HOMOLOGICAL METHODS FOR HYPERGEOMETRIC FAMILIES

LAURA FELICIA MATUSEVICH, EZRA MILLER, AND ULI WALTHER

Journal of the American Mathematical Society **18** (2005), no. 4, 919–941.

In the late 1980s, Gelfand, Graev, and Zelevinsky introduced a class of systems of linear partial differential equations that are closely related to toric varieties [GGZ87]. These systems, now called *GKZ systems*, or A-*hypergeometric systems*, are constructed from a $d \times n$ integer matrix A of rank d and a complex parameter vector $\beta \in \mathbb{C}^d$, and are denoted by $H_A(\beta)$. We assume that the columns of A lie in a single open halfspace. A-hypergeometric systems arise in various instances in algebraic geometry. For example, solutions of A-hypergeometric systems appear as toric residues [CDS01], and special cases are mirror transforms of generating functions for intersection numbers on moduli spaces of curves [CK99], the A-hypergeometric systems there being Picard–Fuchs equations governing the variation of Hodge structures for Calabi–Yau toric hypersurfaces.

The first fundamental results about the systems $H_A(\beta)$ were proved by Gelfand, Graev, Kapranov, and Zelevinsky. These results concerned the case where the semigroup $\mathbb{N}A$ generated by the columns of A gives rise to a semigroup ring $\mathbb{C}[\mathbb{N}A]$ that is Cohen–Macaulay and graded in the standard $\mathbb{Z}$-grading [GGZ87, GKZ89]. In geometric terms, the associated projective toric variety X_A is arithmetically Cohen–Macaulay. The above authors showed that, in this case, the system $H_A(\beta)$ gives a holonomic module over the ring D of polynomial $\mathbb{C}$-linear differential operators in n variables, and hence $H_A(\beta)$ has finite *rank*; that is, the dimension of its space of holomorphic solutions is finite. Furthermore, they showed that this dimension can be expressed combinatorially, as the integer $\mathrm{vol}(A)$ that is $d!$ times the Euclidean volume of the convex hull of the columns of A and the origin $0 \in \mathbb{Z}^d$. The remarkable fact is that their rank formula holds independently of the parameter β.

Even if $\mathbb{C}[\mathbb{N}A]$ is not Cohen–Macaulay or $\mathbb{Z}$-graded, Adolphson showed in [Ado94] that $H_A(\beta)$ is always a holonomic ideal. He further proved that, for all parameters outside of a closed locally finite arrangement of countably many 'semi-resonant' affine hyperplanes, the characterization of rank through volume is still correct.

It came as quite a surprise when in [ST98] an example was given showing that if $\mathbb{C}[\mathbb{N}A]$ is not Cohen–Macaulay then not all parameters β have to give the same rank. Thus the set $\mathcal{E}_A$ of *exceptional parameters* $\beta \in \mathbb{C}^d$, for which the rank does not take the expected value, can be nonempty. Nearly at the same time, the case of projective toric curves was discussed completely in [CDD99]: the set $\mathcal{E}_A$ of exceptional parameters is finite in this case, and empty precisely when $\mathbb{C}[\mathbb{N}A]$ is Cohen–Macaulay; moreover, at each $\beta \in \mathcal{E}_A$ the rank exceeds the volume by exactly 1.

It was shown soon after in [SST00] that the rank can never be smaller than the volume as long as $\mathbb{C}[\mathbb{N}A]$ is $\mathbb{Z}$-graded, and it was established in the same book that $\mathcal{E}_A$ is in fact contained in a finite affine subspace arrangement. More recently, the much stronger fact emerged that $\mathcal{E}_A$ is a finite union of Zariski locally closed sets by means of Gröbner basis techniques [Mat03]. While rank-jumps can be arbitrarily large [MW04], the absence of rank-jumping parameters is equivalent to the Cohen–Macaulay property for $\mathbb{Z}$-graded $\mathbb{C}[\mathbb{N}A]$ when either $\mathbb{C}[\mathbb{N}A]$ has codimension two [Mat01], or if the convex hull of A is a simplex [Sai02], or if $\mathbb{C}[\mathbb{N}A]$ is a polynomial ring modulo a generic toric ideal [Mat03].

Encouraged by these results, which suggest an algebraic structure on the set of exceptional parameters, it was conjectured in [MM05] that the obstructions to the Cohen–Macaulayness of $\mathbb{C}[\mathbb{N}A]$ and the set of exceptional parameters are identified in an explicit manner. To be precise, let $H^{<d}_{\mathfrak{m}}(\mathbb{C}[\mathbb{N}A])$ be the direct sum of all the local cohomology modules supported at the maximal homogeneous ideal $\mathfrak{m}$ of $\mathbb{C}[\mathbb{N}A]$ in cohomological degrees less than d. Then define the set E_A of *exceptional quasi-degrees* to be the Zariski closure in $\mathbb{C}^d$ of the set of $\mathbb{Z}^d$-graded degrees α such that $H^{<d}_{\mathfrak{m}}(\mathbb{C}[\mathbb{N}A])$ has a nonzero element in degree $-\alpha$. With this notation, our motivating result is the following.

Theorem. *For any rank d matrix $A \in \mathbb{Z}^{d\times n}$ as above, the set $\mathcal{E}_A$ of exceptional (that is, rank-jumping) parameters is identical to the set E_A of exceptional quasi-degrees.*

We note that there is no assumption on $\mathbb{C}[\mathbb{N}A]$ being $\mathbb{Z}$-graded. The $\mathbb{Z}$-graded simplicial case of this result was proved in [MM05] using results of [Sai02].

