Central limit theorems on stratified spaces

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joint with Jonathan Mattingly (Duke)
Do Tran (Deutsche Bank (was: Göttingen))

http://arxiv.org/abs/2311.09455
09454
09453
09451

Probability Seminar

Duke University

6 February 2025

Outline

Motivation and history

Mimic ordinary statistics: assume nonlinear M given; want

- averages: measure μ on $M \rightsquigarrow$ mean $\bar{\mu} \in M$
- variance, PCA
- Law of Large Numbers (LLN), confidence regions
- Central Limit Theorem (CLT)
  - smooth M [Bhattacharya and Patrangenaru 2003, 2005]
  - singular M
    - open books [SAMSI Working Group 2013]
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    - phylogenetic tree spaces [Barden, Le 2018, w/Owen 2013, 2014]
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  - stochastic analysis on manifolds e.g., [Malliavin 1978]
  - Brownian motion in manifolds e.g., [Kendall 1984], [Hsu 1988]
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Goals for today

- Gaussians on singular spaces
- $\rightsquigarrow$ stratified CLT

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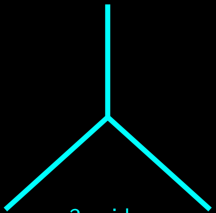

3-spider

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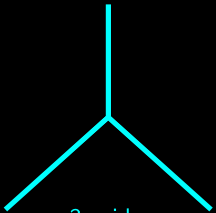

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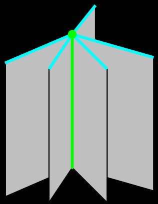

open book

$\mathbb{R}^d \times$ spider

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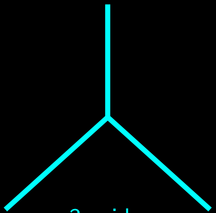

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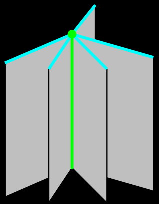

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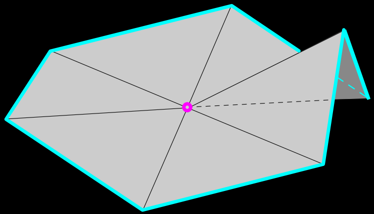

isolated hyperbolic planar singularity

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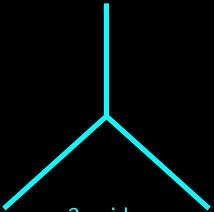

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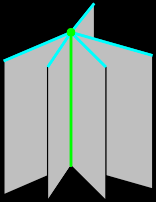

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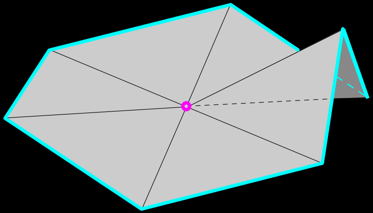

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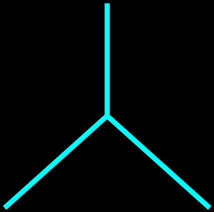
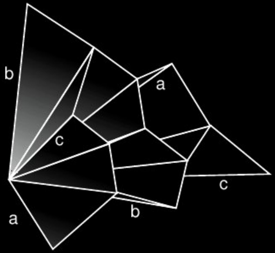


$\mathcal{T}_3$



$\mathcal{T}_4$

from [BHV 2001]

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- M is a complete, locally compact, geodesic space
- $M = \bigsqcup_{j=0}^{d} M^j$ has disjoint locally closed strata M^j
- each stratum M^j
  - is a manifold with geodesic distance $\mathbf{d}|_{M^j}$
  - has closure $\overline{M^j} = \bigcup_{k \in K} M^k$ for some subset $K \subseteq \{1, \ldots, j\}$
- locally well defined exponential maps that are local homeomorphisms
- curvature bounded above by κ: M is CAT(κ)

Examples

- graph (or network): strata are vertices and edges
- polyhedron: strata are (relatively open) faces
- real (semi)algebraic variety: strata $\leftrightarrow$ equisingular loci

Actual examples

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- tree spaces
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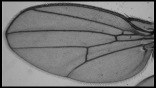

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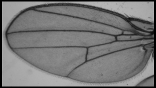

