Local cohomology has finite Bass numbers over Stanley–Reisner rings

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Here are some thoughts about finiteness of Bass numbers of local cohomology modules over Stanley-Reisner rings with support in $\mathbb{N}^n$-graded ideals. Our Čech hull machinery, for which we prove a change of rings statement, is directly and easily applicable to the following theorem. (To get the basic idea of the proof, read Proposition 6 first.) All modules and functors in what follows are $\mathbb{Z}^n$-graded.

Theorem 1. *Let $R = S/J$ be the quotient of the polynomial ring S by the monomial ideal J (not necessarily squarefree). If $I \subset R$ is an ideal and M is a finitely generated R-module, then the Bass numbers (over R) of $H^i_I(M)$ are finite.*

Proof. The local cohomology modules $H^i_I(M)$ are modules over both R and S, since they can be calculated using the Čech complex. It follows from Corollary 5.3 in [HM] along with Lemma 2, below, that $H^i_I(M)(-\alpha)$ is fixed by $\check{C}_S$ for some $\alpha \in \mathbb{Z}^n$. Proposition 5.4 in [HM] implies that this $H^i_I(M)(-\alpha)$ has finitely generated $\mathbb{N}^n$-graded part. The theorem has been recast as a special case of Proposition 6. □

Alternatively, one can simply prove, using the methods for semigroup rings in [HM], that $H^i_I(M)(-\alpha)$ is R-straight for some α, and proceed from there.

Let Q be a semigroup and $\check{C}$ the Čech hull functor over some Q^{gp}-graded ring [HM]. For a module M over this ring, let $M_Q = \bigoplus_{q \in Q} M_q$ be the Q-graded part of M.

Lemma 2. *Suppose A is fixed by $\check{C}$. Then $A(-a)$ is fixed by $\check{C}$ for all $a \in Q$. Equivalently, $(\check{C}A')(-a) = \check{C}((\check{C}A')(-a))$ for all $a \in Q$ and all modules A'.*

Proof. The two claims are equivalent because a module is fixed by $\check{C}$ if and only if it is the Čech hull of something. We prove the second, using uniqueness of adjoints, for which it suffices to verify that $(\check{C}(\_))(-a)$ and $\check{C}(\check{C}(\_)(-a))$ are right adjoints to isomorphic functors. But their adjoints are, respectively, $((\_)(a))_Q$ and $((\_)_Q(a))_Q$, which are obviously isomorphic as functors. □

Let $\check{C}_S$ be the Čech hull over the polynomial ring S and $\check{C}_R$ the Čech hull over $R = S/J$. Use E_S and E_R to denote $\mathbb{Z}^n$-graded injective hulls over R and S.

Lemma 3. *If A is any R-module then $\check{C}_R(A) = \underline{\mathrm{Hom}}_S(R, \check{C}_S(A))$. In particular, $\underline{\mathrm{Hom}}_S(R, E_S(R/\mathfrak{p})) = E_R(R/\mathfrak{p})$ for all primes $\mathfrak{p}$ of R.*

Proof. $\check{C}_R$ is the right adjoint to taking $\mathbb{N}^n$-graded parts, denoted $(\_)_{\succeq 0}$. In other words, $\mathrm{Hom}_R(B_{\succeq 0}, A) = \mathrm{Hom}_R(B, \check{C}_R(A))$, where $\mathrm{Hom}(\_,\_)$ denotes homogeneous homomorphsims *of degree 0*. We have for all R-modules A and B,

$$\begin{aligned}
\mathrm{Hom}_R(B, \check{C}_R A) &= \mathrm{Hom}_R(B_{\succeq 0}, A) \\
&= \mathrm{Hom}_S(B_{\succeq 0}, A) \quad \text{since the } S\text{-action factors through } R \\
&= \mathrm{Hom}_S(B, \check{C}_S A) \\
&= \mathrm{Hom}_R(B, \underline{\mathrm{Hom}}_S(R, \check{C}_S A)) \quad \text{since the image of } B \text{ is killed by } J
\end{aligned}$$

This says that $\check{C}_R(\_)$ and $\underline{\mathrm{Hom}}_S(R, \check{C}_S(\_))$ are both right adjoint to $(\_)_{\succeq 0}$. Therefore, these two functors are isomorphic as functors on R-modules. The "in particular" follows from the fact that $E_S(R/\mathfrak{p}) = \check{C}_S(R/\mathfrak{p})$ and $E_R(R/\mathfrak{p}) = \check{C}_R(R/\mathfrak{p})$. □

More generally, the proof can be appropriately souped up to show $\underline{\mathrm{Hom}}_S(R, \check{C}_S M) = \check{C}_R \underline{\mathrm{Hom}}_S(R, M)$.

