

# COMBINATORICS OF RANK JUMPS IN SIMPLICIAL HYPERGEOMETRIC SYSTEMS

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ABSTRACT. Let  $A$  be an integer  $d \times n$  matrix, and assume that the convex hull  $\text{conv}(A)$  of its columns is a simplex of dimension  $d - 1$ . It is known that the semigroup ring  $\mathbb{C}[\mathbb{N}A]$  is Cohen–Macaulay if and only if the rank of the GKZ hypergeometric system  $H_A(\beta)$  equals the normalized volume of  $\text{conv}(A)$  for all complex parameters  $\beta \in \mathbb{C}^d$  [Sai02]. Our refinement here shows that  $H_A(\beta)$  has rank strictly larger than the volume of  $\text{conv}(A)$  if and only if  $\beta$  lies in the Zariski closure (in  $\mathbb{C}^d$ ) of all  $\mathbb{Z}^d$ -graded degrees where the local cohomology  $\bigoplus_{i < d} H_m^i(\mathbb{C}[\mathbb{N}A])$  is nonzero.

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