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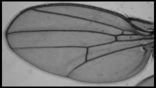

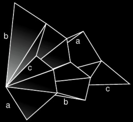

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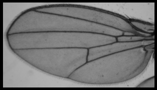

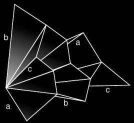

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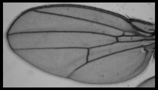

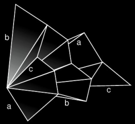

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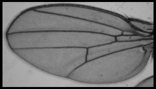

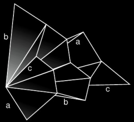

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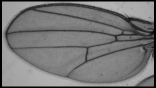

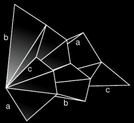

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Def. Probability distribution μ on any metric space M has Fréchet function

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and Fréchet mean $\bar{\mu} = \operatorname*{argmin}_{y\in M} F_\mu(y)$.

Prop. M is CAT(κ)
$\Rightarrow$ M has tangent spaces (cones)

Def. The logarithm map is

$$\begin{aligned}\log_{\bar{\mu}} : M &\to T_{\bar{\mu}}M\\ x &\mapsto d(\bar{\mu}, x)V,\end{aligned}$$

where V = unit tangent to geodesic from $\bar{\mu}$ to x.

Note. M singular at $\bar{\mu} \Leftrightarrow T_{\bar{\mu}}M \ncong \mathbb{R}^d$

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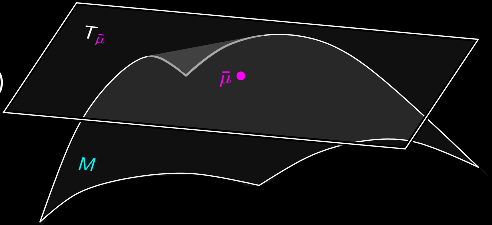

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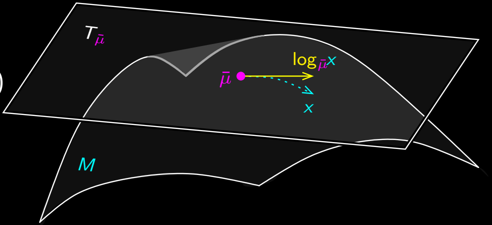

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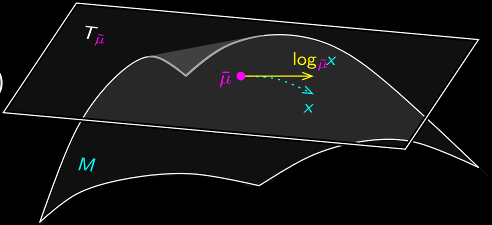


Def. The logarithm map is

$$\begin{aligned} \log_{\bar{\mu}} : M &\to T_{\bar{\mu}}M \\ x &\mapsto d(\bar{\mu}, x)V, \end{aligned}$$

where V = unit tangent to geodesic from $\bar{\mu}$ to x.

Note. M singular at $\bar{\mu} \Leftrightarrow T_{\bar{\mu}}M \ncong \mathbb{R}^d$

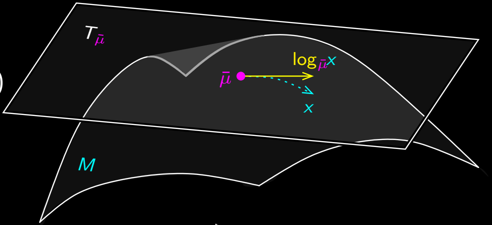
Prop. M smoothly stratified
$\Rightarrow$ $T_{\bar{\mu}}M$ is a smoothly stratified CAT(0) cone.

Fréchet means

Def. Probability distribution μ on any metric space M has Fréchet function

$$F_\mu(y) = \frac{1}{2}\int_M \underset{\substack{\uparrow \\ \text{square} \\ \text{distance}}}{d(x,y)^2}\,\underset{\substack{\uparrow \\ \text{measure} \\ \text{induced} \\ \text{by }\mu}}{\mu(dx)}$$

and Fréchet mean $\bar{\mu} = \underset{y \in M}{\operatorname{argmin}} F_\mu(y)$.



