. $\lim\limits_{n\to\infty} \sqrt{n} \log_{\bar\mu} \bar\mu_n \stackrel{d}{=} \nabla_{F_\mu \circ \exp_{\bar\mu}} \mathfrak{B}(G)$,

- the directional derivative, in the space of continuous maps $\mathcal{C}(T_{\bar\mu} M, \mathbb{R})$,
- of the minimizer map $\mathfrak{B} : \mathcal{C}(T_{\bar\mu} M, \mathbb{R}) \to T_{\bar\mu} M$ that sends $f \mapsto \operatorname*{argmin}_{X \in C_\mu} f(X)$
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Next lecture: proof via continuous mapping thm