Methods and results. The first step in our proof of the Theorem is to construct a homological theory to systematically detect rank-jumps. To this end, we study rank variation in any family of holonomic modules over any base B, and not just A-hypergeometric families over $B = \mathbb{C}^d$. The idea is that under a suitable coherence assumption, holonomic ranks behave like fiber dimensions in families of algebraic varieties. In particular, we prove that rank is constant almost everywhere and can only increase on closed subsets of B. We develop a computational tool to check for rank-jumps at a smooth point $\beta \in B$: since the rank-jump occurs through a failure of flatness at β, ordinary Koszul homology detects it.

The second step toward the Theorem is to construct a homological theory for D-modules that reproduces the set E_A of exceptional quasi-degrees, which a priori arises from the commutative notion of local cohomology. Our main observation along these lines is that the *Euler–Koszul complex*, which was already known to Gelfand, Kapranov, and Zelevinsky for Cohen–Macaulay $\mathbb{Z}$-graded semigroup rings [GKZ89], generalizes to fill this need. Adolphson [Ado99] recognized that when the semigroup is not Cohen–Macaulay, certain conditions guarantee that this complex has zero homology. Here, we develop *Euler–Koszul homology* for the class of *toric modules*, which are slight generalizations of $\mathbb{Z}^d$-graded modules over the semigroup ring $\mathbb{C}[\mathbb{N}A]$. For any toric module M, we show that the set of parameters β for which the Euler–Koszul complex has nonzero higher homology is precisely the analogue for M of the exceptional quasi-degree set E_A defined above for $M = \mathbb{C}[\mathbb{N}A]$.

Having now two cohomology theories, one being a D-module theory to recover local cohomology quasi-degrees for hypergeometric families, and the other being a geometric theory to detect rank-jumping parameters for general holonomic families, we link them in our central result: for toric modules, these two theories coincide. Consequently, we obtain our motivating Theorem as the special case $M = \mathbb{C}[\mathbb{N}A]$ of a result that holds for arbitrary toric modules M. From there, we deduce the equivalence of the Cohen–Macaulay condition on $\mathbb{C}[\mathbb{N}A]$ with the absence of rank-jumps in the GKZ hypergeometric system $H_A(\beta)$.

As a final comment, let us note that we avoid the explicit computation of solutions to hypergeometric systems. This contrasts with the previously cited constructions of exceptional parameters, which rely on combinatorial techniques to produce solutions. It is for this reason that these constructions contained the assumption that the semigroup ring $\mathbb{C}[\mathbb{N}A]$ is graded in the usual $\mathbb{Z}$-grading, for this implies that the corresponding hypergeometric systems are *regular holonomic* and thus have solutions expressible as power series with logarithms, with all the combinatorial control this provides. Our use of homological techniques makes our results valid in both the regular and non-regular cases.

REFERENCES

[Ado94] Alan Adolphson, *Hypergeometric functions and rings generated by monomials*, Duke Math. J. **73** (1994), no. 2, 269–290.

[Ado99] Alan Adolphson, *Higher solutions of hypergeometric systems and Dwork cohomology*, Rend. Sem. Mat. Univ. Padova **101** (1999), 179–190.

[CDD99] Eduardo Cattani, Carlos D'Andrea, and Alicia Dickenstein, *The A-hypergeometric system associated with a monomial curve*, Duke Math. J. **99** (1999), no. 2, 179–207.

[CDS01] Eduardo Cattani, Alicia Dickenstein, and Bernd Sturmfels, *Rational hypergeometric functions*, Compositio Math. **128** (2001), no. 2, 217–239.

[CK99] David Cox and Sheldon Katz, *Mirror symmetry and algebraic geometry*, Amer. Math. Soc., Providence, RI, 1999.

[GGZ87] I. M. Gelfand, M. I. Graev, and A. V. Zelevinsky, *Holonomic systems of equations and series of hypergeometric type*, Dokl. Akad. Nauk SSSR **295** (1987), no. 1, 14–19.

[GKZ89] I. M. Gelfand, A. V. Zelevinsky, and M. M. Kapranov, *Hypergeometric functions and toric varieties*, Funktsional. Anal. i Prilozhen. **23** (1989), no. 2, 12–26. Correction in ibid, **27** (1993), no. 4, 91.

[Mat01] Laura Felicia Matusevich, *Rank jumps in codimension 2 A-hypergeometric systems*, J. Symbolic Comput. **32** (2001), no. 6, 619–641, Effective methods in rings of differential operators.

[Mat03] Laura Felicia Matusevich, *Exceptional parameters for generic A-hypergeometric systems*, Int. Math. Res. Not. (2003), no. 22, 1225–1248.

[MM05] Laura Felicia Matusevich and Ezra Miller, *Combinatorics of rank jumps in simplicial hypergeometric systems*, Proc. Amer. Math. Soc., to appear, 2005. `arXiv:math.AC/0402071`

[MW04] Laura Felicia Matusevich and Uli Walther, *Arbitrary rank jumps for A-hypergeometric systems through Laurent polynomials*, 2004. `arXiv:math.CO/0404183`

[Sai02] Mutsumi Saito, *Logarithm-free A-hypergeometric series*, Duke Math. J. **115** (2002), no. 1, 53–73.

[SST00] Mutsumi Saito, Bernd Sturmfels, and Nobuki Takayama, *Gröbner deformations of hypergeometric differential equations*, Algorithms and Computation in Mathematics, vol. 6, Springer-Verlag, Berlin, 2000.

[ST98] Bernd Sturmfels and Nobuki Takayama, *Gröbner bases and hypergeometric functions*, Gröbner bases and applications (Linz, 1998), London Math. Soc. Lecture Note Ser., vol. 251, Cambridge Univ. Press, Cambridge, 1998, pp. 246–258.