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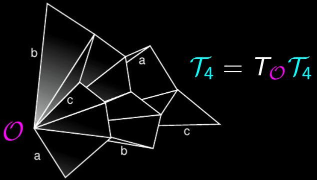

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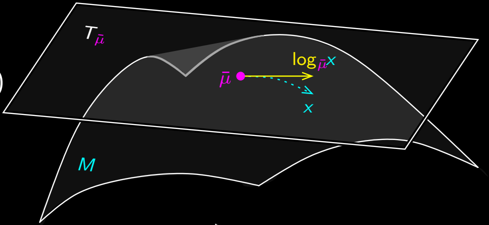

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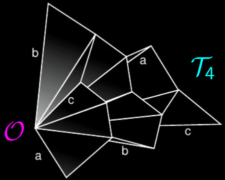

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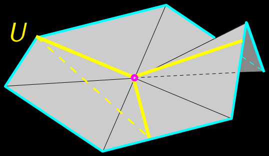

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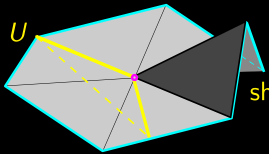

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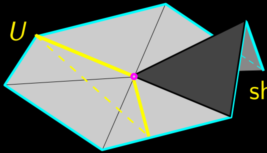

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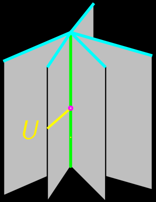

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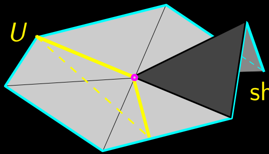

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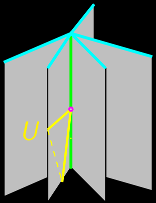

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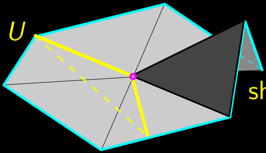

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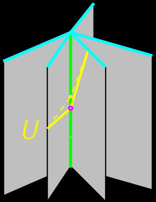

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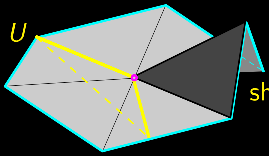

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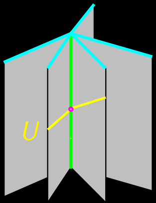

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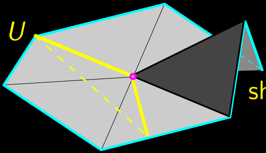

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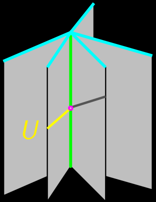

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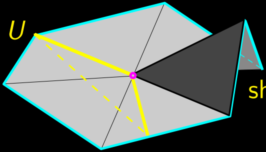

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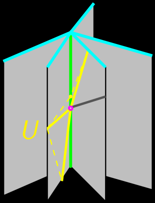

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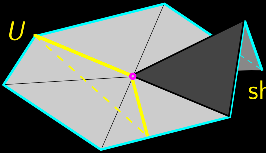

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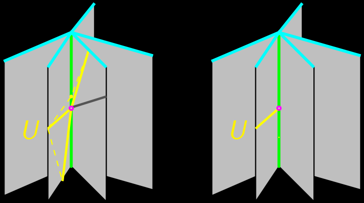

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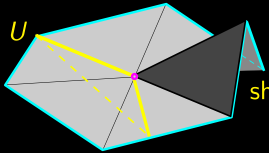

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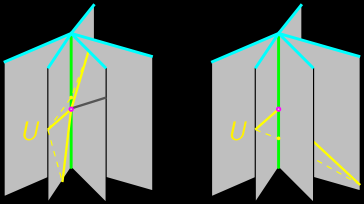

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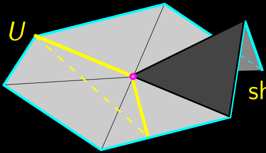

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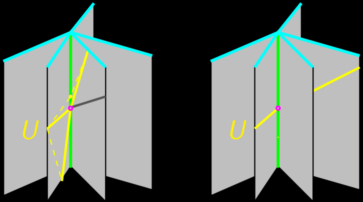

