Binomial Exercises

Lecture I. Affine semigroups and prime binomial ideals

1. Let ω be a primitive cube root of unity, so $\omega^3 = 1$ and $\omega \neq 1$. Find a character $\rho : L \to \mathbb{C}^*$ and $J \subseteq \{1, \ldots, 5\}$ such that $\langle bc - \omega^2 ad, b^2 - \omega ac, c^2 - \omega bd, e\rangle = I_{\rho,J} \subseteq \mathbb{C}[a, b, c, d, e]$, where

$$L = \text{ column-span of } \begin{bmatrix} -1 & -1 & 0 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

2. Exhibit a point configuration A whose affine semigroup $\mathbb{N}A$ does not consist of the intersection of the lattice $\mathbb{Z}A$ spanned by the columns of A with the real cone generated by A.

3. Show that the affine variety of the semigroup ring $\mathbb{C}[\mathbb{N}A]$ for

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

is the affine cone over a product $\mathbb{P}^1 \times \mathbb{P}^1$ of two projective lines. What does this (toric) geometry have to do with the geometry and combinatorics of a square?

4. Let A be any matrix for the quotient of $\mathbb{Z}^6$ modulo the sublattice $L \subseteq \mathbb{Z}^6$ spanned by

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}.$$

Show that the real cone $\mathbb{R}_{\geq 0}A$ generated by the affine semigroup $\mathbb{N}A$ is the cone over a triangular prism. Use any method you can think of (perhaps with the aid of a computer) to write down generators for the toric ideal I_A.

5. Let $L \subseteq \mathbb{Z}^n$ be an unsaturated sublattice; that is, $(\mathbb{R} \otimes_{\mathbb{Z}} L) \cap \mathbb{Z}^n \supsetneq L$.

   (i) Can $I_\rho \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be prime for some choice of character $\rho : L \to \mathbb{C}^*$?

   (ii) What would happen if $\mathbb{C}[\mathbf{x}]$ were replaced by $\Bbbk[\mathbf{x}]$ for an algebraically closed field $\Bbbk$ of positive characteristic?

   (iii) What if $\Bbbk$ is allowed to have arbitrary characteristic, but is not required to be algebraically closed?

Binomial Exercises

Lecture II. Primary binomial ideals

1. Compute the minimal monomial primary decomposition of the monomial ideal
$$I = \langle a^5, c^5, ab^2c^3d^4, a^2b^3c^4d, a^3b^4cd^2, a^4bc^2d^3\rangle \subseteq \mathbb{C}[a,b,c,d]$$
by first computing an (irredundant) irreducible decomposition. Does your answer change if you do the computation in $\mathbb{C}[a,b,c,d,e]$?

2. Let $B = \begin{bmatrix} 1 & -5 & 0 \\ -1 & 1 & -1 \\ 0 & 3 & 1 \end{bmatrix}$ and set $I = \langle a-b, bc^3-a^5, c-b\rangle$. Enumerate all of the connected components of the graph G_I with vertex set $\mathbb{N}^3$. Can a computer help you do this? Draw a picture of every component.

3. Fix $B = \begin{bmatrix} -1 & 2 \\ 2 & -1 \\ -1 & -2 \end{bmatrix}$ and $I = I(B) = \langle b^2 - ac, a^2 - bc^2\rangle$. Set $J = \{3\}$ and $\bar{J} = \{1,2\}$.

   (i) What are all of the components of the graph on $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}} = \mathbb{N}^2 \times \mathbb{Z}$ determined by I? (Hint: Project every component onto $\mathbb{N}^2$ and try to draw it there.)

   (ii) Determine every $u \in \mathbb{N}^3$ that lies in a finite component of $\mathbb{N}^2 \times \mathbb{Z}$.

4. In the semigroup ring $\mathbb{C}[Q] = \mathbb{C}[a,b,c,d]/\langle ad - bc\rangle$ for the semigroup $Q = \mathbb{N}A$, where
$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$
prove that the principal monomial ideal generated by $a^ib^jc^kd^\ell$ has primary decomposition
$$\langle a^ib^jc^kd^\ell\rangle = \langle a,b\rangle^{i+j} \cap \langle c,d\rangle^{k+\ell} \cap \langle a,c\rangle^{i+k} \cap \langle b,d\rangle^{j+\ell}.$$

5. Find a pointed affine semigroup $Q \subseteq \mathbb{Z}^2$ and a monomial $m \in \mathbb{C}[Q]$ such that the unique maximal (proper) monomial ideal is an associated prime of the principal ideal $\langle m\rangle \subseteq \mathbb{C}[Q]$. Make sure your semigroup Q has rank 2 (contains a pair of linearly independent vectors).

Binomial Exercises

Lecture III. Binomial primary decomposition and Combinatorial Games

1. Use everything we learned in Lectures I, II, and III to describe the combinatorics of the entire primary decomposition of the ideal $\langle xz - yz, x^2 - x^3 \rangle \subseteq \mathbb{C}[x, y, z]$.

2. (i) Find a binomial prime $I_{\rho,J}$ and an irreducible $I_{\rho,J}$-primary ideal that is not binomial.

   (ii)* Characterize the irreducible $I_{\rho,J}$-primary binomial ideals in $\mathbb{C}[x_1, \ldots, x_n]$.

3. Fix $I = \langle b^2 - ac, a^2 - bc^2 \rangle$. Let $I_{\rho,J} = \langle a, b \rangle$, so that $J = \{3\}$ and $\rho : L \to \mathbb{C}^*$ for the lattice $L = \{0\}$. Note that $I_\rho = \langle 0 \rangle$. On Tuesday you found all of the equivalence classes on $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}} = \mathbb{N}^2 \times \mathbb{Z}$ determined by I, and you determined every $u \in \mathbb{N}^3$ that lies in a finite equivalence class on $\mathbb{N}^2 \times \mathbb{Z}$.

   (iii) Use this information to down the $I_{\rho,J}$-primary component of I.

   (iv) What is the primary decomposition of I?

   A computer could give you hints by finding the answers to (iii) and (iv) directly. (A free beer to anyone who can get a computer to find the answers to (i) and (ii) directly, without using the combinatorial primary decomposition theorem. A published paper to anyone who can write down an algorithm to do it in general.)

4. Fix an affine semigroup Q.

   (a) Characterize all binomial prime ideals in the affine semigroup ring $\mathbb{C}[Q]$

   (b) For an arbitrary binomial prime ideal $\mathfrak{p} \subseteq \mathbb{C}[Q]$ minimal over a binomial prime ideal $I \subseteq \mathbb{C}[Q]$, provide a combinatorial characterization of the $\mathfrak{p}$-primary component of I.

5.* Let $\Bbbk$ be a field of characteristic $p > 0$. Assuming that $I_{\rho,J}$ is minimal over a binomial ideal $I \subseteq \Bbbk[\mathbf{x}]$, find a combinatorial characterization of the $I_{\rho,J}$-primary component of I. (It is known [Eisenbud & Sturmfels, *Binomial ideals*, Duke Math. J. **84** (1996), no. 1, 1–45] that this primary component is a binomial ideal.)

6. Find a winning strategy for misère Nim.

7. Calculate the misère quotient monoids for normal play Nim and ordinary misère play Nim with heaps of size at most 2.

8. Provide a complete expression of the rule set for the lattice game representing Dawson's chess played on collections of 3-row chess boards of width at most d.