Definition 4. *The* $\mathbb{Z}^n$-graded regularity $\underline{\mathrm{reg}}(M)$ *of a finitely generated* S*-module* M *is the join of the Betti degrees of* M*.*

Equivalently, if $M \leftarrow \mathbb{F}_\bullet$ is a minimal free resolution and $\mathbb{F}^\bullet = \underline{\mathrm{Hom}}_S(\mathbb{F}_\bullet, S)$, then $\underline{\mathrm{reg}}(M)$ is the smallest vector α in $\mathbb{Z}^n$ such that $\mathbb{F}^\bullet(-\alpha)$ is $\mathbb{N}^n$-graded.

Proposition 5. *If* A *is an* S*-module that is fixed by* $\check{C}_S$*, then* $\underline{\mathrm{Hom}}_S(R, A)(-\alpha)$ *is fixed by* $\check{C}_S$ *for all* $\alpha \succeq \underline{\mathrm{reg}}(R)$*.*

Proof. By Lemma 2, below, we may assume $\alpha = \underline{\mathrm{reg}}(R)$. Now use the fact that $\underline{\mathrm{Hom}}_S(R, A) = H^0 \underline{\mathrm{Hom}}_S(\mathbb{F}_\bullet, A) = H^0(\mathbb{F}^\bullet \otimes_S A)$, where $\mathbb{F}_\bullet$ is a minimal free resolution of R over S and $\mathbb{F}^\bullet = \underline{\mathrm{Hom}}_S(\mathbb{F}_\bullet, S)$. By definition of $\underline{\mathrm{reg}}(R)$, the complex $\mathbb{F}^\bullet \otimes A$ is, as a module, a direct sum of *nonnegative* shifts of A. Thus Lemma 2 says that this complex is fixed by $\check{C}_S$. Exactness of $\check{C}_S$ completes the proof. □

Proposition 6. *If* H *is an* R*-module fixed by* $\check{C}_S$ *and having* $H_{\succeq 0}$ *finitely generated, then the Bass numbers of* H *over* R *are finite.*

Proof. The injective hull $E_S(H)$ is fixed by $\check{C}_S$, because applying $\check{C}_S$ to the inclusion $H \hookrightarrow E_S(H)$ yields an inclusion $H \hookrightarrow \check{C}_S E_S(H)$ in which any purported indecomposable summands of $E_S(H)$ not fixed by $\check{C}_S$ are erased [Lemma 3.2, HM]. Since H is an R-module, the inclusion $H \hookrightarrow E_S(H)$ factors through $\underline{\mathrm{Hom}}_S(R, E_R(H))$, which equals $E_R(H)$ by Lemma 3. It follows that the zeroth Bass numbers of H are finite.

Proposition 5 implies that $E_R(H)(-\underline{\mathrm{reg}}(R))$ is fixed by $\check{C}_S$, while Proposition 5.4 of [HM] implies that its $\mathbb{N}^n$-graded part is finitely generated. By Lemma 2 and the same proposition in [HM], the same holds for $H(-\underline{\mathrm{reg}}(R))$. Therefore $(E_R(H)/H)(-\underline{\mathrm{reg}}(R))$ is fixed by $\check{C}_S$ and has finitely generated $\mathbb{N}^n$-graded part, as well. The proof is complete by induction on the cohomological degree. □

We believe our proof can be made (without much trouble) to work with "polynomial ring" replaced by "simplicial semigroup ring". Observe, for instance, that both Lemmas are completely general for the Čech hull, and both Propositions have reformulations in terms of straight modules.

References

[HM00] David Helm and Ezra Miller, *Bass numbers of semigroup-graded local cohomology*, Preprint (math.AG/0010003), 2000.