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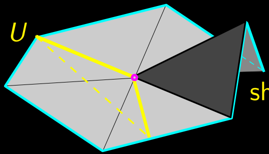

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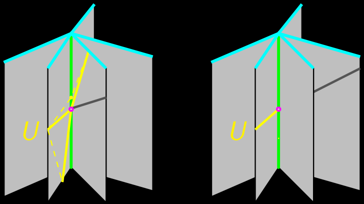

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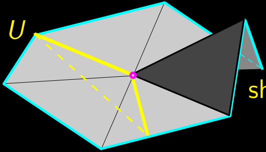

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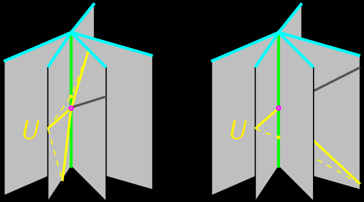

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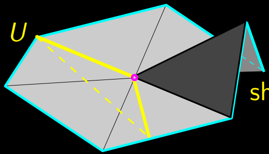

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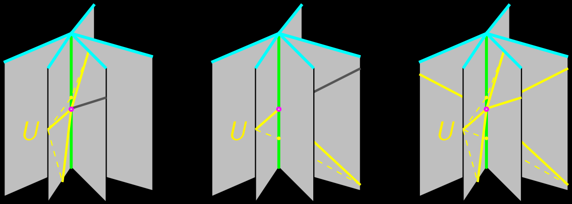

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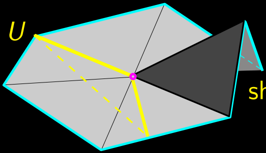

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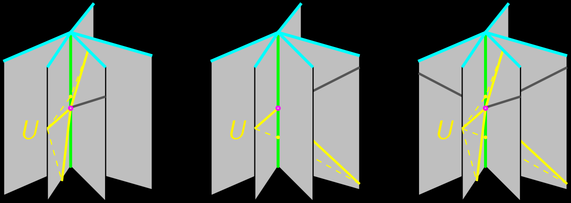

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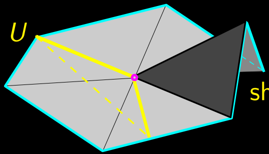

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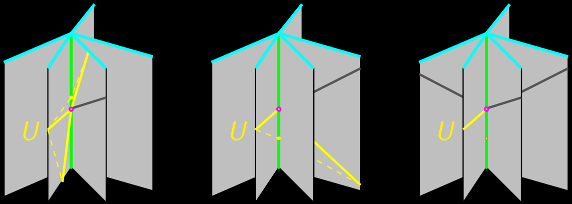

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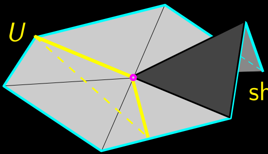

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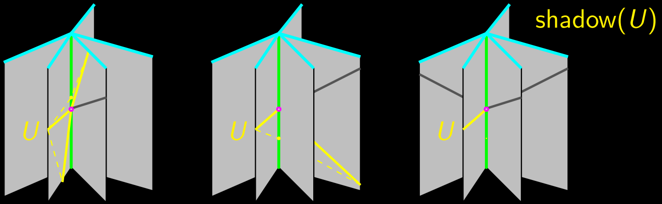

Random tangent fields

Def. A random tangent field on $T_{\bar{\mu}}M$ is a stochastic process $f : \Omega \times S_{\bar{\mu}}M \to \mathbb{R}$, so $f(V) : \Omega \to \mathbb{R}$ for each $V \in S_{\bar{\mu}}M$.

- Gaussian if $\big(f(V_1), \ldots, f(V_n)\big)$ is multivariate Gaussian $\forall\ V_1, \ldots, V_n \in S_{\bar{\mu}}M$
- covariance $\Sigma(U, V) = \mathbb{E}\big[f(U)f(V)\big]$

Def. An M-valued random variable $x = x(\omega) : \Omega \to M$ with law μ yields

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- covariance $\Sigma(U, V; \mu) = \mathbb{E}\Big[\big(g(x, U) - \mathbb{E}g(U)\big)\big(g(x, V) - \mathbb{E}g(V)\big)\Big]$, where $\mathbb{E}(\cdots) = \int_M \cdots d\mu$

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Def. Localized μ on smoothly stratified M has fluctuating cone

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Thm [Mattingly, M–, Tran 2023]**.** M smoothly stratified $\Rightarrow$ some sequence of limit log maps, followed by convex projection to the relevant smooth stratum, is a tangential collapse: a continuous map $\mathcal{L} : T_{\bar\mu}M \to \mathbb{R}^m$ that is

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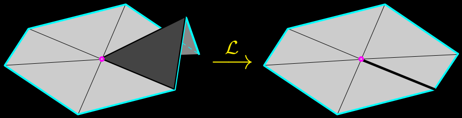
- kale: 

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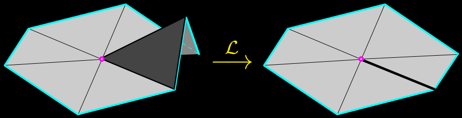
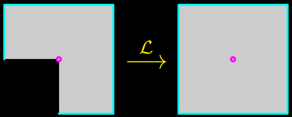
- kale: 

- nonconvex quadrants: 

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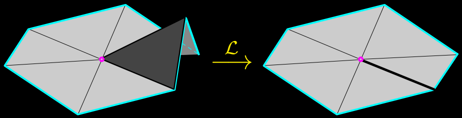

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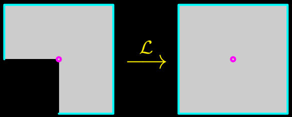
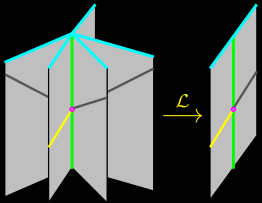


- open book:



Stratified Gaussians

Smooth M: $T_{\bar\mu}M \cong \mathbb{R}^m$ already

Singular M: use tangential collapse $T_{\bar\mu}M \xrightarrow{\mathcal{L}} \mathbb{R}^m$

Lemma. The map $\mathcal{L}$ has a measurable section over $\mathbb{R}^\ell = \mathrm{conv}(\mathrm{image}\,\mathcal{L})$,

$$\Delta : \mathbb{R}^\ell \to \text{discrete measures on } \mathbb{R}_+ \operatorname{supp}(\mu \circ \log_{\bar\mu}^{-1}) \subseteq T_{\bar\mu}M$$

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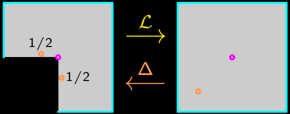
Example. 

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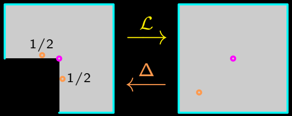
Example. 

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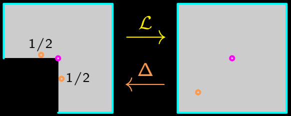

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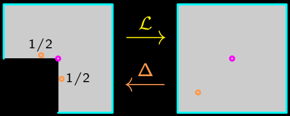

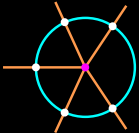

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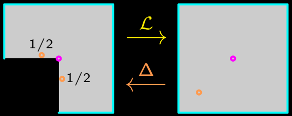
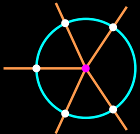
Example. 



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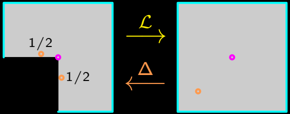
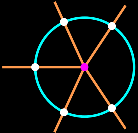
Example. 



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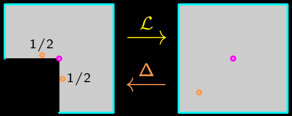
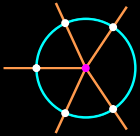
Example. 



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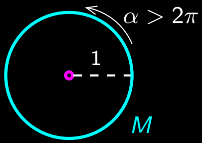

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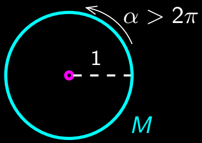

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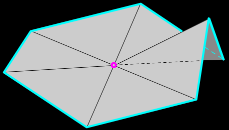
embedded in $\mathbb{R}^3$: 

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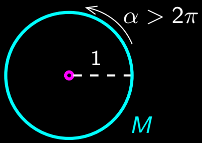

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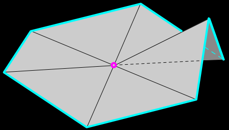

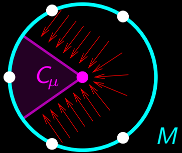

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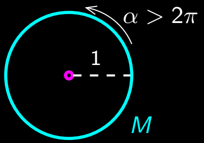

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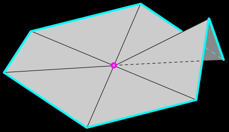

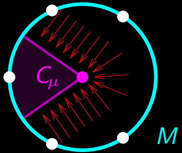

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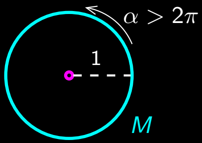

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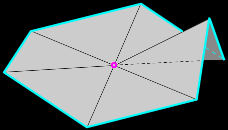
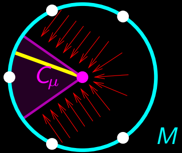
embedded in $\mathbb{R}^3$: 



- $\mathscr{E}$ is convex projection to the fluctuating cone $C_\mu = \{V \in \operatorname{hull}\mu \mid \nabla_{\bar\mu}F(V) = 0\}$

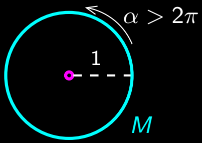
Escape vectors

Def. Fix $\Delta = \lambda_1\delta_{Y^1} + \cdots + \lambda_j\delta_{Y^j}$, a discrete measure on $T_{\bar\mu}M$. If $\delta = \lambda_1\delta_{y^1} + \cdots + \lambda_j\delta_{y^j}$ with $Y^i = \log_{\bar\mu} y^i$ then Δ has escape vector

$$\mathscr{E}(\Delta) = \lim_{t\to 0} \frac{1}{t}\log_{\bar\mu}(\overline{\mu + t\delta})$$

Note. Γ_μ is a random discrete measure of the form Δ.

Example [Huckemann, Mattingly, M–, Nolen 2015]



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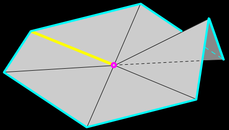

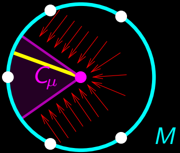

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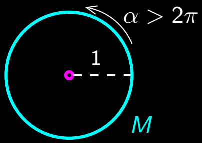

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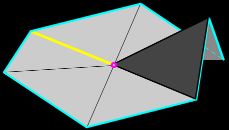

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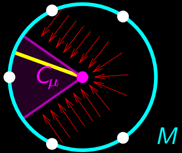

Distortion

Perturbative CLT [Mattingly, M–, Tran 2023]. $\lim_{n\to\infty} \sqrt{n}\log_{\bar{\mu}} \bar{\mu}_n \xrightarrow{d} \mathscr{E}(\Gamma_\mu)$

Def. The distortion map is

$$\mathcal{H} = \mathscr{E} \circ \Delta : \mathbb{R}^\ell \to T_{\bar{\mu}} M$$

Prop. Distortion $\mathcal{H}$ does not depend on choice of section Δ

Geometric CLT [Mattingly, M–, Tran 2023]. $\lim_{n\to\infty} \sqrt{n}\log_{\bar{\mu}} \bar{\mu}_n \xrightarrow{d} \mathcal{H}_\sharp N(0,\Sigma)$

Cor. Smooth CLT [Bhattacharya and Patrangenaru 2003, 2005], etc., where

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is the inverse Hessian of the Fréchet function

Note. Hessian not defined in singular settings, but inverse Hessian is

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Central limit theorems

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Variational CLT in a space of measures

CLT 3 [Mattingly, M–, Tran 2023]. $\lim\limits_{n\to\infty} \sqrt{n} \log_{\bar{\mu}} \bar{\mu}_n = \nabla_\mu \mathfrak{b}(\Gamma_\mu)$,

- the directional derivative, in the space $\mathcal{P}_2 M$ of L^2 measures on M,
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- at μ
- along any Gaussian tangent mass Γ_μ

Variational CLT in a space of functions

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